

Phenomenology, Praxis, and the Question of Mathematical Objects

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In memoriam
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Abstract: In this article, I discuss some aspects of the manner in which phenomenology has dealt with the question of the nature of objects of knowledge and the knowability of such objects. I focus on Kant's phenomenology and consider in particular some ontological presuppositions that make Kant's phenomenology both Platonist and anti-Platonist. Then, I make a brief incursion into Hegel's phenomenological approach and Marx's critique of Hegel, Kant, and German idealism in general. In the last part of the article, I comment on Marx's idea of *praxis* as an entirely new path to tackle the question of the nature and knowability of objects of knowledge. I discuss some of the implications of such an idea for mathematics education. I end up sketching a Hegelian dialectic materialist concept of knowledge that provides room for understanding knowledge as something ineluctably embedded in cultural praxis.

Keywords: phenomenology, sense, sensation, mathematical objects, praxis, dialectical materialism.

Resumen: En este artículo discuto algunos aspectos de la forma en que la fenomenología se ha ocupado de la cuestión de la naturaleza de los objetos de conocimiento y la posibilidad de conocer este tipo de objetos. Me centro en la fenomenología de Kant y considero, en particular, algunos presupuestos ontológicos que hacen a la fenomenología de Kant a la vez platónica y anti-platónica. Después hago una breve incursión en el enfoque fenomenológico de Hegel y la crítica de Marx a Hegel, Kant y el idealismo alemán en general. En la última parte del artículo comento la idea marxista de la *praxis* como una nueva ruta para abordar la naturaleza de los objetos de conocimiento y la posibilidad de conocer dichos objetos. Además, discuto algunas de las implicaciones de tal idea en la educación matemática. Termino esbozando un concepto materialista dialéctico hegeliano de conocimiento que proporciona el espacio para entender el conocimiento como algo ineluctablemente inserto en la praxis cultural.

Palabras clave: fenomenología, sentido, sensación, objetos matemáticos, praxis, materialismo dialéctico.

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1. INTRODUCTION

In his book, *The birth of mathematical objects*, Enrico Giusti (2000) admits that “one of the most embarrassing questions for a mathematician is to ask him what he deals with.” Of course, we could say that mathematics deals with numbers, functions, shapes, and so on. But what kind of objects are they? Giusti’s point is that, indeed, it remains difficult to find a straight, clear, and concise answer to that question.

The question, of course, is of interest to mathematics educators. We cannot talk about the teaching or learning of functions or numbers, for instance, if we do not have a clear understanding of the *nature* of functions or numbers, that is, of what functions or numbers are. Mathematics education discourse provides itself with a series of metaphors that somehow reveal an imaginary dimension to talk about mathematical knowledge. One is the “manufacturing” metaphor: knowledge is expressed as construction. Students *construct* knowledge. Another metaphor is the one of “reaching,” which comes in an assorted variety of images; students try to *discover* or *uncover*, or to get *access* to mathematical objects, as if these objects were there, hanging in a kind of Platonic world of ideas.

These metaphors are not banal. On the contrary, they are concrete scars of the legendary painful struggle of Western thought to understand the question of knowledge. More precisely, these metaphors are an attempt to come to grips with the question of the nature of knowledge and the relationship of subject and knowledge. While the question of the nature of knowledge is an *ontological* question, the question of the relationship between subject and knowledge is an *epistemological* question: it deals with the extent to, and the manner in which, things are knowable. Naturally, the ontological and the epistemological questions are interrelated: one does not make sense without the other.

In this article, I discuss some aspects of the manner in which phenomenology investigated the aforementioned ontological and epistemological questions. I focus on Kant’s phenomenology and consider in particular some ontological presuppositions that make Kant’s phenomenology both Platonist and anti-Platonist. Kant’s phenomenology is of particular interest to mathematics education not only because of the influence his phenomenology has had on Piaget’s (1970) work and on contemporary educational constructivism (e.g., Cobb, Yackel and Wood, 1992; Yackel and Cobb, 1996), but also because, as we shall see later, Kant’s work brings the knowing subject to the fore. In doing so, Kant faces the problem of the link between the subjective dimension introduced by the individual in the act of knowing and the objective dimension of knowledge that transcends the knowing subject *as such*. Kant reflects profoundly on the epistemic role of the senses and the sensual. In light of the recent “return of the body” and the “embodied cognition” in psychology, linguistic, anthropology, and education, Kant’s reflections appear very interesting and hard to dismiss.

After discussing Kant’s views on the senses and the sensual, I make a brief incursion into Hegel’s phenomenological approach, and end up with Marx’s critique of Hegel, Kant, and German idealism in general. In the concluding section, I comment on Marx’s

idea of *social practice* –more precisely, *praxis*) as an entirely new path to tackle the aforementioned ontological and epistemological questions. I discuss the implications of Marx's idea of *praxis* for mathematics education and mention some problems that still remain open.

2. KANT

Phenomenology is usually associated with thinkers such as Edmund Husserl (1900/2001; 1931) and Maurice Merleau-Ponty (1945). Both of them have indeed been instrumental in the development of phenomenology as a domain of inquiry. In *Logical Investigations* (Husserl, 1900/2001), Husserl talks about the phenomenology of the experiences of thinking and knowing. This phenomenology, he says, “has, as its exclusive concern, experiences intuitively seizable and analysable in the pure generality of their essence, not experiences empirically perceived and treated as real facts” (Ibid., p. 86). Husserl is interested in understanding the various modes of lived experiences as defined by the manner in which we relate to things as they “appear.” For this reason, he tells us in Draft B of the 1927 article on phenomenology he writes with Heidegger for the *Encyclopaedia Britannica*, “the lived experiences are called phenomena” (Husserl, 1997, p. 110), and the turning of the gaze to those phenomena is where the “phenomenological attitude” resides (Ibid., p. 110).

