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Teaching Mathematics through Problem Solving

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Resumen En este trabajo se ofrecen dos formas de combinar la enseñanza de las matemáticas y la resolución de problemas: enseñanza de las matemáticas para la resolución de problemas o enseñanza de las matemáticas a través de la resolución de problemas. Por medio de ejemplos de actividades se propone una enseñanza a través de la resolución de problemas.

Palabras clave Enseñanza matemáticas, resolución de problemas

Abstract This paper offers two ways to combine the teaching of mathematics and problem solving: teaching mathematics to solve problems or teaching mathematics through problem solving. By means of examples of activities, teaching through problem solving is proposed.

Keywords Teaching mathematics, Problem solving

There are two broad goals for teaching mathematics. One is the goal of introducing students to the fine intellectual cultural achievement of mathematics, and its beauty, elegance and intrigue. The other goal is that students will be able to use mathematics to solve problems, from simple arithmetic problems encountered in everyday life, to the most complex of problems from science, engineering or indeed within mathematics itself. Because problem solving relates to both the practical use of mathematics and to its intra-mathematical structure, learning to solve problems is the most important goal of mathematics teaching.

Helping students to become good problem solvers requires attention throughout the years of school. Good problem solvers need:

- A strong understanding of mathematics, and procedural skills
- Experience of solving problems, both routine and non-routine
- Knowledge of some general problem solving strategies
- Metacognitive ability to regulate the problem solving process
- A productive disposition, with the confidence to try things out and deal with the emotions of not having immediate success.

How are these skills developed? There are two essential components: teaching FOR problem solving and teaching THROUGH problem solving.

Teaching for problem solving presents students with problems to solve using any mathematics that they know. In a supportive environment, students tackle these problems, and through this they

X ESCUELA DE EDUCACIÓN MATEMÁTICA
MIGUEL DE GUZMÁN

broaden their experience of problems of many types, learn important general strategies for problem solving, get some understanding of the metacognitive aspects of problem solving (knowing its phases and pitfalls etc), and see how a solution to one problem often leads to the posing of many related problems. It is essential that students experience some teaching for problem solving.

Teaching through problem solving refers to the teaching of ordinary mathematical content by having students work on problems. It has the aim of meeting standard curriculum requirements from the national syllabus, but meeting them in a deeper way. It also aims to increase students' ability to think mathematically, including their capacity to ask mathematical questions, solve problems and learn to conduct inquiry using mathematics. Because it is integrated with the teaching of content, teaching through problem solving can make a contribution to developing problem solving skills every day.

Learning mathematics is a long journey for students. Many of the parts of this journey can be achieved by teaching through problem solving. Using the metaphor of a journey, this article presents four different ways of teaching through problem solving, from a wayside stop taking only a few minutes to whole units of work that cross the mountains and the plains.

WAYSIDE STOP (a few minutes)

Teachers who approach mathematics with a spirit of inquiry will often see opportunities to open even routine exercises to a problem-solving approach. It is like pausing on a long journey to look out at a scenic view or to taking a quick side track to see a beautiful waterfall. Students' own observations and questions often initiate wayside stops.

Example 1: Sets and subsets

In a very first introduction to the notation of sets and the definition of subset, the textbook asked Year 7 students to list all the subsets of several sets, including {a, b} and {a, b, c}. While reviewing the answers and without making any special comment, the teacher organised the subsets of the board to highlight how each subset of {a, b} produces two subsets of {a, b, c} as shown in Figure 1. By looking at the list of subsets, most students observed how they could make all the subsets of the three-element set from the subsets of the two-element set with and without the "c" included. This made their work more systematic. They felt a sense of achievement as they generalised to four-element subsets from three-element sets. One student observed that this means that the number of subsets doubles when another element is added and all the students checked that this was true with the examples that they had from the exercise. After a few minutes, the class moved on to the next exercise, already primed for some of the learning to come.

