TEACHING AND LEARNING MATHEMATICAL MODELLING: A BROAD AND DIVERSIFIED BUT SPECIFIC RESEARCH FIELD

Enseñar y aprender modelización matemática: un campo de investigación amplio y diversificado pero específico

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Abstract

As a field of research, the teaching and learning of mathematical modelling and applications has been rapidly growing over the last decades, thanks to a significant involvement of researchers, presenting papers in specialized and broad-spectrum conferences in mathematics education. The number of theoretical perspectives has increased as well as the number of theoretical concepts developed. In fact, diversity and plurality are seen as distinctive features of this young field of research. This conference focuses on reviewing this theoretical expansion and, more particularly, on the work produced at the CERME conferences promoted by the European Society for Research in Mathematics Education. Then I will suggest the possibility of devising a common conceptual ground in the existing research that may play a key role in further conceptual advances and possible combinations of theoretical concepts: the development of the modeller thinking.

Keywords: *modelling and applications, theoretical perspectives, theoretical concepts, modeller thinking development.*

Resumen

Como campo de investigación, la enseñanza y el aprendizaje de la modelización matemática y las aplicaciones ha crecido rápidamente en las últimas décadas, gracias a la significativa participación de los investigadores, presentando artículos en conferencias especializadas y de amplio espectro en educación matemática. El número de perspectivas teóricas ha aumentado y también el número de conceptos teóricos desarrollados. De hecho, la diversidad y la pluralidad se consideran características distintivas de este joven campo de investigación. En esta conferencia me centraré en revisar esta expansión teórica y me focalizaré, en particular, en el trabajo producido en las conferencias CERME promovidas por la Sociedad Europea para la Investigación en Educación Matemática. A continuación, como conclusión de este trabajo, propondré la posibilidad de trabajar en una base común derivada de la investigación existente que pueda contribuir a una teorización más general y combinar distintas conceptualizaciones teóricas: el desarrollo del pensamiento modelador.

Palabras clave: *modelización matemática y aplicaciones, perspectivas teóricas, conceptos teóricos, desarrollo del pensamiento modelador.*

INTRODUCTION

The integration of *mathematical modelling and applications* in the teaching and learning of mathematics has been advocated for several decades and it is currently an international curricular trend leading curricular reforms (e.g. Common Core State Standards for Mathematics, in the USA), as it entails some of the essential skills to be developed in students from primary school to higher education. Some impulse for its inclusion in school practices has also been given by the PISA

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international tests and framework (OECD, 2013, 2019; Wake, Foster, & Swan, 2015). Nonetheless, despite a history of development and growth that is far from negligible, several researchers, including many of its most influential advocates, acknowledge that the spread of mathematical modelling in classes around the world is far from corresponding to what was expected and needed (Blum, 2011; Blum & Borromeo-Ferri, 2009). This does not mean that the situation is identical in all countries and that the presence of mathematical modelling in schools and higher education has the same consistency and the same permeation in curricula, assessment, textbooks and in the mathematics teaching practices.

The history of the international research dedicated to the teaching and learning of mathematical modelling is also rich. It would be an inglorious task to summarize this history in scanty pages and that is obviously not my intention in this article. There are, however, some nodes of the history that I would like to come up with in a concise way so as to reflect in some detail on the direction of progress, especially with regard to the evolution of theoretical perspectives in the research on teaching and learning mathematical modelling and applications. I will start by mentioning some elements that I believe are essential in the evolution of the research and development community which today constitutes an internationally active Affiliated Study Group of the International Commission on Mathematical Instruction (ICMI): The *International Community of Teachers of Mathematical Modelling and Applications* (ICTMA) and its series of bi-annual conferences. Then I will talk about some contributions from researchers who have been working to achieve a mapping of the trends, advances and perspectives that are most fertile in the current research. For this, I will refer to an article by Geiger and Frejd (2015), then to a review made by English, Ärlebäck, and Mousoulides (2016) and finally to the contributions of Stillman, Blum, and Kaiser (2017), and Stillman (2019), who jointly offer a fairly complete view of the state of the art of recent research.

The next step leads me to the Thematic Working Group dedicated to Applications and Modelling, which has been part of the biannual conferences organized by ERME. This is a context in which I have been highly involved over the last few years, particularly in the organization, coordination and empowerment of the group, either as a leader or as a co-leader. In this context, I will try to give my reading of the work of some of its most active and productive participants, in terms of their theoretical perspectives and elected concepts, and try to envisage a certain guideline that allows us to launch an expectation of a future course.

This future direction can continue in many different ways as the field is becoming well established and sustained; in my view, one way of advancing has to do with keeping track of the essence and specificity of the modelling process in the context of mathematics education while developing and synthesising promising theoretical concepts to guide our research and practice.

ICTMA CONFERENCES – MEETINGS OF RESEARCHERS AND PRACTITIONERS ON MATHEMATICAL MODELLING

My first participation in the ICTMA series of conferences took place in 1991, at ICTMA 5, which was held in the Netherlands under the auspices of the Freudenthal Institute. At that time, I was doing my master's degree in mathematics education and teaching in secondary education. At the conference, in collaboration with colleagues from the University of Lisbon, I organized a workshop on modelling physical phenomena related to oscillations and waves, using the computer as a tool for the representation and mathematical exploration of such physical phenomena. Later I would conclude my master's thesis on the teaching of trigonometry in a context of mathematical modelling and applications, based on a teaching approach with two classes of 10th grade students. In my study, I defined three research aims that I here present in an abridged way: i) to characterize students' cognitive processes in real world problems involving mathematical models of trigonometry; ii) to understand the influence of extra-mathematical contexts in the learning of trigonometric models; and iii) to identify the role of the spreadsheet in mathematical modelling tasks. I will return to this point later to reflect on the possibilities that are currently open to various alternative ways of investigating issues similar to these. In the meantime, I would like to emphasize that my first impression of that conference, which has been reinforced in many other ICTMA conferences over the years, was that of a tuneful and fruitful interaction between mathematicians, applied mathematicians, and mathematics educators, including teachers from primary to tertiary, and researchers strongly committed to advocating the introduction of applications and modelling at all levels of education.

Kaiser, Blum, Borromeo-Ferri, and Stillman (2011) present a brief history of these conferences in one of the books that have been published after each of them, which currently constitute the book series offered by the Springer publishing house, under the name of *International Perspectives on the Teaching and Learning of Mathematical Modelling*. In this chapter, the authors speak of the special flavour of this conference, effectively created by attracting researchers from all over the world, both from mathematics and mathematics education. In the first case, the contributions of prominent applied mathematicians who are enthusiasts and proponents of the teaching of mathematical modelling stand out. Such contributions are invaluable in bringing the thinking of the mathematical modeller who acts in professional scenarios in doing mathematical modelling, that is, they constitute a vision from within the mathematical modelling activity brought by their protagonists. This vision of mathematical modelling continues to inspire the educational thinking and to be a strong reference for how it may be introduced in mathematics teaching and learning. Nevertheless, the various actors seem to agree that professional mathematical modelling cannot be traced back to the educational context, particularly because students do not experience the actual practice of professional settings and because their learning is situated in a school context (Jablonka, 2007; Schwarzkopf, 2007). In a way, the *modeller thinking* is one of the important influences of the ICTMA community on how to approach mathematical modelling in the context of mathematics education. It is this form of thinking that, although seen as specific, can cross multiple communities and boundaries, as Stillman et al. (2017, p. 2) put it:

> The teacher, researcher and mathematician are from distinct communities of practice and social worlds with different viewpoints, but all could be involved in the resolution of the same scientific problem, for example, when modelling how a modeller gets at the nub of a problem so the essence of a real-world situation becomes tractable mathematically.