Yet phenomenology goes back considerably earlier, as far back as Parmenides and his famous distinction between knowledge and opinion, or reality and appearance, or to put it in Platonic terms, ideas and things. Since then, the ontological status of reality and the explanation concerning the manner by which reality shows itself to us has been a matter of debate, conflict, and disputation. Plato, as we know, suggests that things are related to ideas through “participation.” Thus, a thing is beautiful because of its participation in the idea of beauty. In more general terms, appearances are caused by “participation” in their corresponding *eidōs*.

If it is true, hence, that phenomenology goes back to the pre-Socratic thinkers and that it underwent a substantial development in Plato's theory of forms, it is also true that phenomenological concerns were alive and well in the philosophical circles of empiricists and rationalists of the 17th and 18th centuries.

In Kant, we find a meticulous discussion of phenomenology. Kant's *Critique of Pure Reason* can even be considered a treatise on phenomenology—one that inquires into the structures of appearances and the limits of what is knowable to humans. To distinguish between appearance, reality, and objects of knowledge, Kant, much as Husserl, has to elaborate a complex conceptual terminology.

Perhaps our best starting point is a passage in the Preface that Kant adds to the 1781 original version of the *Critique of Pure Reason*. In the reworked version (called version B, published in 1787, when Kant was 64 years old), he says:

A new light flashed upon the mind of the first man (be he Thales or some other) who demonstrated the properties of the isosceles triangle. The true method, so he found, was not to inspect what he discerned either in the figure, or in the bare concept of it, and from this, as it were, to read off its properties; but to bring out what was necessarily implied in the concepts that he had himself formed *a priori*, and had put into the figure in the construction by which he presented it to himself. If he is to know anything with *a priori* certainty he must not ascribe to the figure anything save what necessarily follows from what he has himself set into it in accordance with his concept (Kant, 1787/1929, p. 19; B xi-xii).

The first part of the passage reveals Kant's acknowledgment and dismissal of the British empiricist approach to knowledge formation (as elaborated, in particular, by Hume and Berkeley). The second part ("but to bring out...") reveals Kant's alignment with the rationalist tradition (as epitomized by Descartes, Leibniz and others). In fact, Kant's theory of knowledge and its concomitant phenomenology is a desperate battle to reconcile both camps, as this passage already intimates. But the previous passage tells us about more than this battle in which Kant participated over the course of his entire life. The passage also reveals a set of central themes of his theory of knowledge:

1. the distinction between a) *experimental* concepts (i.e. concepts derived from experience) and b) *a priori* concepts (by which Kant means concepts independent of all experience and all impression of the senses);
2. the *sensuous construction* of representations (e.g., a particular triangle in the previous example) as a manner to bring out the logical properties of concepts;
3. the idea of the human mind as architectonically built so as to *logically* reason through the *sensible* with apodeictic certainty.

These three themes develop around two chief ontological and epistemological assumptions:

The *ontological* assumption concerns the *nature* of the objects of knowledge. For Kant there is a mind-independent reality, which lies beyond appearance. The world, he says, is divided into a world of the senses and a world of the understanding, the former populated by *phenomena* and the other by intelligible entities or things in themselves, which he terms *noumena*. It would indeed be ridiculous, Kant argues, to claim that "there can be appearance without anything that appears" (Kant, 1787/1929, p. 27; Bxxvii), "For if the senses represent to us something merely as it appears, this something must also in itself be a thing, and an object of a non-sensible intuition, that is, of the understanding" (Ibid., pp. 266-267; B306). Naturally, the aprioristic stance that Kant adopts brings him close to Platonism and its crucial dichotomic distinction between objects of knowledge (ideas) and phenomena.

The *epistemological* assumption concerns the manner in which, according to Kant, we come to know. As the example of the triangle suggests, in coming to know something, we resort to a *sensuous construction or representation*; we represent not *the* thing to be known (for instance, the concept of dog) but *a* particular one. What we learn of the thing to be known is not read from the particular representation thus drawn, but from something general that, for him, is beyond the sensuous realm: something not sensual but intellectual.

Kant's epistemology rests indeed on the claim that human cognition will always need *both*, a sensible, hence, experiential component, and an intellectual part. The sensible part he theorizes through the concept of *sensibility*, that is, our capacity for being affected by material things, and the intellectual part through the concept of *understanding*. He says:

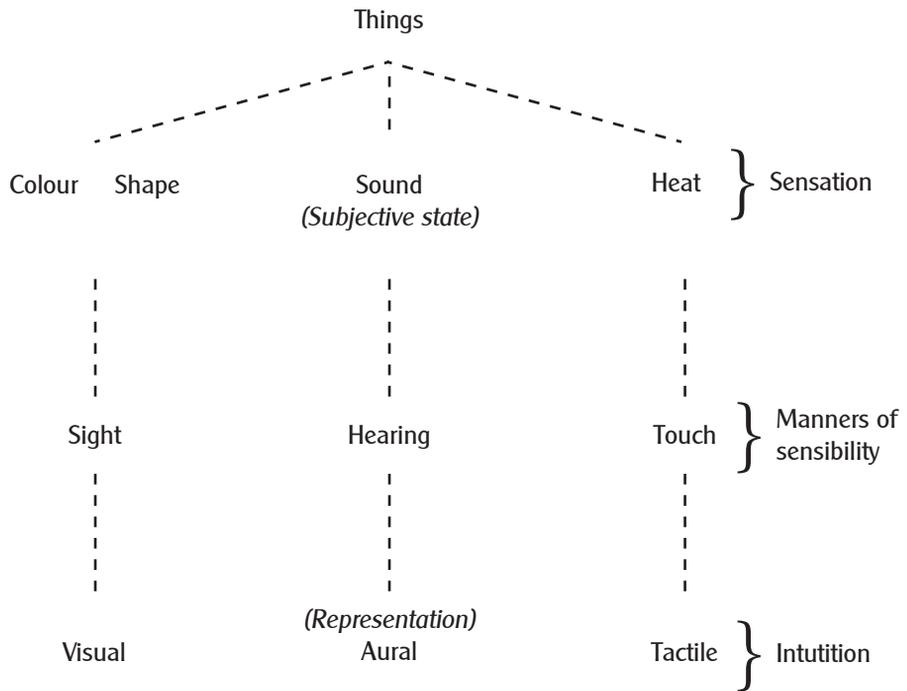
Without sensibility no object would be given to us, without understanding no object would be thought. Thoughts without content are empty, intuitions without concepts are blind. It is, therefore, just as necessary to make our concepts sensible [...] as to bring them under concepts. These two powers or capacities cannot exchange their functions. The understanding can intuit nothing, the senses can think nothing. Only through their union can knowledge arise (Kant, 1787/1929, p. 93; B75-76).