Subsets of [a, b]	Subsets of {a, b, c}	
{a}	{a}	{a, c}
{b}	{b}	{b, c}
{a, b}	{a, b}	{a, b, c}
{ }	{ }	{c}

Figure 1. Organising subsets to highlight relationships

Example 2: Patterns in Multiplication Tables

Students in Year 2 were learning the 6 times multiplication table. A student noted that the first answers [6, 12, 18, 24, 30] are numbers from the 3 times multiplication table, but the others are not (see Figure 2).

	× 1	× 2	× 3	× 4	× 5	× 6	× 7	× 8	× 9	× 10
3	3	6	9	12	15	18	21	24	27	30
6	6	12	18	24	30	36	42	48	54	60

Figure 2. Patterns in multiplication tables

Through discussion and manipulating groups of 3 counters, the teacher and class discuss how:

- a group of 6 is two groups of 3; 2 groups of 6 is equal to 4 groups of 3; 3 groups of 6 is equal to 6 groups of 3, etc. The repeated numbers in the 3 and 6 times tables (e.g. 6, 12, 18) reflect this.
- the 3 times table can be extended beyond 3 × 10 by successively adding 3. Some students did this and they saw that the larger numbers in the 6 times table did indeed appear there.
- this feature occurs in some other multiplication tables. Some students made a generalisation and tested it with another pair of multiplication tables (e.g. I will see if the numbers in the 8 times tables are from the 4 times tables).

The teacher returned to the lesson, reminding the class how patterns can help them learn multiplication tables.

About wayside stops

Wayside stops are unplanned, so a teacher has to decide in the moment on the potential mathematical richness and whether the class as a whole or perhaps just some individuals will benefit from it. This requires deep pedagogical content knowledge. The teacher must also balance time that might be spent against the time pressures of the day's agenda. On the other hand, wayside stops have many advantages, especially when these mini-investigations arise from the spontaneous observations of students. This sends a strong message that students can find interesting mathematical properties for themselves if they look for them, and it promotes problem posing. Listening to students 'discoveries' gives insight into how they see mathematics and what further teaching they need. Wayside stops can add a spirit of inquiry to any curriculum.

A DAY TRIP (one lesson)

On a long journey, it is good to plan some special side trips to important sites, perhaps to see an ancient city or a fabulous volcano. They add something special to the main journey. The day trips for teaching through problem solving use carefully engineered problems to promote mathematical inquiry and understanding of target knowledge.

Example 3: Prime Dice Game

In Year 7 in the Australian Curriculum: Mathematics, students are to learn how to represent whole numbers as products of powers of prime numbers. The prime dice game was designed for reSolve: Mathematics by Inquiry (www.resolve.edu.au) as part of a lesson show why prime factorisation is useful (i.e. the factors show some of the properties of the number) and to point students towards informally appreciating the uniqueness of prime factorisation. It is played like the game Yahtzee with four dice. One every face, there is one of the four prime numbers {2, 3, 5, 7}. For example, one dice might have faces labelled 2, 3, 3, 5, 7, 7. The rules are given in Figure 3.

At the beginning, students will probably multiply out the numbers to see if the product fits in a category (e.g. even), so it is good if they have a calculator available so they can work quickly. But they should later be able to look at the numbers thrown (recorded on the accompanying worksheet) to see the properties of the product without calculation, including:

- one way (later seen to be the only way) to get an even number is if there is a factor of 2;
- one way (later the only way) to get a fourth power is if there are four identical factors;
- the only way to get multiple of ten is to have a 2 and a 5.

In playing this game, students are presented with many small problems to solve, and the motivation to solve them. An example of the thinking required is provided in Figure 4. Teachers need to consolidate the learning from the game by encouraging students to discuss their strategies and their reasoning, to critique the reasoning of others, and to encourage students to make and test conjectures.

About mathematical day trips

There are many different types of mathematical day trips. The Prime Dice Game immerses students in a mathematical situation (in this case, products of primes) and a purpose for finding regularities. There is the opportunity to make conjectures and test them, although it is likely that these conjectures will initially be very specific. Class discussion is needed to reinforce the important principles that emerge, as students will probably struggle to articulate their discoveries and may not see the generality of what they find.