The knowledge that has been generated in the ICTMA conferences reflects the view that the modeller thinking can be recognized, discussed, investigated and promoted in different communities of practice. And this connects to the fact that a relevant space is often dedicated to the analysis, discussion and sharing of mathematical modelling problems, rich situations for the construction and development of models, interesting tasks to explore and use with students, and modelling contexts involving several disciplines and subject domains. In short, at these conferences, opportunities are offered to explore and engage with materials, or to examine tasks and problems and ways of implementing them in the educational context. In my judgement, this openness to the generation and sharing of worked examples of mathematical modelling tasks is one of the undeniable benefits that this community has been able to bring about and to nurture. And this wealth of ideas is essential to illustrate and realize what I am naming as the modeller thinking.

Another fundamental idea is that in this series of conferences, as it has happened in several other international forums dedicated to the teaching and learning of mathematical modelling, the diversity of theoretical perspectives, concepts and research approaches has continuously and quickly increased. Kaiser et al. (2011) refer to this diversity as a striking feature of the scientific debate: a proliferation of foci, aims, methods and also theoretical orientations.

In one of the books in the ICTMA series, Geiger and Frejd (2015) presented an analysis of the evolution of the contributions contained in these conferences over a period of 10 years (2002-2011), also including in that collection the important 14th ICMI Study. The authors aimed to identify the diversity of theories and the way in which they arise in this research field. For their analysis they made a distinction between local theories and general theories. This distinction echoes other authors' ideas, such as Bikner-Ahsbahs and Prediger (2010), who refer to the theory of didactical situations as an example of a general comprehensive theory and to the modelling cycle as a local conceptual tool.

For the group of local theories, the classification of theoretical perspectives, proposed by Kaiser and Sriraman (2006), led to the following list:

- Modelling cycle
- Modelling competency
- Emergent modelling
- Models and modelling perspective
- Approaches for authenticity and defining modelling tasks

For the group of general theories, some of the approaches deemed to be well-established theoretical approaches in the wider field of mathematics education research were considered:

- Radical constructivism
- Social constructivism
- Embodied cognition
- Discursive and semiotic approaches
- Sociological approaches
- Neuroscience approaches
- Other approaches (such as critical mathematics education, feminist approaches)

One of the conclusions drawn by Geiger and Frejd was that the so-called local theories showed a much broader presence than the so-called general theories. In addition, a tendency has been noted, over the years, for a greater theoretical foundation of the research work, both as a way of justifying the research aims and as a means of putting the findings against previous results in the research literature. The authors interpret this result as an evidence of a greater maturity of the research field. They also considered that the preponderance of local theories over general theories suggests that theoretical positions coming from within the field of study (home-grown theories) can already be established. But they also add that these local theories are strongly condensed around two approaches: the modelling cycle and modelling competencies, which arise more often than all the general theories endorsed. This result therefore seems to suggest a dominance of a perspective that can be synthesised as the modelling cycle and modelling competencies.

In this regard, we may find in Frejd's doctoral thesis (2014) a similar analysis but there he also mentions that several studies adopting general theories do not explicitly use the modelling cycle, with the exception of the research drawing on the Anthropological Theory of the Didactic (ATD) and on Critical Mathematics Education. Thus, it seems that the concepts of modelling cycle and modelling competency tend to gradually give rise to an autonomous perspective.

In their chapter, Geiger and Frejd (2015) show yet another trend, that of a decrease in the number of papers more related to the concerns and contributions of practitioners. They presume that this trend may be the result of a greater pressure from the academy, driven by the growing need for publication of research-oriented papers throughout the academic world. At the same time, they observe a scarcity of purely theoretical articles and conclude that the generation of new theory seems to be hampered, although they identify white spots that offer potential for further theoretical developments.

In drawing a lesson from these results, it seems possible to say that the large diversity of approaches and perspectives is becoming less obvious and that this is the result of two pressing causes: some local theories tend to weaken and coalesce around one or two theoretical conceptualizations and the contributions of practitioners (whether mathematicians, scholars or school teachers) are being overlooked in favour of the focus on applied research. In other words, the inclination towards research and development is no longer as strong as it was in the genesis of ICTMA, given the decrease in contributions seeking to inform and support practitioners, that is, the kind of "empirical, descriptive or polemic/discussion chapters that aim to inform teaching and learning practice at any level of education" (Geiger & Frejd, 2015, p. 165).

PME AND ICME – MATHEMATICAL MODELLING IN THE MIDST OF OTHER RESEARCH TOPICS AND RESEARCH GROUPS

The annual conference promoted by the International Group for the Psychology of Mathematics Education (IGPME) attracts researchers from all over the world, being one of the most prestigious meetings in mathematics education research. PME is a broad-spectrum conference, so it is normal that research related to the teaching and learning mathematical modelling represent only a small part of the total. Recently, English et al. (2016) conducted an extensive review of the PME Proceedings covering 10 years of PME conferences (2005 to 2015). They selected a total of 37 papers for their review and analysis. The categories defined to organize this review were: perspectives on models and modelling; curricular and instructional approaches in fostering modelling competence; the inclusion of generic processes; and approaches to models and modelling in teacher education.

One of their first remarks is that they found a great diversity of perspectives on the notions of model and modelling over the 10-year period considered. Before starting their analysis of the collected works, the authors refer the three theoretical perspectives that they recognize as most popular in the literature: modelling cycles; modelling competency or modelling competencies; and the perspective labelled as models and modelling, which has in its genesis the idea of model-eliciting activities (MEAs).

Overall, they conclude that there is a substantial presence of articles that have adopted the models and modelling perspective (MMP) proposed and advanced by Lesh and colleagues for several years (Lesh & Doerr, 2003; Lesh & English, 2005). These articles addressed different topics and research questions but all of them have shown the use of MEAs in the studies they reported. The second perspective they discern is related to the modelling cycle, which according to the authors is a conceptualization of the modelling process that has shown to be highly productive. They found that although there are several different schematizations of the modelling cycle available, some have emerged as more commonly used: those proposed in Blum and Leiß (2007) and in Galbraith and Stillman (2006).

The concept of modelling competency as an analytical framework is also observed in the papers reviewed and makes a third theoretical trend. Others have also been acknowledged, including: Realistic Mathematics Education (RME), Mathematical and Cognitive Perspectives, and Anthropological Theory of the Didactic (ATD). Here, unlike the classification used by Geiger and Frejd (2015), the authors do not make a distinction between local theories and general theories, but rather try to signal the popularity and dissemination of some theoretical perspectives, observing that two perspectives seem to lead the field: on the one hand, the framework of the modelling cycle and modelling competencies and, on the other hand, the models and modelling perspective based on the notion of model-eliciting activities.

Also within the work comprised in PME, an overview of the state of the art on modelling and applications can be found in a Research Forum carried out in PME 38. It was dedicated to the identification of perspectives on the teaching and learning of mathematical modelling in terms of: mathematical, cognitive, curricular, instructional, and teacher education aspects (Cai et al., 2014). The introductory piece of this forum touches on the idea of a "conceptual fuzziness" (Lesh & Fennewald, 2013) due to the lack of a decisive specification of what counts as mathematical modelling and what cannot be included under that label. This is accentuated by the fact that the researchers working in the field apparently do not agree on a conceptualization of the mathematical modelling process. According to the authors, there is, nevertheless, a common understanding on the premise that mathematical modelling involves the formulation of a question and/or a problem originated in a real-world situation. However, much seems yet to be known about the cognitive processes, that is, the type of thinking and reasoning that students produce when they are solving modelling problems. And this was one of the themes addressed in the forum, within the cognitive perspective presented and characterized by Borromeo-Ferri and English (in Cai et al., 2014).