The chapter on "Transcendental Aesthetic" is central to our understanding of Kant's phenomenology, for it is there that he introduces the concept of sensibility and other related concepts that ground the subject's phenomenological experience. Sensibility, as previously mentioned, is for Kant the embodied manner in which we sense the world. It is in our nature as human beings to be affected by things around us in specific manners. Sight, hearing, and touch are *manners* of sensibility, which may be different from other forms of life. Sensibility and its various manners are distinguished from *sensation*, which Kant defines as "The effect of an object on the capacity for representation, insofar as we are affected by it" (Ibíd., p. 64; B34). Sensation (e.g., colour, sound, heat) is, hence, a state caused in the subject by the presence of an object, and as such is specifically subjective.

In his dealing with phenomenology, Kant also introduces another important concept: the concept of *intuition*. Intuition is a mode of cognition, which "relates immediately to the object, and is single" (Ibíd., p. 314; A320-B377). In other words, intuition is the manner in which the immediate representation of the object is furnished by the corresponding manner of sensibility. In semiotic terms, intuition is a form of indexicality mediated by sensibility.

Let me illustrate these concepts here through Kant's example of wine as an object of knowledge. I could resort to a mathematical example, but the example of wine is good enough. Wine, in its phenomenological experience, affects us thanks to our faculty of sensibility. Furthermore, in affecting us, it appears to us not as such but as mediated by modes of sensibility: colour, taste, smell, etc., that modify our physical sense organs; the raw sensuous data constitute the *matter* of intuition (see Figure 1). Notice, however, that

Figure 1 Kant's structure of sense and sensation



we intuit not wine in general but a *specific* wine (it might be a specific 2009 Dolcetto D'Alba from Piemonte or a Nero d'Avola from Sicily). This is what Kant means by intuition relating to a single object.

Now, it is not really clear in Kant's work whether intuition is located in the sensible or the intellectual realm. As Buroker (2006, p. 40) notes, "intuition' sometimes refers to the pre-conscious data received through sensibility, and sometimes to a conscious, intellectually processed perception." The fact that Kant calls intuition a *mode of cognition* seems however to put it on the side of the intellectual while allowing it to still be a carrier of sensations so to speak. What seems plausible though, given Kant's insistence on the need for empirical data to conform to the a priori concepts of the intellect, is that intuition requires a prior intellectual processing in order for us to become conscious of intuitive data. Kant's theory and contemporary theories of schematism rest on this crucial idea of prior intellectual processing.

Be that as it may, the sensation of the wine taste, for instance, Kant tells us, "does not belong to the objective determinations of the wine, not even if by the wine as an object we mean the wine as appearance, but to the special constitution of sense in the subject that tastes it" (Kant, 1787/1929, p. 73; B45). For Kant, taste and other

sensations produced on us by wine, such as colour and smell, “are connected with the appearance only as effects accidentally added by the particular constitution of the sense organs” (Ibid., p. 73; B46). Sensations are unable to yield true knowledge; they do not constitute an objective determination of the object, as they pertain to the subjective dimension of the sensing subject, and change from one individual to other. As a result, although the various modes of sensation of an object are unavoidably attached to their corresponding intuitions, they are not, according to Kant, necessary conditions of the object’s appearance. And as such, sensations are not a constitutive part of the process of knowing the concept of wine. They are kinds of side effects of the manner in which we come to know things in the world. In a marginal manuscript note to his copy of the first edition of his *Critique of Pure Reason*, Kant wrote: “Sensation is that which is really empirical in our cognition [...] Sensation therefore lies outside all a priori cognition.” (Inserted in Kant’s copy of the First Edition in the chapter of Schematism, see p. 274; Guyer and Wood 1998 edition). Because of the Kantian duality between the sensible and the intellectual, sensations remain decidedly within the realm of the concrete and the organic.

But there is an interesting corollary that derives from Kant’s phenomenology that we must not omit. The corollary deals with the limits of human reason. Let me mention it *en passant*. For Kant, the faculty of understanding cannot work in a spatial-temporal vacuum; it needs things to be presented to us through sensibility—for instance through a figure, like in the aforementioned instance of the triangle or a glass of wine in our second example. What the understanding can know, hence, is the objects of experience, not the objects as such, the *noumena*—those things in themselves that are not and cannot be objects of sensible intuition. In other words, because noumena correspond to a realm where our sensible faculty of intuition has no relation whatsoever, we cannot escape the “startling” consequence (to use Kant’s term) that we do not and cannot have knowledge of things as they are, i.e., things in themselves.

This is, in a nutshell, the limit of human reason, according to Kant. The limit results from two innovative moves that Kant makes. The first concerns the question of *subjectivity* and the new role with which Kant endows the knowing subject. The second one concerns the manner in which sensibility is conceptualized.

As to the first move, I have mentioned previously that, in the introduction of the distinction between phenomena and noumena, Kant goes back to Platonism. Yet, Kant’s approach brings forward a new conception of the subject. As Rockmore (2011, p. 46) mentions in his recent book *Kant and Phenomenology*, “Early Greek philosophy often approaches the general problem of knowledge without taking the subject into account.” Yet, the subject of Kant’s *Critique of Pure Reason* is not a psychological subject, but an abstract one that we may term the *epistemic subject*. Rockmore goes on to say:

All the main British empiricists are concerned with human knowledge from the perspective of finite human beings. Kant, on the contrary, avoids any hint of philosophi-

cal anthropology because he is concerned to avoid the difficulty that, since Husserl, is known as psychologism, roughly the confusion between the analysis of epistemological conditions and their psychological description (Rockmore, 2011, p. 46).