Perhaps a more common type of mathematical day trip is setting students to solve a challenging problem: for teaching through problem solving it may prepare the way for a main idea or it may provide opportunities for students to use the idea in a novel way. This can be used to introduce or consolidate concepts, to develop fluency, and to develop strategic thinking, as well as to deepen knowledge that students may have learned superficially. For problem solving like this to be effective, students need a supportive classroom environment, problems that are accessible at some level to all students, and structured reflection so that they learn from the experience.

re(Solve) MATHS BY INQUIRY

Prime Dice Game

Reproduced from www.resolve.edu.au

In this game you will be rolling prime dice and trying to get the biggest product possible in each of 6 categories.

Categories	Score for the Categories
A number to the 4th power	The product of the prime numbers showing on the dice
A number ending in 00	This roll scores 500 points
A number ending in 0	The product of the prime numbers showing on the dice
Square	The product of the prime numbers showing on the dice
Odd	The product of the prime numbers showing on the dice
Even	The product of the prime numbers showing on the dice

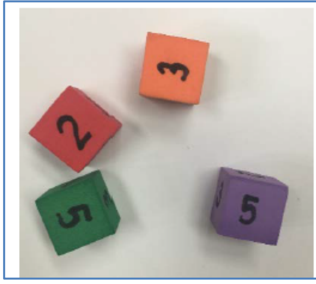
How to Play

1. When it is your turn, roll the dice. Think about the product of the numbers you have rolled.
   - What categories will the product be in?
   - Which category will give you the biggest score?
   - If you rolled one of the dice again could you get a different category or a bigger score?
2. If you decide to try to improve your score, roll one of the dice again.
3. Decide on a category and work out your score.
4. The game ends when all the players have filled the six sections on the scorecard.

Rules

- If your product doesn't fit any of the remaining categories you will have to put a zero score instead.
- You can only record one number when it is your turn.
- You can only score once in each category.

Figure 3. Rules for Prime Dice lesson from reSolve:Mathematics by Inquiry (www.resolve.edu.au)



Amy's first roll was 2, 3, 5 and 5.

She knows that this is an even number and ends in a zero.

She decides to re-roll the 3 to try and get a number ending in a double zero or a perfect square number.

Only re-rolling the 3 means that her new number will still be even and that it will still end in zero.

Figure 4. Extract from Prime Dice lesson of reSolve: Mathematics by Inquiry (www.resolve.edu.au)

CLIMBING THE MOUNTAINS (a unit of work)

In any long journey, there is likely to be some terrain that is very difficult to cross, such as getting over a mountain range moving up through narrow mountain passes. In learning mathematics there are many such 'mountain ranges', where students have to work hard to build new concepts and develop new skills, and might lose track of the direction in which they are heading. Teaching through problem solving can be very effective here if there is a very carefully designed sequence of tasks. This approach has been developed to a high degree in Japan, and forms the basis of some Japanese school textbooks. The example below is from a presentation by Akihiko Takahashi of DePaul University Chicago drawing on material from the Lesson Study Alliance (http://www.LSAlliance.org) following the Japanese curriculum (Takahashi, 2018). The overarching goals are that students should learn the ideas deeply and have experience of using them flexibly.

Example 4: Area of polygons

In Year 4, students learn about the area of shapes by imposing them on a grid and counting the squares, and they learn how to calculate the area of a rectangle. For Year 5, the goals are to learn to calculate areas of triangles, parallelograms and trapezia by using formulas, and to use these formulas to find areas of other shapes by:

- decomposing a shape into components which are shapes with known area formulas, or
- moving parts of the shape to construct a shape with a known area formula, or
- seeing a shape as half of a shape with a known area formula.