A special emphasis was placed on the mathematical perspective of modelling. It is interesting to know what mathematicians think about the way to help students becoming competent mathematical modellers. Regarding this last point, Pelesko (in Cai et al., 2014) begins by giving his definition of mathematical modelling, which means a practitioner's view: "mathematical modeling is the art or process of constructing a mathematical representation of reality that captures, simulates, or represents selected features or behaviors of that aspect of reality being modeled" (p. 150). He then offers a number of clues about what is relevant to the modeller, which are grouped together in the designation of thought tools for modelling. These cognitive tools consist of mathematical thought tools, observational thought tools, and translational thought tools. A mathematical thought tool is any mathematical tool that allows the modeller (or student) to think mathematically or do something in the field of mathematics. The observational thought tools derive from the experience of working with diverse problems, namely allowing the modeller (or student) to find patterns, to establish causal relationships, etc., and therefore are related to thinking about the real world. Finally, the translational thought tools include knowing various ways of mathematically translating phenomena, which has to do with the ability of the modeller (or student) to translate assumptions about reality into mathematics and also mathematical results into assertions about the real world. According to Pelesko, one of the limitations that may be noticed in the modelling cycle (in this case, the cycle that is mentioned in the *Common Core State Standards for Mathematics* -CCSSM- in the USA) is that it conceals or buries much of the work that is carried out with the use of observational thought tools and translational thought tools, while it very explicitly considers the need for mathematical tools. This leads to his recommendation that the thinking tools of the mathematical modeller should be explicitly considered and valued in the teaching and learning of modelling: "Identifying, unpacking, and learning how to equip our students with these sets of tools is an essential step in learning how to teach mathematical modeling" (p. 151).

In accordance with this view, Borromeo-Ferri and English (in Cai et al., 2014) emphasize the analysis of students' cognitive processes in the course of solving modelling problems. Analysing their forms of reasoning, their interpretations or conceptualizations of the real situation, the blockages they face in different steps of the modelling process, and their individual modelling routes allows to better understand the cognitive processes of the student modeller and to find ways of helping him/her in handling the modeller's tools. For example, in her study, Borromeo-Ferri (2010) found that mathematical thinking styles have a strong influence on the modelling behaviour of students and teachers since this implies a greater tendency to focus on the real world side or on the mathematical side of the modelling process.

Thus, one conclusion that can be withdrawn from these two contributions is that the *development of the modeller thinking* seems to be an idea to be pursued in the discussion about the learning of mathematical modelling, in particular, within a cognitive approach.

Another forum shared by multiple research groups in mathematics education is the International Congress on Mathematical Education (ICME). Over time, several Working Groups that make a relevant part of the congress program have proposed some of their outcomes for publication in the form of books. Very recently, a policy of publishing the work emanating from the ICME Working Groups has resulted in the ICME Monographs that are published by Springer.

The recent volume entitled *Lines of Inquiry in Mathematical Modelling Research in Education* (Stillman & Brown, 2019) includes a state of the art of mathematical modelling in education along with the lines of inquiry that are being followed. In that chapter, Stillman (2019) starts by making a distinction between mathematical modelling and mathematical application, and initiates a discussion of the modelling process in which she puts considerable emphasis on the modeller. The author reviews the most recent papers published in the ICTMA book series and in major mathematics education journals. On the plane of the most consistent theoretical perspectives, Stillman initially identifies three local theories and one general theory. The first three are described as: prescriptive modelling, modelling frameworks/cycles, and modelling competencies. The general theory is presented as anticipatory metacognition.

The notion of prescriptive modelling is connected to the awareness that the modelling cycle better captures the process of constructing descriptive and explanatory models than other kinds of models that entail action and decision-making, and may contribute to make changes in the world. As for frameworks based on the modelling cycle, the main idea emanating from her analysis is that empirical studies continue to confirm its logic of progression. Still, some forms of reformulation and fine tuning of the modelling cycle are also coming about, such as the Dual Modelling Cycle Framework introduced by Saeki and Matsuzaki (2013), which combines two representations of the cycle suggested by Blum and Leiß (2007). In any case, other alternative or discordant ways of conceiving the process of mathematical modelling are not outlined. For example, the author states that the use of technology in mathematical modelling activity should not modify the essentials of the modelling process and therefore not change the traditional modelling cycle, since the idea of using technology is probably driven by the logic of the modelling process. Finally, the third local theory considers not only modelling competency, in a holistic way, but also a set of sub-competencies based on the successive phases of the modelling cycle. In regard to a general theory, Stillman points to metacognition and, in particular, anticipatory metacognition, which is strongly associated with verification and validation processes of the modelling results and, consequently, of the mathematical model. It should also be noted that the perspective of modelling competencies as well as the theory of anticipatory metacognition are seen by the author as theories that are under empirical test and confirmation phase.

As for the inquiry lines distinguished in this revision work, three paths are pointed out: the study of the student modeller, the study of the teacher when teaching modelling, and the research on the design of modelling tasks.

In short, Stillman (2019) mentions several aspects that characterize the theoretical and empirical approaches of the latest research in the field. In these approaches, the primacy of the modelling cycle framework seems to reappear, to which a number of other notions were added, such as modelling competencies and to some extent the design of modelling tasks. It therefore reveals a certain consensus around the core nature of the modelling cycle as a theoretical and analytical framework in several research approaches, be it with the focus on the student as a modeller, or the focus on the teacher as mediator of the learning of the student modeller, or the focus on the tasks that will be used to promote the learning of the student modeller.

Only the perspective of prescriptive modelling, which, from my point of view, can be aligned with a socio-critical perspective on the role and purposes of mathematical modelling in mathematics teaching and learning, is in doubt as to its fit with the framework of the modelling cycle.

Lastly, in a somewhat surprising way, the models and modelling perspective, based on model-eliciting activities, also seen as a contextual modelling approach, does not appear in the state of the art presented in this ICME Monograph.

CERME – A RESEARCH GROUP ON APPLICATIONS AND MODELLING WITH STRONG EUROPEAN ROOTS

The *European Society for Research in Mathematics Education* (ERME) was founded in 1988. The ERME conferences (CERME) are held every two years and aim to promote communication, cooperation and collaboration in mathematics education research in Europe and all over the world. One of its distinguishing characteristics is the creation of thematic working groups (TWG) whose participants work and discuss together in a given area of research.

Since CERME4, held in 2005, in Spain, a thematic group on applications and modelling in mathematics teaching and learning has been active. For some years this has been the TWG6 - Applications and Modelling. Over time, the group has been increasing in productivity, namely in the number of contributions submitted by the participants and number of papers presented. The total number of papers and posters submitted in the sessions of this TWG amounts to about 160, from CERME4 to CERME11, the last of which took place in the Netherlands, in 2019 (Carreira, Barquero, Kaiser, & Cooper, 2019).

An important aspect of the work presented is the usual plurality of themes, theoretical and empirical perspectives, research aims, and also the different views on mathematical modelling and applications that coexist within the group. Such diversity was captured early on by some of its leaders and co-leaders, who started a systematic discussion about trends and perspectives in the midst of the diversity of the group and across the field of mathematical modelling in mathematics education. Those efforts have culminated with the publication of the essential article by Kaiser and Sriraman (2006) in the ZDM International Journal of Mathematics Education.

In each of the CERMEs the group has been presenting syntheses of the developments, advances and issues under analysis, as well as open questions for future research. These syntheses are usually shared in a final session of the work of CERME and have been published, in the respective Proceedings, as articles that describe the work produced. For a brief glance at this group's activity in the last CERMEs, from 2009 to 2019, I visit each of those introductory articles, written by the leading team in each conference.