As to the second move, Kant's conceptualization of sensibility draws the epistemic contours that we bring with us as we experience the world in peculiar ways. Our sensibility, according to Kant, enables us to be affected by things and to produce appearances or representations of things in a concrete manner. In doing so, sensibility *filters* what we come to know.

To sum up, while Kant moves into new directions through the insertion of an abstract sensing constructive epistemic subject, and elaborates a sophisticated phenomenological account around the manner by which we intuit things, he remains, as ontology is concerned, a Platonist who, although acknowledging the importance of sensuous experience, makes sure to keep phenomena and noumena separate. Although controversial, his epistemology is clearly innovative. His ontology, by contrast, is squarely traditional. In his ontology, Kant is a Platonist. In his epistemology, he is anti-Platonist.

Let me now turn to Hegel's phenomenology.

3. HEGEL

For Kant, as we just saw, things in themselves are part of a mind-independent world. Things in themselves transcend the subject. Hegel suggests a different route. He considers Kant's critical philosophy a form of empiricism, although not equal to empiricism in general, as practiced by Hume and others. For Hegel, Kant's critical philosophy remains bounded to the "actual" and, hence, is unable to understand the necessary theoretical dimensions of concepts. For Hegel, Kant's approach remains empiricist in that the perceptual and actual realm of experience is considered simply received, grasped, and then elevated to the realm of the intellectual. He says: "The precise stage of consciousness at which the Kantian philosophy grasps spirit [i.e., mind] is perception" (Hegel, 1978, p. 27). As we shall see later, for Hegel, consciousness comes into being in a sensuous form, what he calls *sensuous consciousness* (Ibíd., p. 19).

Hegel criticizes Kant for holding too strong a distinction between phenomena and noumena. Like Fichte (1994), Hegel turns away from the idea of noumena or thing-in-itself. To the static ontology of Kant, Platonists and Idealists, Hegel opposes an ontology of *movement*, in which things and beings react in a mutual or dialectical manner, and articulates an approach in which the distinctions between falsity, appearance, and truth are considered as relative categories. Thus, appearance, for instance, is not the index of a noumenon, but a moment on the way to an always moving historical truth.

In Hegel's account, subjects come to know through interaction with things. Knowing is not merely something that the subject "constructs" or the intellectual "grasping" of a

mind-independent object. For, as Hegel argues, “consciousness simultaneously distinguishes itself from something, and at the same time relates itself to it” (Hegel, 1977, pp. 52-53). That is, consciousness is self-relation and relation to otherness. It is in this context that he distinguishes *being-for-another* from *being-in-itself*. Being-for-another is the manner in which the individual comes to relate to a thing, or comes to know it. Being-in-itself is the knowledge of the thing in all its determinations, as exterior to the subject and “posited as existing outside of being-for-another.” This being-in-itself, he says, is called *truth*.

It is in this movement between being-for-another and being-in-itself that individual consciousness and historical-cultural consciousness develop and subject and object determine each other. Here Hegel refutes solipsism. In Kant’s critical philosophy, like in all forms of idealism, the subject can know only what it has put in the object and the subject remains ineluctably a prisoner of its own mind.

In Hegel’s account, the development of consciousness is ensured by a dialectical critical stance towards itself. Consciousness finds the conditions of possibility of its development in the inner contradictions of previous formations and concepts. Without contradiction there would not be development. There would be mere identity—the identity of subject and object, of mind and thing, of consciousness and itself.

In his well-known and dense book, *The Phenomenology of Spirit*, Hegel offers an account of consciousness’s development. In its development, consciousness first finds sensuous immediacy, followed by perception, then understanding. In its first stage—that of sensuous immediacy—the object reaches us through our senses. “The object regardless of its being external or internal, is still devoid of any *thought determination* other than, firstly, that it simply *is*” (Hegel, 1978, p. 23). Sensuous consciousness knows *sense-certainty* only.

Hegel argues that, at this stage, consciousness appears as the richest in content and the poorest in thought. Consciousness is filled with the *determinations of feeling* and with the *indeterminations of thought*. Here the object is a wholly immediate being. Immediacy is in fact devoid of truth and filled with being. Rather than constituting a unity, here the object is “an independent being over against me [...] a singleness confronting my single immediacy” (Ibid., p. 23). Here consciousness is sensed otherness and abstract relation between subject and object.

Perception is a more developed relation between subject and object. In perception, the immediate and underdeveloped unity of mind and thing is overcome. That which is sensuous becomes *other*. But in perception this other is a *thing*, not another singleness, and as such has many properties or predicates. In short, it is a *universal*:

Perception starts with the observation of sensuous material. It does not remain confined to smelling, tasting, seeing, hearing and feeling however, but necessarily proceeds to relate what is sensuous to a universal which is not a matter of immediate observation. (Hegel, 1978, p. 27)

In perception, there is a recognizance of the sensuous material as being something else. The sensuous singleness becomes a sign. "Consequently, the many single beings of sensuousness become a range, a multiplicity of relations [...] and universalities" (Ibid., p. 25).

Rather than an identity between the singularity of the object of perception and its universality, perception is its *contradiction*. "Its truth is rather that the general object is *appearance*" (Ibid., p. 309). Sensuous immediacy does not know the truth. Perception begins to *demonstrate* the truth of things. But, Hegel claims, this demonstration is inadequate.

Consciousness moves beyond the inherent dialectical contradiction of perception and reaches the level of understanding, in which objects are considered as subjected to laws (in Hegel case, laws of necessity). "For the understanding consciousness [...] the world is a general object consisting of a realm of laws" (Ibid., p. 311).