Target tasks for the Year 5 unit include simple and complex applications of these ideas, including the two tasks shown at line 1 of Figure 5. In the first task, the two shaded triangles meet at an arbitrary point in the parallelogram. The task is to find their combined area (always 12 cm^2). A second task is to cut the L-shape into two parts of equal area using just one straight line. There are several ways to do this – in each case the key idea is that any point through the centre of a parallelogram divides it into two parts of equal area. Both these tasks are very demanding for unprepared students.

These lessons are guiding students through difficult mathematical terrain and so the path needs to be carefully mapped out, and developed through careful experimentation. In this case a lesson study approach was used. The unit begins with the area of a parallelograms. The initial task is to find the area of a parallelogram presented on 1 cm grid paper (see line 2, Fig. 5). The lengths need not be given – they can be measured on the grid paper. In the previous year, students would have counted the squares. They can do this again (to see that the area is about 30 cm^2) but the intention is for students to demonstrate what the area is exactly, using their earlier knowledge of the area of a rectangle. Some students will over-generalise this formula, and argue that the area is 7 x 5 cm^2, whereas others will choose 7 x 4 cm^2, so that the need for a better argument is highlighted. There are multiple ways that students may discover to find this area. Two methods that are important for teachers to discuss with the whole class are shown in Figure 5 (lines 3 and 4), but there are others. For example, some students may add two 3-4-5 triangles (one to the upper left-hand corner and one to lower right-hand corner) making a 4 x 10 rectangle by adding the area of a 4 x 3 rectangle. Of the two important methods shown, one drops the perpendicular from the corner, and the other from any point inside. They both produce the same rectangle. The intention is that students discover such decompositions and rearrangements for themselves and the selected students share their methods in a discussion orchestrated by the teacher. When they critique the solutions of others, students develop soundly-based understanding of the formula for the area of a parallelogram, and also experience the strategies

X ESCUELA DE EDUCACIÓN MATEMÁTICA MIGUEL DE GUZMÁN

of decomposing, rearranging and adding to shapes to find areas. Practice includes cases, such as that at shown on line 5, where some students find it hard to see the perpendicular height shown, and cases where students must choose what line segments to measure.

