STRUCTURES AND GENERALISATION IN A FUNCTIONAL APPROACH: THE INVERSE FUNCTION BY FIFTH GRADERS

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STRUCTURES AND GENERALISATION IN A FUNCTIONAL APPROACH: THE INVERSE FUNCTION BY FIFTH GRADERS

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Against the backdrop of functional thinking to early algebra, this paper discusses an initial study of how 24 fifth year elementary school students (10 to 11 years old) perceived the inverse function. This notion has been scarcely tackled in the context of early algebra and, particularly, within the functional thinking approach. Based on structure and generalisation notions, we analysed the responses of a group of 24 students when solving a problem, which contains questions involving the direct and inverse forms of a problem involving a linear function. Ten of the 24 students were observed to establish structures involving inverse function and five to generalise that form of the function.

INTRODUCTION

Generalisation is a crucial element in research on algebraic thinking in early schooling (hereafter, early algebra). Some authors (Schifter, Monk, Russell, & Bastable, 2008) contend that children are naturally inclined to perceive and discuss regularity, which is the key to generalisation, even when they lack the resources needed to represent general relationships. The present study was conducted in the context of early algebra, to which generalisation and the way it is expressed are core aspects (Kaput, 2008).

Functional thinking is a vehicle for introducing algebra in the early years of schooling. This type of algebraic thinking focuses on the relationship between two or more covarying quantities. Relationships maybe identified for specific cases or in general (generalisation) (Smith, 2008). Given that functions constitute the prime mathematical content in functional thinking, some researchers recommend focusing on how students perceive both direct and inverse forms of functional relationships (Oehrtman, Carlson, & Thompson, 2008). Whilst several researchers are interested in generalisation at elementary school (e.g., Carraher & Schliemann, 2016; Pinto & Cañadas, 2017), very few studies have been published on how such students perceive and generalise inverse functions, the issue addressed in this article.

Regularities from a functional approach to early algebra have to do with the relationships between the dependent and independent variables involved in a given situation. Specifically, the notion of structure is associated with how regularity between variables is organised, as perceived by students through different representations when working with specific cases as well as when generalising. Some researchers note that before being able to generalise, students must 'see' the structure in a mathematical situation (Mason, Stephens, & Watson, 2009). Structure, although seldom dealt with in the context of functional thinking, provides a way to describe generalisation as

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engaged in by lower year students. Authors like Mulligan and Mitchelmore (2009) exemplify the notion of structure through rectangular grids represented in Figure 1.

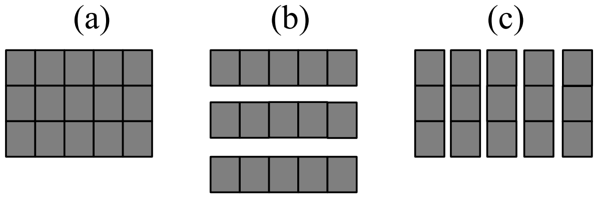


Figure 1: Rectangular grid perceived as (a) 3x5, (b) 3 rows of 5, (c) 5 columns of 3 (Mulligan, & Mitchelmore, 2009, p. 34)

Often students at elementary grades have difficulties identifying the pattern 3x5 in the three grids of Figure 1 because they are not able to recognize the implicit structure of three rows of five squares each. In this case, the way in which the squares are organized is one important characteristic of the structure.

In the present study about generalisation with fifth graders within a functional thinking approach, in Spain, we focus on the inverse function. According to MacGregor and Stacey (1995) inverse function poses greater difficulty than direct function to students, and is barely addressed in the literature (Carraher & Schliemann, 2016). Fifth year elementary school was chosen because whilst research has been conducted on the inverse function with older ages, it has not been studied in elementary school.

The general aim of the paper is to describe how fifth year students (10- to 11-year-olds) perceive the inverse function when working with a problem involving a linear function in the context of early algebra. The two specific aims pursued are: (a) to identify the structures detected by students; and (b) to describe the students' generalisation based on notion of structures.

FUNCTIONS

This study focuses on linear functions, a type recommended for elementary school students, for instance, $f(x) = mx+b$, in which constants m and b, as well as x and y are natural numbers (Carraher & Schliemann, 2016). A function is a rule that establishes a relationship between two variables, with the emphasis on how the changes in one are related to changes in the other (Thompson, 1994). The direct and inverse forms of a function are consequently related to the roles played by each variable involved. The independent variable in the direct form of the function is the dependent variable in the inverse form, and vice-versa.

Prior research on inverse functions has been conducted with post-elementary school students. MacGregor and Stacey (1995), for instance, explored how 143 14 to 15 years old perceived functional relationships in exercises involving direct and inverse functions. They reported 63 % of correct answers for the direct, but only 43 % for the indirect, function. The authors gave no information on how students perceived regularity when the inverse function was involved.