It is interesting to note that Hegel's phenomenology rests on a philosophy of language that is alien to the thought systems of Kant, empiricists and idealists. It is in language, Hegel tells us, that the forms of thinking find their conditions and possibilities. Through language we talk about things, but, reflectively, language also talks about he/she who speaks. We see here the dialectical reciprocity of things and beings as reflected through language. Dialectics is the discourse that forges the development of a universal consciousness. In this context, singular consciousness is at the same time universal. A singular consciousness is, in fact, a *concrete universal*, an instance of the universal through its various determinations. There is no such a thing as an "I" as pure individuality. "I" is unavoidably related to a form of universal consciousness. Explaining Hegel, French philosopher Jean Hyppolite says:

La conscience qui prétend vivre la singularité pure sans la penser ou la signifier ne peut en fait que se dissoudre, c'est en vain qu'elle refuse le langage et le discours, et prétend atteindre un absolu ineffable. Ce qu'elle dit est le contraire de ce qu'elle vise, et c'est le langage qui a raison; ou si, par entêtement elle renonce au langage, elle ne peut que se perdre, se dissoudre. (Hyppolite, 1961, p. 14)

Hegel's philosophy of language helps also to clarify the nature of sensuous consciousness by realizing that the sensibly intuited thing of immediate apperception remains unsayable. For the characteristic of the singular is its ineffable nature. As soon as we utter *this*, we utter a universal, not a singular (it can be this table, this chair, this pen, etc.). Through the term 'this' we are already making a cognitive distinction between particulars; hence, we have already moved beyond the immediacy of immediate apperception. Hegel says: "Of course, we do not envisage the universal This or Being in general, but we utter the universal; in other words, we do not strictly say what in this sense-certainty we mean to say" (Hegel, 1977, p. 60). The finesse or delicacy of the particular cannot be grasped through the overly thick granularity of language.

Hyppolite comments as follows:

Nous croyons bien saisir l'être singulier immédiat comme singulier, mais ce que nous disons c'est ce qu'il y a de plus universel, un ceci, un celui-ci, mais tout est un ceci, tout moi est un celui-ci. Nous croyons saisir la richesse même, il ne nous reste de cette expérience que la conscience de notre pauvreté. Nous voyons le singulier se transformer en universel, et l'être unique passer dans le néant comme néant de toutes les déterminations. (Hyppolite, 1961, pp. 15-16)

Hegel says:

When I say "this Here", "this Now", or a "single item", I am saying all Thises, Heres, Nows, all single items. Similarly, when I say "I", this singular "I", I say in general all "Is"; everyone is "I", this singular "I" (Hegel, 1977, p. 62).

Through language, things and beings lose the horrifying silent nothingness of pure singleness and become community.

Hyppolite continues:

[...] certes nous pouvons reprendre ces déterminations dans leurs connexions et retrouver alors l'être comme déterminé, mais nous entrons dans le discours qui s'amorce avec le geste par lequel nous désignons les choses, et si l'universel se particularise, ou se détermine de proche en proche, nous restons cependant toujours dans l'universel sans jamais pouvoir dire autre chose que de l'universel (Hyppolite, 1961, pp. 15-16).

Rather than naming the particular, the word negates it. For, as Vygotsky would say later on, "The basic and central feature of any word is generalization. All words generalize" (Vygotsky, 1987, p. 249). As such, a word does not point to *this* or *that* particular object, but rather to a class of objects. Yet, a word cannot help but keep the particular as an echo. In its truly dialectical nature, the word signifies what is not there by signifying what is there, and signifies what is there by signifying what is not there (Hyppolite, 1961, p. 34).

To sum up, Hegel (1977) elaborated a phenomenology based on a dialectical relationship between universals and particulars, between things and beings. In this phenomenology, things and beings exist as phenomena. One distinctive trait of this phenomenology is the fundamental role that is ascribed to language. Language is not merely considered a system of signs. It constitutes the universe of sense, where things and beings are reflected by and reflect each other. Rather than being a medium of communication, language is the material in which things, thought, and consciousness come into being in a unitary manner. As Feuerbach puts it, in Hegel being in language is the

being of beings, much as being in water is the being of fish (Feuerbach, 1843, Part 2: Critique of Hegel, Section 27).

As mentioned previously, for Hegel, the particular exists as something intended (*visé*) and ineffable, whose truth is its immediacy. In this context, thought is not something that goes from the silence of the immediate and unspeakable particular to a linguistic expression that would illuminate the particular as it were. For thought is not a form of intellectual elaboration of sensible intuitions. Thought is movement; thought is sense development. "Sense deploys itself and find its determinations without being previously given in an ineffable manner" (Hyppolite, 1961, p. 26). It is this extremely interesting aspect of Hegel's phenomenology that Vygotsky articulates in *Thinking and Speech*, in particular in the last chapter, in which the Russian psychologist deals with the relationship of word to thought. In that chapter Vygotsky suggests that "thought is not *expressed* but *realized* in the word" (Vygotsky, 1985, p. 328; see also Vygotsky, 1987, p. 250). In Hegelian terminology, it is in language that thought finds its determinations and effectuates its descent to the concrete. In a truly Hegelian spirit, Vygotsky articulates the relationship between thought and speech not as a thing, but as a process of contradictory units, subsumed into a dialectical unity behind which lies consciousness, which Vygotsky understands no longer in a pure Hegelian sense but in Marx's Hegelian version of it, that is to say, as a lived and emotional relation of the individual to its concrete life.

4. THE QUESTION OF BEING AND REALITY

Feuerbach, and Marx after him, recognizes in Hegel the merit of presenting a philosophy in which the question is no longer about representing the world. Yet, for Feuerbach as well as for Marx, the questions of being and reality remain abstract in Hegel. They remain urgent questions to be elucidated. What are the determinations of the real? What are the determinations of being? Although it is true that we are beings in language, this Hegelian insight remains very abstract. "*The question of being is indeed a practical question; it is a question in which our being participates – a question of life and death.*" (Feuerbach, 1843, part 2). For, Feuerbach argues, "I owe my existence by no means to the verbal or the logical bread – to the bread in itself – but always only to *this* bread," and went on to assert that "Where words cease, life begins and reveals its secret [...] Existence has meaning and reason in itself, without being verbalized" (Feuerbach, 1843, Part 2: Critique of Hegel, Section 28).

Feuerbach summarizes his critique of Hegel as follows: "Hegel is a *realist*, but a *purely idealistic* realist, or rather an abstract realist; namely, a realist abstracting from all reality" (Feuerbach, 1843, Part 2: Critique of Hegel, Section 29).