In CERME6, Blomhøj (2010) stated that the group was aiming to identify and discuss the various theoretical perspectives in the research on teaching and learning mathematical modelling and applications. In the work plan of the group, the plenary presentation made by Morten Blomhøj and Gabriele Kaiser was intended to create the scenario for the subsequent work: "A survey of theoretical perspectives in research on teaching and learning of mathematical modelling" (p. 2042). In this conference, the various papers and posters revealed a set of more salient themes and theoretical perspectives. Among the theoretical perspectives that stood out, some became quite persevering and consistent throughout the successive CERMEs, namely ATD and MMP. Thus the papers presented were distributed by the following 5 themes: (1) Teachers' professional development for teaching and assessing mathematical modelling, (2) The role of ICT in teaching and learning mathematical modelling, (3) Researching the teaching and learning of mathematical modelling within the Anthropological Theory of Didactics, (4) Researching the teaching and learning of mathematical modelling within the framework of Realistic Mathematics Education, and

(5) Researching the teaching and learning of mathematical modelling under the Models and Modelling Perspective.

Turning to CERME7, held in 2011, Kaiser, Carreira, Lingefjärd and Wake (2011) made a summary of the central themes addressed in the various papers and sessions of the group. Three focuses of discussion were approached around the conceptualization of mathematical modelling in education: (1) The difference between mathematical modelling and applications; (2) The difference between problem solving and mathematical modelling; and (3) The role and influence of technology on the modelling process. During the conference there were several theoretical perspectives embraced and topics addressed. The MMP perspective had a relevant presence and introduced the notion of model development sequence, an extension of the concept of model-eliciting activities by adding other type of activities that seek to explore and apply previously developed models. The perspective of the modelling cycle and modelling competency was also very relevant in several studies, both related to the professional development of the mathematics teacher and to assessment. There was also the presentation of research that brought the extended cycle of modelling regarding technology, which includes a third world – the world of technology. The Anthropological Theory of the Didactic was also very much dealt with, mainly based on the idea of praxeology and the idea of study and research paths. Some studies have shown to be more oriented according to a cognitive perspective although others involved epistemological and curricular discussions.

Moving on to CERME8, although the report by Lingefjärd, Carreira, Kaiser and Wake (2013) did not show great changes compared to the previous CERME, it was noted that a good amount of papers did not explicitly include a theoretical perspective and also that empirical research seemed to be lacking more breath and consistency. The question of the use of technology has returned to the debate and the number of studies that focused on teacher beliefs and teacher education has increased. The MMP and ATD perspectives and the modelling cycle perspective were represented with relative balance in the works produced.

In the Proceedings of CERME9, Carreira, Barquero, Kaiser, Lingefjärd and Wake (2015) mention the fact that the group is gaining maturity and cohesion, one of its distinguishing features being its openness to the diversity of theoretical, methodological and philosophical perspectives taken by its researchers. Thus, diversity is placed as a touchstone of the group. One of the themes that emerged from the work was the perspectives and conceptualizations for mathematical modelling. In that CERME some objection was raised to the idea of promoting a firm distinction between mathematical modelling and applications, under the claim that “modelling means not only to create models, but also integrate and coordinate models somehow contradicting the idea that applications are simply a limited portion of the modelling cycle” (p. 790). Another emphasis found in the proposed papers refers to the relations of mathematical modelling with other neighbouring areas, such as ethnomathematics, project-based learning and inquiry-based learning, for example. In all these instructional modes there seems to be the common idea that there is an active learner seeking for a solution to a real-world problem in a realistic situation. Thus, a broad question started to appear: whether it makes sense to propose and develop a comprehensive theory of teaching and learning mathematical modelling and whether such a comprehensive theory is desirable and useful to the community of researchers and mathematics educators. The topic of the design of mathematical modelling tasks has also shown a clear rise in the group, namely in bringing to the discussion the concept of authenticity of modelling problems in education.

While the authors highlight the plurality of perspectives, they also point out some prevailing tendencies, which seem to suggest a certain reliability of some of the approaches. In that regard, they state: “the modelling cycle and the modelling phases represent the most common conceptual tools in informing the research. There are also shared concerns related to the contexts of mathematical modelling problems and the authenticity of the tasks” (p. 792). In short, in this congress the TWG6 faced its diversity and its wealth of theoretical perspectives, valuing an inclusive and open facet but, at the same time, frontally expressing some doubts. In particular, the group wondered about the possibility that this diversity could be detrimental to the growth of the research if it gives rise to a conceptual fuzziness as other authors have already pointed out (Cai et al., 2014; English et al., 2016).

Regarding CERME10, the summary of the main contributions begins again with acknowledging the great diversity of themes, theoretical perspectives and research purposes. Barquero, Carreira and Kaiser (2017) affirm: "The contributions discussed at the congress are characterized by a strong and fruitful diversity in the research questions considered, the school levels addressed and the theoretical approaches taken" (p. 877). Therefore, unanimity or uniformity seems to be relatively far off in this field. Several theoretical perspectives and several lines of empirical research continue to exist. In this case, the following themes were analysed through the various contributions presented in 20 papers and 4 posters: (1) Interdisciplinary modelling activities (including in engineering teaching); (2) Connection between problem solving and mathematical modelling; (3) Developing modelling strategies and competencies; (4) Tools and methodologies used to analyse modelling processes; (5) Teachers' beliefs in relation to the teaching of mathematical modelling; (6) Teachers' interventions in mathematical modelling classes; (7) Experimental materials and technology in modelling; and (8) The assessment of modelling activities.

Finally, at CERME11, which took place in February 2019, the team leaders, Barquero, Carreira, Ärlebäck, Jessen, Vorhölter, and Wake produced a report of the main contributions, advances and open questions resulting from the work presented, which will be part of the forthcoming Proceedings. In the report it was reiterated the great diversity and plurality of themes and perspectives. Still, it was possible to identify 5 overarching themes: (1) Analysis of modelling processes when solving modelling problems; (2) Mathematical modelling and simulations in connection to other disciplines; (3) Strategies to support design and implementation of modelling; (4) Use of resources to support teaching and learning of modelling; and (5) Teacher education for modelling and its implementation. A review was also made of the different theoretical perspectives and it led to the conclusion that the modelling cycle (and its several variations) was prominent in the group but other ways to conceptualise modelling were also contemplated. Some theoretical concepts acquired more visibility, namely, Model Activity Diagrams, Mathematical Working Space, and the Study and Research Path proposed by the ATD. In the concluding remarks, the authors allude to the fact that there was an opportunity to discuss possible complementarities between different approaches, which indicates an attempt to a concerted action or a combination of different approaches, especially in terms of theoretical constructs and more relevant theoretical perspectives.

It seems possible to say that the evolution of CERMEs over the period 2009-2019 in relation to the trends and perspectives of the Applications and Modelling research group converges, in many respects, with what has been seen in other research forums. There is a clear concomitance of theoretical perspectives that are affirming themselves and solidifying. In addition, qualitative empirical studies prevail, including grounded theory, case study designs or other forms of applied research, based on pedagogical experiences or with a design-based research orientation; a few studies using a mixed methodology that analyse data emanating from wider research projects are also seen. One of the conclusions that can be drawn from this analysis is that there are some more represented theoretical perspectives, which are the perspective of the modelling cycle and modelling competencies, the MMP perspective, and the ATD. With less representativeness there is still the perspective of Realistic Mathematics Education and, in a certain way, the problem-solving perspective.