GENERALISATION AND STRUCTURES

Generalisation "involves deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly and exposing commonality across cases" (Kaput, 1999, p. 136). From the functional approach to early algebra, generalisation is related to the various ways students expressed a general functional relationship involving two variables. According to Radford (2002), we assume that generalisation at elementary grades can be expressed through different representations (natural language, numerical, tabular, for instance).

A pattern can be defined as a spatial or numerical regularity and its structure as the relationships among its components (Mulligan, Mitchelmore, & Prescott, 2006). The distinction between pattern and structure is considered to be pertinent here, for the former is associated more with recurrence than with the establishment of a functional relationship such as covariation between two quantities. From the functional thinking approach and further to the literature (such as Kieran, 1989), we assume here the notion of structure as the numbers and numerical variables (expressed via different representations), operations and their properties present when students identified a regularity. Previous authors have argued that to generalise, students must previously identify the structure of the relationships observed (Mason et al., 2009). Some studies have shown that when they identify structures in mathematical tasks, students experience mathematics more deeply (Mason et al., 2009).

Structure and generalisation were exemplified here in a problem involving the linear function $y=2x+6$. To find the number of grey tiles (g) that can be laid around a given row of white tiles (w) (see Küchemann, 1981), students might identify the relationship between variables as: (a) "three tiles on the right, three on the left, eight at the top and eight at the bottom" (specific case, eight white tiles); or (b) "double the number of tiles plus six" (general case). In the first, non-generalised description (a), the structure would be $3+3+x+x$ (if applied to more than one specific value) and in the second generalised response (b) it would be $2x+6$. These two structures are equivalent with the function given in the problem.

METHOD

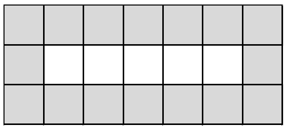
This study forms part of a broader teaching experiment on functional thinking in different years of elementary education, focused on fifth year students (10 to 11 years old).

Students and tools

A group of 24 Spanish 10 to 11 year olds participated. They had previously studied the four arithmetic operations with natural numbers, integer and rational number sets. Their only experience with the type of problems proposed here in the session at issue was during three prior sessions, in which they were introduced to functions involving addition, multiplication and both.

The fourth session was divided into three parts. In the first part, the students were shown the tiles problem (see Figure 2) and asked questions to ensure they understood it. In the second part, each student worked individually on the worksheets. In the third part, students shared orally their answers to some of the questions. This paper deals only with students' written answers in the worksheets.

A school wants to re-floor its corridors because they are in poor condition. Its administration decides to use a combination of white and grey tiles, all square and all the same size, to be laid as in the drawing.



The school contracts a company to re-floor the corridors. We want you to help the workers answer some questions before they get started.

Q1. How many grey tiles do they need for a corridor with 5 white tiles?

Q2. As some corridors are longer than others, the workers need a different number of tiles for each. How many grey tiles do they need for a corridor with 8 white tiles?

Q3. How many grey tiles do they need for a corridor with 10 white tiles?

Q4. How many grey tiles do they need for a corridor with 100 white tiles?

Q5. The workers always lay the white tiles first and then the grey tiles. How can they calculate how many grey tiles they need in a corridor where they've already laid the white ones?

Q6. In some corridors, the workers mistakenly laid the grey tiles before the white tiles. They laid 20 grey tiles. How many white tiles do they need?

Q7. In another corridor where they laid the grey tiles before the white, they laid 56 grey tiles. How many white tiles do they need?

Figure 2: The tiles problem

Questions Q1 to Q5 involved the direct form of the function (particular and general cases) and questions Q6 and Q7 the inverse form of the function (only particular cases).

Data analysis

Taking all the students' answers, structures were observed in the answers to all the questions (involving either specific cases or the general case). A structure was regarded to have been identified when the same student answered two or more questions with the same regularity or generalisation. In other words, identification of the structure consisted in the use of the same regularity in at least two questions involving the direct function (Q1-Q5) or of generalisation in one of those questions, and analogously for inverse function questions Q6 and Q7. Students' perception of the inverse function (Q6 and Q7) was explored in depth in this study.

The structures detected are represented here using algebraic symbolism, although some students used other representations. For instance, in Table 1 we show different possible students' answers and structures inferred from the answers.

Questions	Students' responses	Underlying structures
Q2 (8 white tiles)	(2·8)+3+3	2*x*+3+3
Q3 (10 white tiles)	(2·10)+3+3	2*x*+3+3
Q4 (100 white tiles)	(2·100)+3+3	2*x*+3+3
Q6 (20 grey tiles)	(20-6)/2	(*x*-6)/2
Q7 (56 grey tiles)	(56-6)/2	(*x*-6)/2

Table 1: Examples of possible students' responses

In Table 1 we observe that in Q2, Q3, and Q4 the structure was identified as the double of the number of tiles plus three on the right and three on the left, it is inferred here as 2*x*+3+3. However, in Q6 and Q7, students subtracted the tiles from both sides to the total number of grey tiles and divided this quantity by two, which can be inferred as (*x*-6)/2.