Feuerbach's critique moves, hence, around the dissatisfaction with the idea of reality that remains abstract and thematized along the lines of language only. Feuerbach makes a plea for a philosophy of the concrete and the sensuous:

Taken in its reality or regarded as *real*, the real is the object of the senses – the *sensuous*. Truth, reality, and sensuousness are one and the same thing. Only a sensuous being is a *true and real* being. Only through the senses is an object given *in the true sense*, not through thought *for itself* (Feuerbach, 1843, Part 3, Section 32).

Within Feuerbach's materialist approach, the real is equated to the sensible and practice becomes the sensorial relation of sensible beings to this sensuous real.

Up to the 1844 Paris Manuscripts, Marx remains a follower of Feuerbach. Yet, Marx starts finding some holes in Feuerbach's account. Against Feuerbach he argues that the real is not merely the real objects of sense. The real is certainly something much more complex than a heap of sensuous things. Moreover, Marx reminds us, the objects of our senses are *historical* objects that bear in themselves traces of human activity. If Hegel is right in suggesting a dialectical determination of things and beings, then we still have to make explicit how the historicity of things and knowledge come to affect the manner in which we come to be the subjects that we are through our interaction with things.

For if the nature of things resides in their human objectification, as Hegel contends, the nature of beings, we must accept, resides in their material subjectification.

To a large extent, the 1844 Paris Manuscripts and *The German Ideology* are Marx's efforts to achieve a more coherent view of humans and things as mutually constitutive. In particular, within the dialectical view opened up by Hegel, Marx goes beyond the traditional anthropological conception of five human senses and thinks of human senses as the whole range of historically constituted powers of subjectivity and intentionality. Thus, while materialism characterized itself through its opposition to (abstract) thought, Marx's materialism includes superior forms of sensing that appear as a result of individuals' cultural and political forms of life. These superior forms of sensing are thus cultural-historical modalities of individuals' relations to their sensuous historical objects. It is within this context that Marx argues in *The German Ideology* that our biological senses are transformed and that they become theoreticians. My FLM paper (Radford, 2010) about the eye as a theoretician is little more than a footnote to this interesting insight by Marx.

It is this same interesting insight that Vygotsky (1978) articulates in the book *Mind in Society*, in which he talks about the phylogenetic merging of biological and cultural lines of development.

But Marx's greatest insight, I think, is the following. Reality, Marx comes to realize, does not reside in the mind's objectification of a pre-existing world; nor is reality constituted by an objectifying mind either. To accept the first idea would amount to remaining in the confines of the causal version of transcendental idealism that assumes that we can reach things in themselves through their phenomenal appearances. To accept the second idea would amount to remaining in the confines of Hegel's realist idealism. The historical abruptness of Marx's *Theses on Feuerbach* (1998), written as sketchy notes in Brussels in January 1845, is a sudden rupture with both Hegel's idealism and Feuerbach's sensuous materialism.

Marx had then to answer the question of what constitutes the nature of beings and things whose conditions of possibility cannot be located in the intuitions and workings of the mind. As French philosopher Michel Henry (1976) notes, one after the other the *Theses on Feuerbach* (Marx, 1998) tirelessly give us the answer: the conditions of possibility of things and beings, of knowledge and consciousness, reside in *human praxis*—an entirely new conception of practice.

Marx' concept of *praxis* is a concrete response to the abstract reality of Hegel and the sensuous one of Feuerbach—in which sensuous subjects and objects still remain *general* categories. With Marx, objectification and subjectification are anchored in a view in which individuals are really concrete beings, “apprehended in a situation that is theirs, embedded in a web of relations, in the effective fabric of a pre-existing totality” (Henry, 1976, pp. 316-317). Individuals find their idiosyncratic and contingent determinations in history and in their specific society. This new conception of practice, to which Marx refers as *praxis*, is not, hence, a kind of ancillary background where people get in touch or gather for some purpose (as in modern versions of interactionism). Praxis is rather the foundation from which consciousness arises; *praxis is the ultimate ontological founding category in which individuals live their concrete lives, in which they labour, suffer, struggle, love, think, and hope*. Through the introduction of this new materialist and historical conception of practice, Marx moves away from the idealist and abstractionist conceptions of practices of his time.

5. IMPLICATIONS FOR MATHEMATICS EDUCATION

Marx' concept of *praxis* is certainly helpful in rethinking the key concepts of mathematics education. It invites us to move away from classical accounts that cast knowledge in terms of representations or individual constructions or embodiments.

Certainly, the subject of *praxis* is a concrete subject. As such it is a subject who, in the act of knowing, acts with her entire body. The subject of *praxis*, however, is much more than an embodied subject. It is someone who interacts with material culture and other subjects, and in doing so, resorts to signs, language and other elements of culture and society. Yet, as mentioned in the previous section, it is not body only, or interaction, or signs, or language or discourse, or material culture, not even all these elements taken together, that count as the ultimate ontological founding category. The ultimate founding category is *praxis*.

A. N. Leont'ev (1978) is credited with having developed from a psychological viewpoint Marx's idea of *praxis* in his activity theory. However, as Leont'ev's introduction to his book makes it clear, his intention was not to develop a theory of activity. Actually, in the 1978 translation in English there are only 12 pages dedicated to discuss the structure of activity. The rest of the book is about something else, something that he considered the important thing: to understand how consciousness and personality emerge from

activity. Not in the sense that the human psyche is merely a derivative of activity, but rather how the human psyche arises out of the contradictions, the transformations, and all the elements that come into play in activity. He was led, at least, to describe certain elements and certain layers of activity, but not to theorize activity in general.

Given the large spectrum of human activities, it may be very well the case that a general theory of activity is not achievable or useful. Be it as it may, we can try to reduce the scope of the spectrum of activities and focus on educational activities, those that are of interest to mathematics education, for example, teaching and learning classroom activities.

There are three points that derive from Leont'ev's concept of activity and Hegel's concept of knowledge that I would like to stress in this section.