The group integrates in its activity all the lines of inquiry that are suggested by Cai et al. (2014): mathematics, cognitive, curricular, instructional and teacher education. However, it should be noted that the mathematical perspective is very poorly included or almost absent. The academic nature and the standards that are imposed on the accepted research work lead to the impossibility of discussing articles based on exploration of modelling problems *per se*. In spite of this, theoretical articles are welcome and allow us to create interesting points of discussion around the conceptualization of mathematical modelling. The community has evolved substantially in the quality standards of the accepted research and in the greater conceptual and theoretical wealth of the articles proposed. However, more recently, the proliferation of concepts and perspectives that coexist in the group and in the mathematical modelling research community has started to be under attention.

It seems indeed intriguing that the researchers who constitute the community do not share a consensual view of the bulk of their scientific activity: the notion of mathematical model and modelling. In this respect, the classification of different perspectives on teaching modelling and applications proposed by Kaiser and Sriraman (2006) has often been used as a way of mapping the field. The mapping shows, among other things, that addressing mathematical modelling in mathematics education is dependent on the perspective adopted. This is a unique element that actually brings into the research substantial richness, diversity, plurality and openness. But it may also be seen as an element of division, confusion, and disintegration. Apparently, as reported in the last CERMEs, there has been some space for reflection on that matter and even a subtle speculation about possible links and articulation among theoretical perspectives.

Tracing the path of active researcher at CERMEs

I will now seek to look in more detail at the progress that took place in the context of the CERMEs, in the period 2009-2019, by analysing the successive contributions of some of its most active participants. To do this, I listed the amount of papers that each researcher, as a single author or as co-author, presented in the CERMEs, between the 6th edition and the 11th edition. Then I elected those who had a greater number of papers, also considering their commitment, participation and involvement in the debate, in the promotion of dialogue and in the exchange of ideas and arguments. I will now present a very abbreviated summary of the work of five European researchers in the most recent CERMEs, where I will also include my own activity developed there: Jonas Ärlebäck, from Sweden, Berta Barquero, from Spain, Gilbert Greefrath, from Germany, Rita Borromeo-Ferri, from Germany, and Susana Carreira, from Portugal. The two first researchers have participated in the aforementioned period with 8 and 7 papers respectively; the later three researchers have participated with 5 papers each.

I must make a point of order regarding my intention to look at the paths of these researchers. It is clear that the papers presented at the CERMEs are not expected to give a complete picture of all the work developed by the researchers in this field. I am not assuming in any way that those papers can be a representative sample of each researcher’s scientific production. Obviously, many of their other publications, interventions and academic products are missing from this picture. So this is only a very simplified but also very accessible way of arriving at a reading of how the research has been taking place, just by looking at the more assiduous and regular contributions in the working group on Applications and Modelling.

From all the articles of the aforementioned authors I have collected the aim of the study as well as the theoretical perspectives and/or the theoretical concepts that were embraced. The information was summarized in simple and short terms for the sake of space and briefness. What interests me is, first of all, to obtain a certain temporal perception of their work frame within the various perspectives endorsed in the literature. Intentionally, I have dismissed the idea of considering also the methods and methodological approaches since I will restrict myself to create a view of the theoretical perspectives that researchers have used and developed. In the following tables (Tables 1 to 5) this information is summarized for each of the researchers, according to the above order.

Table 1. A snapshot of the range of contributions of active researcher 1

Active researcher 1 (8 papers)			
Author(s)	*Title*	*Aim of the research*	*Theoretical perspective / theoretical concepts*
Ärlebäck, J. CERME 6	(1) Towards understanding teachers' beliefs and affects about mathematical modelling	To capture and conceptualize teacher's beliefs about mathematical models and modelling	Beliefs and belief systems; Modelling process as described in the modelling cycle
Ärlebäck, J. CERME 7	(2) Exploring the solving process of groups solving a realistic Fermi problem from the perspective of the anthropological theory of didactics	Finding the questions that the students formulate and the mathematics they use in solving a Fermi problem	Modelling process and sub-processes as described in the modelling cycle; Problem Solving; Modelling Activity Diagram Framework (MAD); Praxeologies – Anthropological Theory of the Didactic
Ärlebäck, J. Doerr, H. O'Neil, A. CERME 8	(3) Students' emerging models of average rates of change in context	To study and support the continued development of pre-university students' emerging models of average rate of change	Contextual modelling perspective; Model development sequence
Doerr, H. Ärlebäck, J. O'Neil, A. CERME 8	(4) Teaching practices and modelling changing phenomena	Examining the characteristics of teaching in a Model Development Sequence on modelling changing phenomena	Contextual modelling perspective; Model development sequence
Ärlebäck, J. Doerr, H. CERME 9	(5) At the core of modelling: Connecting, coordinating and integrating models	Towards a common conceptualization of modelling that bridges the research field	Applications and Modelling; Models and Modelling Perspective (MMP) Model Development Sequence
Doerr, H. Ärlebäck, J. CERME 9	(6) Fostering students' independence in modelling activities	Elaborating on teaching practices that foster student independence in modelling activities	Models and modelling perspective; Model Development Sequence
Ärlebäck, J. Albarracín, L. CERME 10	(7) Developing a classification scheme of definitions of Fermi problems in education from a modelling perspective	To define and characterize Fermi problems; To describe the connection between Fermi problems and modelling, according to different perspectives	Fermi problems; Modelling Perspectives

Ärlebäck, J. Albarracín, L. CERME 11	(8) An extension of the MAD framework and its possible implication for research	Investigating the potential in extending the Model Activity Diagrams as an analytical tool for studying students' activity in modelling problems	Individual Modelling Routes; Model Activity Diagrams

Table 2. A snapshot of the range of contributions of active researcher 2

Active researcher 2 (7 papers)			
Authors	*Title*	*Aim of the research*	*Theoretical perspective / theoretical concepts*
Barquero, B. Bosch, M. Gascón, J. CERME 6	(1) The 'ecology' of mathematical modelling: Constraints to its teaching at university level	To study the institutional constraints that hinder the implementation of modelling activities in education	Barriers, obstacles and dilemmas in teaching mathematical modelling 'Ecology' of didactic organisations Anthropological Theory of the Didactic
Barquero, B. Serrano, L. Serrano, V. CERME 8	(2) Creating the necessary conditions for mathematical modelling at university level	Exploring some of the essential characteristics or principles of Study and Research Course as a didactic device for the integration of mathematical modelling	'Ecology' of mathematical modelling practices; Praxeology Anthropological Theory of the Didactic
Barquero, B. Bosch, M. Romo, A. CERME 9	(3) A study and research path on mathematical modelling for teacher education	To illustrate the phases of the SRP-TE design and some preliminary results of the implementation	Study and research paths for teacher education (SRP-TE)
Sala, G. Barquero, B. Font, V. Giménez, J. CERME 9	(4) A multi-disciplinary approach to model some aspects of historical events	Report on the design, implementation and analysis of a sequence of tasks based upon a historical context	Inquiry-based learning; Study and research paths (SRP)
Barquero, B. Monreal, N. Ruíz-Munzón, N. CERME 10	(5) Levels of analysis of a mathematical modelling activity: Beyond the questions-answers dialectic	Report on the design, implementation and analysis of Study and Research Path	Study and Research Paths (SRP) as a teaching proposal for mathematical modelling
Sala, G. Font, V. Barquero, B. Giménez, J. CERME 10	(6) Mathematical modelling in an archaeological context: Their complementarity as essential tool for inquiry	Report on the design, implementation and analysis of a teaching sequence to promote inquiry and modelling competences	Study and Research Paths (SRP); Relationship between modelling and inquiry learning; Modelling competences
Barquero, B. Bosch, M. Wozniak, F. CERME 11	(7) Modelling praxeologies in teacher education: The cake box	Addressing teachers' lack of discursive tools to teach modelling processes	Praxeologies Anthropological Theory of the Didactic