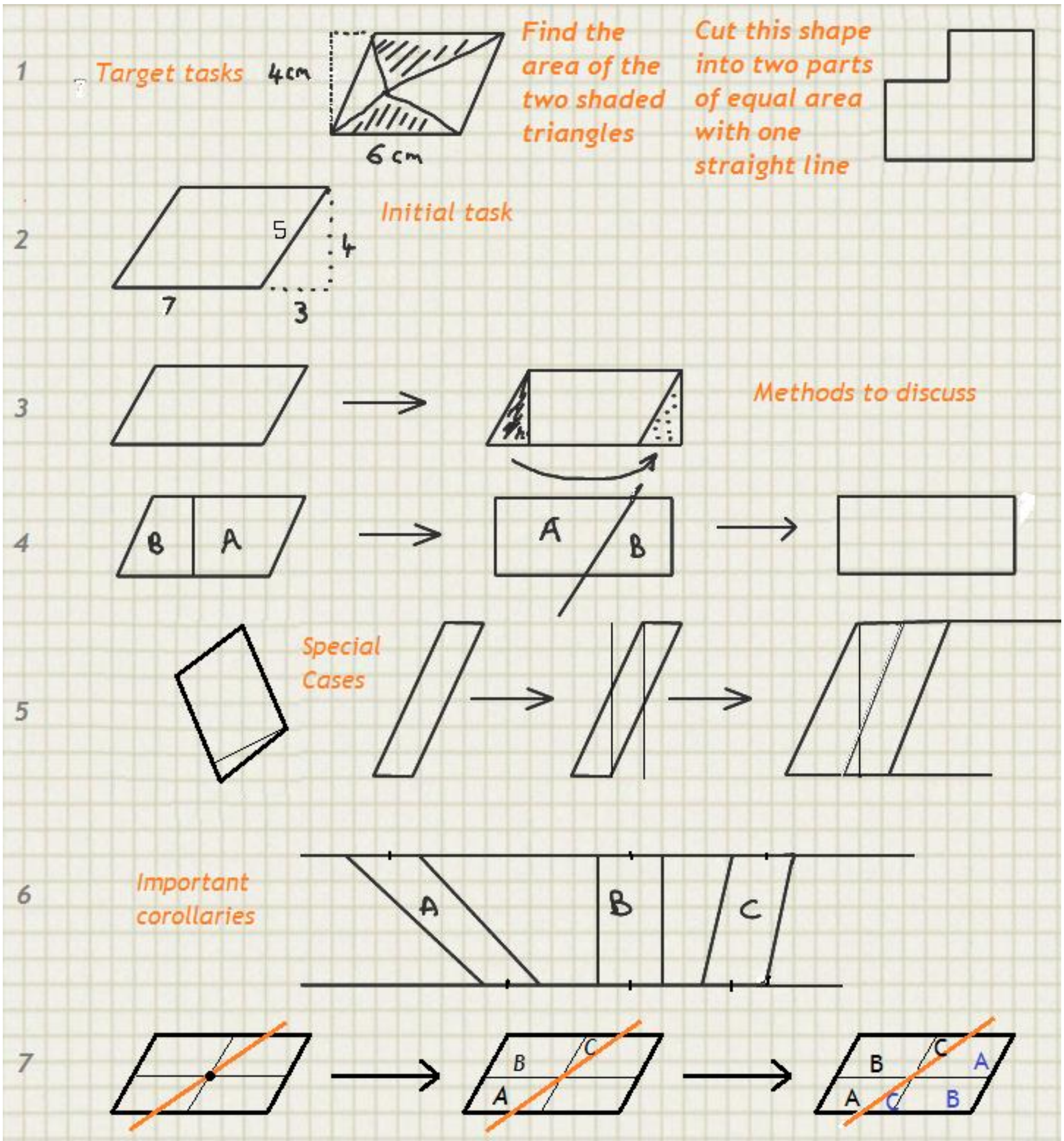


Figure 5. Sample tasks for teaching the area formula of parallelograms. (Derived from Takahashi, 2018)

The next phase involves more teaching through problem solving. Students now find the area of the special parallelogram shown on line 5. Because the perpendicular height does not lie within the parallelogram, the earlier methods shown in lines 3 and 4 do not work. Putting several of the parallelograms together to make one where the perpendicular height is inside the figure is one way of establishing the basic formula in this special case. This is an example of a 'shaking lesson', where students' growing understanding is strengthened by being used again in a more hidden, difficult or complex situation.

Students undertake further problem solving to meet some common corollaries of the formula, and also of the decomposing and reconstructing strategies that have been employed (line 6 and 7) . For example, knowing that the areas of parallelograms of constant width between parallel lines are equal is often, as is the fact that any line through the centre of a parallelogram divides the area into halves. It is this fact that is the key to the second target task on line 1. Further lessons take the students in a similar way through the formulas for areas of triangles (especially using the fact that every triangle is half a parallelogram), and trapeziums and special cases such as rhombuses.

About climbing mathematical mountains

Developing a robust, firmly grounded and flexible knowledge of these ideas is difficult mathematical terrain to traverse, like finding your way across a mountain range. Hence, tasks need to be very carefully selected to build a connected body of concepts, facts and skills step by step. The best sequences have been carefully refined by trialling with students and careful observation by teachers. The teaching through problem solving approach helps students to understand why the formulas work (they have probably been able to discover at least a partial method themselves) and prepares them to use the knowledge flexibly. Using unusual cases helps students generalise their thinking and scrutinise its validity.

The usual plan of the lesson is for the teacher to present a problem and ensure that students understand what is required. Students then try to solve the problem either individually or in groups. During this time, the teacher moves around the classroom, looking for students whose work demonstrates some important learning points. In the following class discussion, selected students show their work and the teacher formalises and consolidates the main points of the lesson. The general approach of Japanese lesson study, where teachers work together to test potential problems, has resulted in the creation of many productive sequences for teaching through problem solving.