5.1. OBJECT/MOTIVE AND THE OVERCOMING OF DUALISM

First, Leont'ev provides us with a general schema about activity that makes it overcome the dualism of other approaches. Indeed, for me, the most significant element of Leont'ev's view on activity is the Hegelian dynamic connection between subject and society, that is, the connection between the individual and the sociocultural world. This connection appears in Leont'ev's account as a couple, consisting of *object* and *motive*.

The *object* is that which drives and sets the activity into motion. An object, "(fishing, for instance) is what endows the activity with a particular *intent*" (Roth and Radford, 2011, p. 6). The *motive* is both sociocultural and subjective (or individual): it is sociocultural in the sense that fishing has a sociocultural signifying valence attached to it; it is subjective in the sense that a motive is "determined by the sense of the child [or the individual] for a given task, a given situation" (Leont'ev, 1978, p. 178). The motive responds to a need that is not merely organic; it belongs also to the emotional realm.

The connection between the individual and the sociocultural at the basis of Leont'ev's concept of object/motive, is cast in such a way that it allows for a reconceptualization of the relation between the individual and the social. Human activity does not constitute a relationship that opposes the individual and society; As Leont'ev (1978) contends, "This must be stressed because psychology is now being flooded with positivist conceptions that are in every way imposing the idea of opposition of the human individual to society" (Ibíd., p. 51). Leont'ev criticizes positivist conceptions for reducing society to a mere "external environment to which [the individual] is forced to accommodate, in order not to appear 'nonadapted,' and to survive in exactly the same way as an animal is forced to adapt to an external, natural environment" (Ibíd., p. 51). This positivist perspective misses precisely the main point: "the fact that in society a man finds not simply external conditions to which he must accommodate his activity, but that these same social conditions carry in themselves motives and goals of his activity" (Ibíd., p. 51).

Within this context, the idea of teaching and learning can be formulated as an activity in which the student is not opposed to the teacher, nor is the teacher a simple coach, or

assistant, or a transmitter of knowledge. The teacher is part of the joint activity through which learning and teaching occur. The teacher takes part in the teaching-learning activity; she enjoys and also suffers with the students. For activity in Marx's and Leont'ev sense is not just a gathering together, nor is it an ensemble of actions led by a contractual, utilitarian outlook: activity is a form of life, what Hegel and Marx called *labour* (Radford, 2012).

5.2 KNOWLEDGE

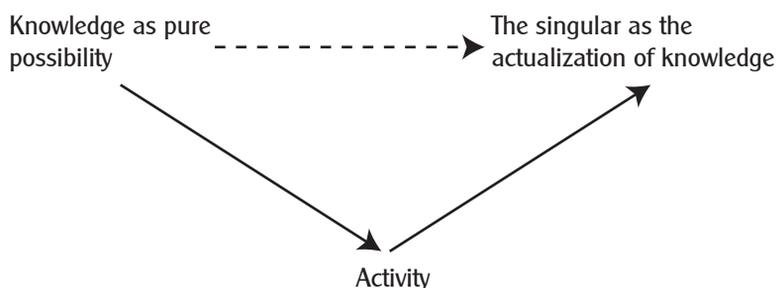
Hegel's concept of knowledge as movement opens up new avenues in which to theorize teaching and learning. From a Hegelian perspective, knowledge is not something that we represent. Actually, knowledge cannot be represented, for knowledge is always in motion. Knowledge is *pure possibility*. It is constituted of culturally and historically encoded forms of reflection and action that, instead of lending themselves to representation, are sources for action (Radford, 2013). Numbers, for instance, are not things or essences to be represented. They are possibilities for action (e.g., to count or to carry out complex calculations).

As pure possibility, knowledge cannot be an object of consciousness. To become an object of consciousness and thought, knowledge has to be set into motion. Knowledge has to be filled up with concrete determinations. And this can only happen through activity—sensuous and material activity. This is what students and teachers do when they participate in classroom activity.

Let us refer to a short example to illustrate these ideas. The example is about pattern generalization. Pattern generalization is a cultural activity at the heart of many ancient civilizations. The Pythagoreans and the Babylonians, for instance, practiced it, and thus, it started as an endeavour motivated by concrete counting processes or sense-making investigations. These endeavours became encoded ways of reflecting and acting that were refined in the course of cultural history (Diophantus, Fermat, etc.).

In contemporary curricula, in particular in English-speaking countries, pattern generalization appears often as a road to algebra. It is within this pedagogical intention that we have resorted to pattern generalization (see, e.g., Radford, 2011). As an object of knowledge, pattern generalization is not something to be represented. It is something to be known. However, from the students' viewpoint, pattern generalization (in fact all mathematical content to be known) appears, first, as pure possibility (a possibility to do something, to solve some problems or to argue about something). And in order for pattern generalization to be known, it has to be set into motion. Knowledge has to evolve and to *appear* in concrete practice. By being filled up with some conceptual content, what appears is not knowledge in its entirety, but a concrete instance of it. Hegel (2009) called it the *singular*. We have, then: 1) the *general*, which is knowledge as such (in this case pattern generalization), 2) the *activity* through which knowledge is actualized, and 3) knowledge in motion, filled up with conceptual content, that is, the *singular*. Figure 2 provides a diagram of these three elements.

Figure 2 The singular as knowledge actualized in activity



What Figure 2 expresses is the mediated nature of knowledge. We do not have access to knowledge but through mediation. As pure possibility, knowledge cannot be fully accounted for by anyone of its instances (the singulars). Not even the most perfect triangle reveals the depth of the concept of triangle, not because we will always make imperceptible mistakes in drawing a triangle or because there would be triangles with other shapes different from the one we drew. The reason is this: The concept of triangle cannot be revealed in its representation, because the concept is not representable. The concept is knowledge. That knowledge is possibility, and as such cannot be represented; it can only be actualized in the activity that fills it up with particular conceptual content.

The singular as actualization of knowledge in activity should not be seen as something static or as an end point, but as an *event*. It is rather an “unfinished and inherently open-ended event” (Roth, 2013). It is a process—a *semiotic process* through and through. Not only because in the activity that actualizes knowledge and transforms it into an event, students and teachers resort to discursive, embodied, and material signs and artifacts, but also overall (and indeed this is the real reason), because in mobilizing signs students and teachers engage in processes of signification. The singular is an activity-bound semiotic event.