Table 3. A snapshot of the range of contributions of active researcher 3

Active researcher 3 (5 papers)			
Authors	*Title*	*Aim of the research*	*Theoretical perspective / theoretical concepts*
Siller, H-S. Greefrath, G. CERME 6	Mathematical modelling in class regarding to technology	Discussing the specifics of modelling with computers and handheld technology in mathematics classes	Extended Modelling Cycle regarding technology when modelling; Phases of the modelling cycle
Greefrath, G. CERME 7	Modelling problems and digital tools in German centralised examinations	Discussing the simultaneous use of realistic tasks and digital tools in examinations	Modelling cycle; Extended Modelling Cycle concerning technology
Greefrath, G. Riess, M. CERME 8	Solution aids for modelling problems	Knowing more about the students' processes while working on a modelling task with a solution plan	Modelling competencies; Solution plan; Problem solving
Greefrath, G. Siller, H-S. Ludwig, M. CERME 10	Modelling problems in German grammar school leaving examinations (Abitur) – Theory and practice	Examining the relevance, the authenticity of the context, the openness of the task and the partial competencies in examination problems	Partial competences of modelling; Authenticity; Openness
Wess, R. Greefrath, G. CERME 11	Professional competencies for teaching mathematical modelling – Supporting the modelling-specific task competency of prospective teachers in the teaching laboratory	To study the development of modelling-specific task competencies in teacher education.	Modelling competency and sub-competencies; Design of modelling tasks

Table 4. A snapshot of the range of contributions of active researcher 4

Active researcher 4 (5 papers)			
Authors	*Title*	*Aim of the research*	*Theoretical perspective / theoretical concepts*
Borromeo-Ferri, R. Blum, W. CERME 6	(1) Mathematical modelling in teacher education – Experiences from a modelling seminar	Design and implementation of a university course on teaching modelling at school; Observing the students' progress on learning and understanding mathematical modelling	Modelling competencies; Modelling process as described in the modelling cycle

Borromeo-Ferri, R. Blum, W. CERME 7	(2) Are integrated thinkers better able to intervene adaptively? – A case study in a mathematical modelling environment	Identifying connections between teachers' mathematical thinking styles and types of teacher interventions in mathematical modelling environments	Thinking styles; Teacher interventions; Phases of the modelling cycle
Borromeo-Ferri, R. Blum, W. CERME 8	(3) Barriers and motivations of primary teachers for implementing modelling in mathematics lessons	Investigating central barriers and also motivations of primary teachers for implementing modelling in mathematics lessons	Techer's beliefs about modelling in school mathematics
Borromeo-Ferri, R. Mousoulides, N. CERME 10	(4) Mathematical modelling as a prototype for interdisciplinary Mathematics education? – Theoretical reflections	Discussing core similarities and differences between mathematical modelling and interdisciplinary mathematics education	Integrated teaching of science and mathematics; Modelling process as described in the modelling cycle; Students' modelling routes Models and modelling perspective
Borromeo-Ferri, R. CERME 11	(5) Assessing teaching competencies for mathematical modelling	Developing test items to assess prospective teachers mathematical modelling competencies	Modelling competencies; Competencies for teaching mathematical modelling

Table 5. A snapshot of the range of contributions of active researcher 5

Active researcher 5 (5 papers)			
Authors	*Title*	*Aim of the research*	*Theoretical perspective / theoretical concepts*
Carreira, S. Amado, N. Canário, F. CERME 8	(1) Students' modelling of linear functions: How Geogebra stimulates a geometrical approach	To examine students' approaches to application and modelling situations in the "technology world" of the modelling activity	Conceptual modelling; Contextual Modelling; Co-action between tool and user
Carreira, S. Baioa, A. M. CERME 9	(2) Assessing the best staircase: Students' modelling based on experimentation with real objects	Looking at students' conceptualization of slope in the activity of examining and assessing real-world staircases	Models and Modelling Perspective; Realistic Mathematics Education; Experimental activities
Carreira, S. Baioa, A. M. CERME 10	(3) Creating a colour palette: The model, the concept, and the mathematics	Reflecting on criteria of authenticity for school modelling tasks	Authenticity; Experiential Learning

Baioa, A. M. Carreira, S. CERME 11	(4) Simulations and prototypes in mathematical modelling tasks	Exploring the use of simulations and prototype construction in a STEM context	STEM education; Engineering Model-Eliciting Activities; Simulation
Almeida, L. Carreira, S. CERME 11	(5) The configuration of mathematical modelling activities: A reflection on perspective alignment	Discussion on the configuration of modelling activities as dependent on the modelling perspective adopted	Modelling perspectives; Design of modelling tasks

One conclusion that can be drawn from the analysis of the set of tables is that these researchers seem to fit with relative firmness in some of the so-called theoretical perspectives that are most popular or most widespread in the research community: the modelling cycle and modelling competences, the models and modelling perspective (MMP), and the anthropological theory of the didactic (ATD).

The work of Ärlebäck shows a certain tendency to explore the crossing of boundaries, namely when crossing problem solving and the ATD to undertake the analysis of modelling processes. One of the theoretical concepts that he offers for this analysis is the Model Activity Diagram, which can be considered a powerful analysis tool to study, describe, and understand the student modeller thinking. This researcher is fundamentally involved in MMP and from this perspective he develops an important concept related to promoting the modeller thinking: the idea of Model Development Sequence, which breaks with the division between modelling and application and conveys a more integrated idea of modelling activity. Another important element of his contributions lies in the implementation of the so-called Fermi problems, where he explores some of its features as being pertinent and typical of problems that appeal to a modeller thinking. In this sense, he also follows the line of inquiry of the design of modelling tasks for teaching.

Barquero's research is clearly situated in the theoretical perspective of the general theory named as ATD. The author gives great visibility to the concept of Study and Research Path as an analytical tool to describe and understand the modeller thinking of students and teachers; she also proposes that it may represent a didactic approach, that is, a way to promote the learning and the practice of a modeller thinking (SRP for learning and SRP for teacher education). In several of her studies, she proposes modelling problems that highlight authenticity and realism and contribute to the integration of knowledge from other disciplinary areas. In this sense, interdisciplinarity and the notion of inquiry-based learning seem to appear in her work as privileged approaches for the promotion and implementation of the modeller thinking.

Greefrath seems to place most of his studies in a perspective centred on the theoretical concept of the modelling cycle and the associated competencies and sub-competencies. The author also gives attention to the use of digital resources in the modelling process and proposes an interesting theoretical concept, the Extended Modelling Cycle, in which the modelling cycle includes a third world called the technological world. This brings to the forefront one of the often forgotten aspects that refer to the instruments and tools that are part of the modeller's work, that is, it shows that the modeller's thinking is mediated by tools that transform and influence his/her actions in solving a real world problem.

In the case of Borromeo-Ferri's papers, we perceive a clear alignment with the perspective of the modelling cycle and modelling competencies. Her contributions include considerable work on teacher education, namely on promoting and assessing the development of modelling competencies of future teachers. One of her interventions brings to light the very important connection between mathematical modelling and interdisciplinarity. Thus, the author opens the theoretical field in discussing mathematical modelling as a mathematical practice for promoting not only modelling competencies but also interdisciplinary mathematics education. This work combines the concept of

Individual Modelling Routes with design principles of modelling tasks according to the MMP. The theme of modelling task design is further developed in the work of Borromeo Ferri, in terms of studying the task competency for the teaching of mathematical modelling.

Finally, Carreira follows a direction generally close to the models and modelling perspective in her work. One of her themes of interest is the analysis of mathematical modelling environments that rely on practical experiences involving concrete objects of everyday life. The notion of experiential learning is part of her theoretical framework and the concept of engineering model-eliciting activity has also been a key element in some of her studies. She brings some concepts related to authenticity, based on modelling problems that involve simulation and the construction of prototypes. The integration of STEM also appears in her work as a way of putting interdisciplinarity in connection with mathematical modelling. Carreira focuses, in particular, on the thinking of the student modeller in experimental activities that involve simulation of real-world situations.