CROSSING THE PLAINS (a unit of work)

Another approach to teaching mathematics through problem solving is closer to what is often called ‘inquiry learning’. A situation or problem (usually a real world question) is posed for students to investigate, and responding to this principal question motivates students’ activity throughout the unit. I have likened this to ‘crossing the plains’ because the task is quite open and there is opportunity for students to go in various directions, to get a broader view of the uses of mathematics, and to use different parts of mathematics together in order to find a good solution.

Example 5: Target Ball

This unit, from the *reSolve: Mathematics by Inquiry* website www.resolve.edu.au, is for Australian Year 1 students who are learning to compare distances and measure using informal units. The unit consolidates these early measurement concepts and skills, further develops counting with 2-digit numbers and placing them on a number line, builds a qualitative familiarity with a half and possibly other fractions, and provides an opportunity to organise data into a simple frequency table. The measurement tasks are sequenced according to an established developmental framework (see DET 2010).

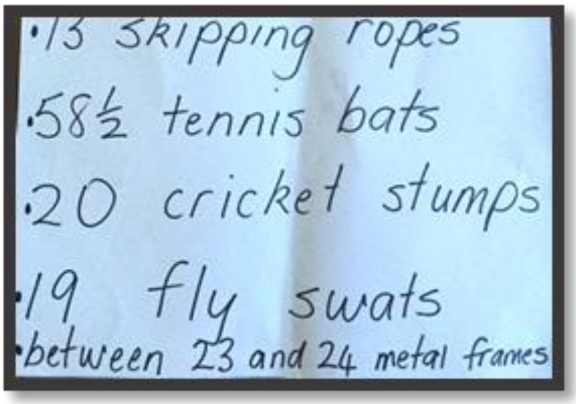
The lessons use the 4D Guided Discovery model of Makar, Allmond and Wells, progressing through phases of Discover, Devise, Develop, Defend. Lessons feature regular checkpoints where the class is brought together for discussion or instruction to support students in their mathematical activity and its real world links. Importantly, there is a strong emphasis on gathering mathematical evidence to justify a solution and this culminates in students presenting their solutions and evidence to the class in the Defend phase. (See also "Mathematical Inquiry into Authentic Problems" Teachers' Guide from www.resolve.edu.au; Allmond, Wells, Makar (2010) and Makar (2010)).

In the Discover phase of the Target Ball unit, students are set the challenge of advising the school sports teacher on how to set up a game of 'Target Ball' for children of their age. In this game, children roll a ball aiming to be closest to the target. The lesson begins with students brainstorming what is required, establishing that the sports teacher will need to know what sort of ball to use and where to put the target. Students test a variety of balls (e.g. football, rugby ball, tennis ball), undertake direct comparison of distances rolled, and review the class results to decide on the best ball to use. They come to appreciate that they will need to be able to describe the distances rolled (i.e. they need to measure).

In the Devise and Develop lessons, students work in groups to gather evidence on how far the chosen type of ball rolls. First, they measure with an informal unit of their own choice (e.g. fly swat as shown in Figure 6). Checkpoints highlight how to reduce error when measuring, and how to record the 'measurements' including recording lengths involving part of the informal unit. The need to make a class decision motivates the use of the same unit (e.g. a cut-out cardboard foot as in Figure 6). Groups repeat the rolling and measuring of the previous lesson, gathering better data. They organise and display their own groups' measurements on a number line, then write mathematical statements to say how far the ball typically rolls. A feature of the unit is the amount of measurement practice that is included: groups are given responsibility for checking all the measurements made to improve their skills, with frequent monitoring by the teachers. All students are helped to advance their mathematical understanding at the regular checkpoints where students share and discuss effective processes and practices and challenges encountered.