From a semiotic viewpoint, there is something extremely important to understand about the activity that actualizes knowledge. This activity is, essentially, an activity of signification. In fact, the activity through which knowledge is actualized is an activity of conflicting significations. The teacher is aware of the aim of the activity. In our example, the aim (or in Leont’ev’s terminology, the *object* of the activity) is to make the students aware of the historically and culturally constituted way of thinking and reflecting about pattern generalization. Before engaging in the activity, the students do not know about such a way of reflecting and thinking—at least not in all the scientific-cultural curricular details. If the students knew, there would not be learning on the horizon. The activity would be an exercise activity—i.e., practicing something already known. The epistemological asymmetry that underpins teaching and learning activity (Roth and Radford, 2011) infuses the

activity with its inherent contradictions. The idea of contradiction has to be understood here in its dialectical sense, namely as precisely what drives the activity further.

Let me turn now to my last point: learning.

5.3 LEARNING

The fact that the students do not yet know the aim of the activity (e.g., how to generalize a pattern algebraically) does not mean that they cannot engage in the activity. In fact, they resort to what they already know. This is why it is not surprising that, when students engage in algebraic pattern activity, they resort to arithmetic generalizations.

The conflicting significations that are at the heart of the activity can be formulated in the following terms. The aim of the activity (knowing how to generalize patterns algebraically) is dynamically and variously refracted in the students' and teachers' consciousness as the activity unfolds. The conflicting significations move (in a dialectical sense), creating tensions that, at moments, may be partially resolved or intensified. Attenuated or not, these tensions do not disappear. They constitute mobile *wholes* made up of different perspectives and positions that each participant in the activity brings.

The attuning of inter-subjective perspectives is the requisite for learning to occur. It does not mean that teachers and students have to agree on, say, the manner in which a pattern can be generalized. Attuning refers also to matters of deep disagreement and unresolved tensions.

In previous work, we have suggested that learning can be studied through *processes of objectification*, that is, "those processes through which students gradually become acquainted with historically constituted cultural meanings and forms of reasoning and action" (Radford, 2010, p. 3). In light of the previous discussion, we want to stress that *acquaintance* does not mean *agreement*. It means *understanding*—a socially responsible and conceptually articulated understanding of something even if we do not agree with it.

6. SYNTHESIS AND CONCLUDING REMARKS

In this article, I argued that mathematics education cannot talk about the teaching or learning of mathematics without having a clear understanding of the *nature* of mathematical knowledge, that is, of what mathematical objects are. In other words, mathematics education theories need to specify the ontologies they resort to.

I started by mentioning two famous metaphors that have been developed to tackle, directly or indirectly, the ontological problem, namely mathematical objects as constructions or as ideas to be accessed. I cast my discussion in phenomenological terms and strived to sketch the path followed by two very different philosophical traditions: the Kantian and the Hegelian. Both traditions rest on different understandings and

assumptions about knowledge and the knowing subject. In the last part, I endeavored to show how Marx's idea of practice, namely Marx's concept of praxis, is an attempt to reformulate the ontological and epistemological questions, that is, the question of the nature of knowledge and the question of the relationship between subject and knowledge, respectively. I argued that, within the Hegelian materialist dialectics as developed by Marx, it is not the individual's body only, or interaction, or signs, or language or discourse, or material culture, or even all these elements taken together, that count as the ultimate ontological founding category. The ultimate founding category is *praxis*. In the last section of the article, I discussed how praxis as an ontological category might serve as the starting point to reconceptualize teaching and learning. Teaching and learning are not two separate things. They constitute *joint-labour*, that is a self-fulfilling-form-of-life-with others (Radford, 2008; Roth and Radford, 2011). Within this context, the teaching and learning of mathematics moves from the social endeavour directed at the diffusion of knowledge that often defined it in the 1990s to a more complex endeavor inscribed in the politics and ethics of education. Teaching and learning is not about content only, but also about being and becoming; it is also about the processes of subjectification and agency that we promote and fail to promote in schools (Radford, 2008).

Drawing on Leont'ev's idea of activity as the foundational ontological category—an idea that, as we saw, Leont'ev borrows from Hegel and Marx—I stressed some aspects of knowledge and learning. Although the picture I offered is certainly very incomplete, it gives, I hope, a glimpse of what these concepts could look like within the dialectic materialist perspective.¹ Yet, the question can be asked: what then are mathematical objects within this account?

In a passage from an 1842 manuscript, Marx writes: "Philosophers do not grow on earth like champignons" (quoted in Henry, 1976, p. 416). Nor grow mathematicians and their ideas. Ideas –philosophical, mathematical, and others– come from people, as they labour together to satisfy their biological, practical, spiritual, and intellectual needs. Ideas are patterns of action that have been generalized, perfected, and encoded in culture and its various activities. Mathematical objects are *historically and culturally constituted ideas*. More precisely, mathematical objects are *crystallized labour* that new generations find and interpret in new ways and modify or expand as, in turn, they come to reflect and act in the always changing societies and cultures of their time. It is in this sense that mathematical objects, I suggested a few years ago, are culturally recognizable mov-

¹There are, of course, within the field of Mathematics Education, other current approaches that rest on different conceptions of practice and mathematical objects. See, e.g., Sfard (2008) and Font, Godino and Gallardo (2013). While Sfard puts emphasis on discursive practices and considers mathematical objects as reification of individuals' discursive deeds, Font, Godino, and Gallardo convey a view of practice aligned with Wittgenstein's social nominalism that features the object-process duality. Although these approaches are undoubtedly interesting, the ideas presented here draw on a different philosophical tradition: the dialectical materialist view of knowledge as potentiality, as something fuzzy whose fuzziness makes it beyond representation: knowledge is rather something purely virtual that is put in *motion* and that acquires its *concrete determinations* through transformative historical-materialist dialectical praxis.

ing patterns of activity embedded in the always changing realm of reflective, semiotic and artefact-mediated social practice (Radford, 2004).

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