RESULTS

A total of 14 students showed no sign of detecting structures as defined in the preceding section. These students: (a) answered the question without giving any information about their used procedure; (b) merely copied the problem wording; (c) drew or referred to illustrations of the problem; (d) performed inconsistent operations; or (e) did not answer the question.

Structures were detected in the remaining 10 students' replies to the questions for both the direct and inverse forms of the function. In their answers to Q5, which involved the direct function, all 10 students generalised the function, while one student's answers to Q1 and Q2 were also a generalisation. The three structures identified in these students' answers, which appeared in the specific and the general case, were: 2*x*+6, 2*x*+3+3, and 2*x*+2. The first two structures matched the situation set out in the problem, but the third did not. The structure 2*x*+6, the most frequent, was identified by eight students.

Five of these 10 students generalised the inverse function structure when we asked for particular instances (Q6-Q7). Broadly speaking, four structures were inferred in their replies to questions Q6 and Q7: (*x*-6)/2; (*x*/2)-3-3; (*x*/2)-6 and (*x*/2)-2. A discussion of these inverse function structures and generalisation follows.

Particular cases of the inverse function

Among the 10 students who described structures in both direct and inverse function questions, five did not generalise in their answers to Q6 and Q7. Three inverse function structures were inferred in this group: (*x*-6)/2; (*x*/2)-6 and (*x*/2)-2. The first is correct with the problem, whereas the second and third are not.

Four students defined the structure (x-6)/2. Mario, is one of them, he answered Q6 (number of white tiles needed for 20 grey tiles) as follows: "they need seven tiles. You have to calculate backwards, which means (20-6):2=7". In this answer, the student took the total number of grey tiles, subtracted the ones on the sides (6) and then divided the remainder by two, to get seven. Like the other three students, Mario started from an identified structure of the direct function (2x+6) to describe the structure for the inverse function. In his response to Q7 (number of white tiles needed when 56 grey tiles are laid) Mario answered: "25 white tiles. (56-6):2=25".

Lara was the only student who described the structure (x/2)-2, which is not equivalent with the function describing the problem. She replied to Q6: "18 white tiles. You divide 10 by 2 and -2". She used the same structure for Q7, failing to consider the constant part of the function (6).

Generalisation of the inverse function

Five students generalised the inverse functional relationship when calculating the number of white tiles from the 20 grey tiles cited in Q6 and the 56 in Q7. The three generalised structures were: (x-6)/2, (x/2)-6 and (x/2)-3-3. A number of examples of students' answers follow.

Juan generalised in his reply to Q6: "(…) You have to subtract six tiles (from the sides) [and divide] by two". This student generalised the structure (x-6)/2. His reply to the next question (Q7) is shown in Figure 3.

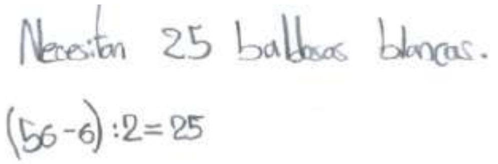


Figure 3: Juan's reply to Q7 (English translation of the first line:
'They need 25 white tiles').

In his reply (Figure 2), Juan applied the structure identified in Q6 and wrote the answer to the problem in natural language. He used symbolic-numerical representation for the specific case (56 grey tiles).

Two students generalised in both Q6 and Q7. In her answer to Q6, for instance, Ana reasoned: "(…) Dividing the grey tiles by two and subtracting the three at the beginning and the three at the end (…)". Here, irrespective of the specific case, the students divided the number of tiles in half and then subtracted the number of tiles on the right (3) and left (3). In this case generalisation was identified as (x/2)-3-3.

DISCUSSION AND CONCLUSION

The present findings contribute to the description of how elementary education students perceive the inverse function, an issue barely researched in the context of functional thinking (Carraher & Schliemann, 2016). By focusing on structure and generalisation, the study provides insight into how students interpret the relationships between variables.

Stacey and MacGregor (1995) explored secondary education students' perception of relationships in direct and inverse functions. Complementing that research by studying elementary education students, this study distinguishes between students who identified structures when working with specific cases and those who generalised and described the elements comprising those structures.

The 10 students who generalised the direct form of the function 'saw' the structure before answering Q5, confirming the ideas put forward by Mason et al. (2009). Only five students also generalised the inverse form of the function, inferring that generalisation is more difficult in this type of question than in the direct form, as contended by Stacey and MacGregor (1995).

In a study on direct functions, Pinto and Cañadas (2017) distinguished between fifth year elementary school students who generalised in questions referring to specific cases (spontaneous generalisation) and those who did so when confronting the general situation (prompted generalisation). Further to that distinction, generalisation observed here in the inverse function was systematically spontaneous, for both Q6 and Q7 refer to specific cases. The most common strategy for generalising the inverse function, implemented by four of the five students who did so, was to 'reverse' the structure generalised for the direct function ($2x+6$). In all five cases, the generalisation was equivalent with the inverse function set out in the problem. These results suggest that a new line of research might address the generalisation strategies used by students in problems involving inverse functions after working with the direct form.

Acknowledgements

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