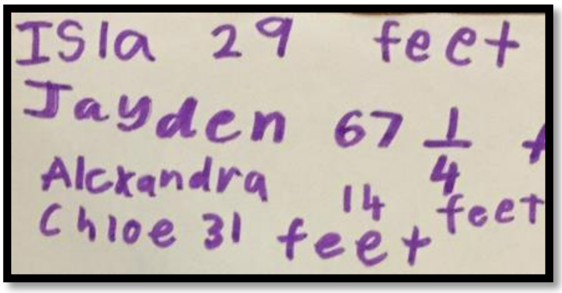
In the final Defend phase, students decide where to position the target to make the game suitable for children of their age. Measurements from the whole class are categorised into intervals and displayed in a table (see Figure 6). Students then make simple inferences to recommend the position of the target to the sports teacher and give their reasons. They test their recommendations by playing the game.

Measuring Rolling distance using fly swat as informal unit

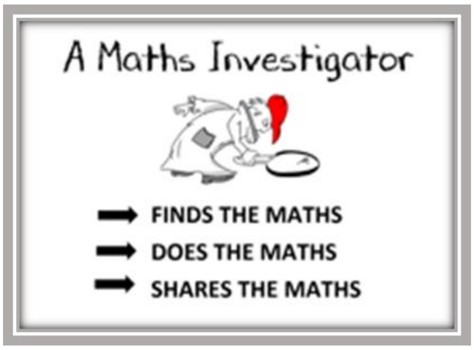


Reports to class of typical rolling distances

Measuring distance using cut-out foot as common informal unit



Groups' recommended distances using common unit



Poster emphasising gathering of mathematical evidence

Class constructing an informal frequency graph of recommended distances

Figure 6. Episodes from Target Ball (www.resolve.edu.au)

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About crossing mathematical plains

This unit provides a real world context and a clear purpose for students to apply old and new mathematical ideas in an authentic way. This creates connections that deepen students' understanding. The unit is built on a well-researched developmental framework for measurement, and the direction of this partially-open inquiry has been designed to provide motivation for new ideas, practice for developing skills and motivation for improving accuracy for students with a broad range of abilities. Compared to the "crossing the mountains" tasks, these tasks are less prescribed and emphasise more generic skills, such as gathering, recording and interpreting data. A special feature of the 4D approach is the focus on using mathematical evidence to make an evidence-based judgement: this distinguishes this approach from some other 'inquiry methods' that are less suited to mathematics teaching. The lessons place strong emphasis on the expectation that all students contribute ideas and explain their thinking during partner sharing and class discussions. This provides each student with an opportunity to demonstrate their ability to reason and justify and to develop conceptual understanding and learn to apply mathematics to make good decisions about authentic problems.

Conclusion

The journey of learning mathematics through schooling is a long one, with many different parts, so problem solving can feature in many ways. In this article, I have outlined four different ways in which teaching through problem solving can be part of this journey. Whilst the focus of teaching through problem solving is on the content to be learned, the intention is that by adopting a spirit of inquiry in all classes, students will simultaneously build both deeper understanding of the content and general problem-solving skills. With a teacher who can identify the mathematical richness of a situation, students can experience asking mathematical questions in the expectation of finding useful answers for themselves, see how mathematical ideas are applied, and using ideas flexibly right from the start. This can make a major contribution to helping build the proficiency in problem solving that is the main goal of mathematics teaching.

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Profesora emérita retirada a finales de 2012, después de ocupar la Cátedra Fundación de Educación Matemática desde 1992. Continúa su interés activo en la práctica e investigación de la educación matemática. La profesora Stacey es autora de aproximadamente 400 trabajos publicados, incluidos artículos de investigación, artículos para profesores de matemáticas y un número cada vez mayor de recursos electrónicos. Sus intereses se centran en el pensamiento matemático y la resolución de problemas, el desarrollo conceptual de los estudiantes, el plan de estudios y las nuevas tecnologías para la enseñanza de las matemáticas. Ha sido Presidenta del Grupo internacional de expertos en matemática para la encuesta PISA 2012 de la OCDE. Ella trabaja como consultora de sistemas educativos en Australia y en el extranjero.