In brief, I believe that there are innovative and promising concepts that show theoretical advances, not so much in terms of new theoretical perspectives, but of ideas, notions, and constructs that broaden, develop, deepen, or extend the frontiers of other previous concepts. Here are some of them:

- Model Development Sequence
- Modelling Activity Diagram
- Study and Research Path for learning modelling
- Study and Research Path for teaching modelling
- Extended Modelling Cycle with Technology
- Interdisciplinary Mathematical Modelling
- Experimental Mathematical Modelling

These are all theoretical and conceptual developments that I find worthy of being continued.

DISCUSSION AND CONCLUSIONS

We must have representations, models, concepts, principles, abstractions, and discourses for the development of theories and surely representations only depict certain aspects of ideas or concepts that form the fabric of theories. Thus, for example, we can point out the claims from Doerr, Ärlebäck, and Misfeldt (2017) who discuss the conceptual insufficiency of each of the numerous representations of the modelling cycle in the literature. Specifically, they indicate that the cycle typically does not give due notice or pay due attention to the following aspects: "the non-linearity of modelling, the role of multiple models and pre-existing models within modelling activity, the social and critical aspects of modelling and the role of computational media in modelling" (p. 74). At the same time, the divergences remain on the advantage of drawing a line between mathematical modelling and applications and of making that partition based on the direction in which the modelling cycle is being followed. Several people continue to maintain the blended A&M (Blum & Niss, 1991; Blum, 1995) as a comprehensive and productive way of thinking about an activity that has several nuances when implemented in the school context. Another issue that is the subject of divergences concerns the influence produced by the introduction of technological devices in the process of mathematical modelling, including, for instance, the possibility of working with simulators or using computational tools that automate the obtaining of approximate models.

Despite the disagreements that different perspectives can bring out on some theoretical concepts, my main interest is to ask whether we are able to trace a possible conceptual common ground with generative power to nourish not only the development of concepts, representations, perspectives, and lines of inquiry, but also to induce combining, coordinating and even synthesising or integrating within the field (Prediger, Bikner-Ahsbahs & Arzarello, 2008; Bikner-Ahsbahs & Prediger, 2010). I tentatively say yes and my conviction is that we need to keep alive the contributions of the different communities of practice that intersect in the field of teaching and learning mathematical modelling.

Ideas like those of Pollak (2007, 2015), Pelesko (in Cai et al., 2014), Galbraith (2015), or Blum (2011), just to mention a few, seem very useful in revealing a certain thread: irrespective of the perspective adopted, we are fundamentally and essentially working on the development of the modeller thinking. This is similar to what other topics in mathematics education have been advocating, namely the learning of algebra through the development of algebraic thinking or the learning of geometry through the development of geometric thinking. It seems indisputable that students learn modelling and applications through the development of a modeller thinking and that teachers need to engage with that thinking, for instance, in designing tasks or supporting students in modelling activities.

One may then ask whether there is any possibility at the present state of combining, coordinating or synthesizing, in a more or less coherent and productive way, the various theoretical concepts and perspectives. My answer to this question is that of an optimistic yes. And I believe that such endeavour could be achieved by a diligent work around the construction, deepening and refinement of theoretical concepts that offer relevant contributions to further reconceptualise the development of the modeller thinking in its multiple facets: cognitive, cultural, critical, pedagogical, etc.

One of the quite robust concepts today is definitely that of the mathematical modelling cycle (with its various schematic representations). Another one, which is closely related, is that of modelling competencies. They are concepts of enormous relevance and have captured the energy and efforts of many researchers. Also very promising is that of Study and Research Path, maybe reconstructed as Study and Research Path for Modelling (learning and teaching). Maybe less developed theoretical concepts are, for example, those of interdisciplinary modelling and of experimental modelling. Besides these, there are other concepts that are important and valuable such as prescriptive modelling and critical modelling, to mention only a few.

In Figure 1, I suggest a rather free representation (surely deficient in many senses) of what I see as the generative role of the modeller thinking development in fostering new advances and possible combination and synthesis. As mentioned before, we have actually the example of the combination between the modelling cycle and modelling competencies. Other possibilities of coalescence will probably require boundary-crossing and further conceptual strengthening.

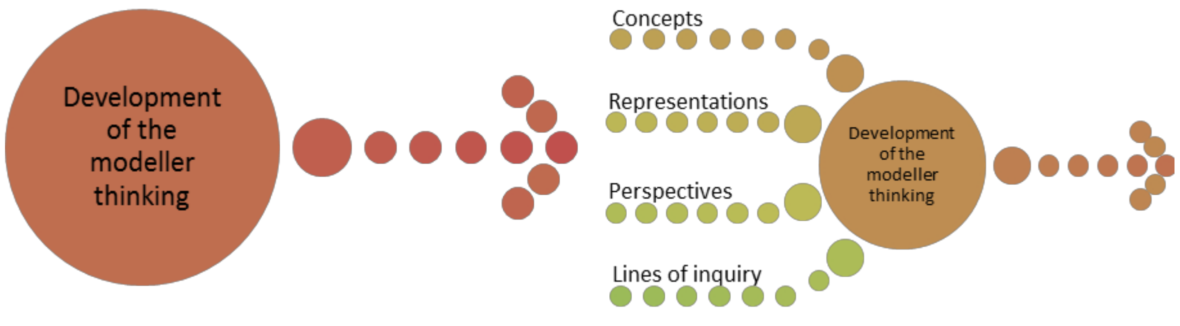

Figure 1. The development of the modeller thinking in fostering the combination of theoretical concepts and perspectives

This suggestion arises from my perception, based on various proposals for the organization, classification, systematization and mapping of the field over a period of at least fifteen years, that we are seeing cross-boundary movements and the search for interactions between different theoretical perspectives (Stillman et al., 2017). Simultaneously, the predominance of some theoretical perspectives in the debate seems to be clearer, and it is also evident that there is a widening/deepening of some theoretical concepts that are being worked out by several active and committed researchers, based on empirical studies across several lines of inquiry.

I conclude my discussion with a simple illustration whose purpose is mainly to be thought provoking. In my master's thesis (Carreira, 1993), I studied the processes of 10^{th} grade students when solving application and modelling problems involving trigonometry, with the use of the spreadsheet. At that time, I used as a theoretical lens the notion of translating between different systems of mathematical representations and the idea of conceptual amplifier to interpret the role of the spreadsheet in the problems proposed. The empirical work involved a sequence of application and modelling activities; for the purposes of the study, I considered that students were modelling physical phenomena or situations when they used their mathematical tools (like sine functions, parameters, etc.) to come up with mathematical models. Moreover, I adopted the modelling cycle as a structuring concept of the pedagogical approach, with special importance given to some processes such as interpreting, representing and mathematizing.

Looking back, I believe that I was proposing something like a Model Development Sequence (it was not a theoretical concept of my research then). One of the activities could be seen as a Model Exploration Activity (MXA), according to Ärlebäck, Doerr, and O'Neil (2013). The students were asked to describe mathematically the physical phenomenon of beats that can be generated by using two tuning forks of slightly different frequencies vibrating simultaneously. The students already had a previous model about the propagation of sound waves and new the meaning of the parameters for amplitude and frequency of the sound. What they did then was to explore, with the help of the spreadsheet, the mathematics of constructive and destructive interference and to mathematically translate the sound that occurs by overlapping sound waves with very similar frequencies, that is, a succession of lower and higher sounds, the sound wave beats, as represented in Figure 2.

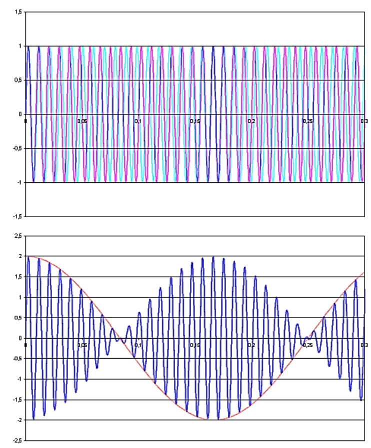

Figure 2. The sum of two sine waves (Retrieved from https://upload.wikimedia.org/wikipedia/commons/9/91/Beat.png (copyrighted free))

At the same time, students worked on graphical and tabular representations of sound waves and examined the effects of interference, both graphically and numerically. For example, they looked for ways to find the maxima and minima of the wave resulting from the sum of two waves and verified that they occurred periodically. They conjectured a possible relation between the frequency of the beats and the frequencies of the two generating waves and were able to interpret this idea in terms of the oscillatory movement of each of the superimposed waves. Thus, this application and modelling task could be an example of interdisciplinary modelling (relating mathematical models and concepts of physics) and illustrates a certain kind of modeller thinking, in which the knowledge and tools of other disciplines are integrated into the exploration of mathematical models.

Students have also worked intensively in the technological world, and in many respects this world of the spreadsheet representational tools has become a conceptual amplifier. The fact that the spreadsheet allows to work on functions and relationships between variables in numerical form, that is, using increments of the independent variable, makes it helpful for understanding mathematical results and concepts and for creating accessible representations of the real phenomenon. Both graphically and numerically, beats emerged expressed in mathematical terms. In this sense, the students were developing their thinking with technology. The extended modelling cycle regarding technology (which of course was not a formal concept at that time) makes perfect sense as a theoretical concept to support this way of thinking and working with mathematical models.

Finally, the students have produced many questions and answers and also some results from an initial generating question (mathematically explaining the beat that is produced by two very close sound frequencies). This could well have been investigated and described using the conceptual tool of Study and Research Path, namely in finding and interpreting interesting differences in the solutions of the various groups in the class.

I used the example of my early 1992 study (completed shortly after ICTMA 5 where I offered a workshop on oscillations and waves) because it was done at a time when theorizing in the field of teaching and learning mathematical modelling was still taking its first steps. Now, it suits me for underlining the point that theoretical concepts are extremely important in stirring the advancement of the field. And I also notice that from that time on, throughout the various lines of inquiry, a major aim of the research continues to be the development of the student modeller and of the teacher modeller thinking. Therefore, it seems admissible that the development of the modeller thinking is a key for some possible conceptual integration.

References

Ärlebäck, J. B., Doerr, H. M., & O'Neil, A. H. (2013). A modeling perspective on interpreting rates of change in context. *Mathematical Thinking and Learning*, *15*(4), 314–336.

Barquero, B., Carreira, S., & Kaiser, G. (2017). Introduction to the papers of TWG06: Applications and modelling. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME10)* (pp. 877-883). Dublin, Ireland: DCU Institute of Education and ERME.

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of Theories – An approach for exploiting the diversity of theoretical approaches. In B. Sriraman, & L. English (Eds.), *Theories of Mathematics Education: Seeking New Frontiers* (pp. 483-506). Dordrecht, The Netherlands: Springer.

Blomhøj, M. (2010). Introduction - Applications and modelling. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 2042-2045). Lyon, France: ERME.

Blum, W. (1995). Applications and modelling in mathematics education: Some important aspects of practice and research. In C. Sloyer, W. Blum, & I. D. Huntley (Eds.), *Advances and Perspectives in the Teaching of Mathematical Modelling and Applications* (pp. 1–20). Yorklyn, USA: Water Street Mathematics.

Blum, W. (2011). Can modelling be taught and learnt? Some answers from empirical research. In G. Kaiser, W. Blum, R. Borromeo-Ferri, & G. Stillman (Eds.), *Trends in Teaching and Learning of Mathematical Modelling (ICTMA 14)* (pp. 15-30). Dordrecht, The Netherlands: Springer.

Blum, W. & Borromeo-Ferri, R. (2009). Mathematical modelling: Can it be taught and learnt? *Journal of Mathematical Modelling and Application, 1*(1), 45-58.

Blum, W., & Leiß, D. (2007). How do students and teachers deal with modelling problems? In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling (ICTMA 12). Education, engineering and economics* (pp. 222–231). Chichester, United Kingdom: Ellis Horwood.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects – State, trends and issues in mathematics instruction. *Educational Studies in Mathematics*, *22*(1), 37–68.

Borromeo-Ferri, R. (2010). On the influence of mathematical thinking styles on learners' modelling behaviour. *Journal für Mathematik-Didaktik*, *31*(1), 99-118.

Cai, J., Cirillo, M., Pelesko, J. A., Borromeo-Ferri, R., Borba, M., Geiger, V., …, & Kwon, O. N. (2014). Mathematical modeling in school education: Mathematical, cognitive, curricular, instructional, and teacher education perspectives. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the 38th meeting of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education, Vol. 1* (pp. 145–172). Vancouver, Canada: IGPME.

Carreira, S. (1993). *A aprendizagem da Trigonometria num contexto de aplicações e modelação com recurso à folha de cálculo* (Master Thesis). Lisbon, Portugal: Associação de Professores de Matemática.

Carreira, S., Barquero, B., Kaiser, G., & Cooper, J. (2019). Introducing CERME's Thematic Working Group 6 – Applications and Modelling. *Newsletter of the European Mathematical Society, Mathematics Education, ERME Column, 111*, 48-49.

Carreira, S., Barquero, B., Kaiser, G, Lingefjärd, T., & Wake, G. (2015). Introduction to the papers of TWG06: Applications and modelling. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (CERME9)* (pp. 790-793). Prague, Czech Republic: Charles University in Prague and ERME.

Doerr, H. M., Ärlebäck, J. B., & Misfeldt, M. (2017). Representations of modelling in mathematics education. In G. A. Stillman, W. Blum, & G. Kaiser (Eds.), *Mathematical Modelling and Applications: Crossing and Researching Boundaries in Mathematics Education* (pp. 71-81). Cham, Switzerland: Springer.

English, L. D., Ärlebäck, J. B., & Mousoulides, N. (2016). Reflections on Progress in Mathematical Modelling Research. In A. Gutiérrez, G. C. Leder, & P. Boero (Eds.), *The Second Handbook of Research on the Psychology of Mathematics Education* (pp. 383-413). Rotterdam, The Netherlands: Sense Publisher.

Frejd, P. (2014). *Modes of Mathematical Modelling: An analysis of how modelling is used and interpreted in and out of school settings* (Unpublished doctoral thesis). Linköping University, Linköping, Sweden.

Galbraith, P. (2015). Modelling, education, and the epistemic fallacy. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical Modelling in Education Research and Practice: Cultural, Social and Cognitive Influences* (pp. 339-349). Cham, Switzerland: Springer.

Galbraith, P., & Stillman, G. (2006). A framework for identifying student blockages during transitions in the modelling process. *ZDM*, *38*(2), 143–162.

Geiger, V., & Frejd, P. (2015). A reflection on mathematical modelling and applications as a field of research: Theoretical orientation and diversity. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical Modelling in Education Research and Practice: Cultural, Social and Cognitive Influences* (pp. 161-171). Cham, Switzerland: Springer.

Jablonka, E. (2007). The relevance of modelling and applications: Relevant to whom and for what purpose? In W. Blum, P. L. Galbraith, H-W. Henn, & M. Niss (Eds.), *Modelling and Applications in Mathematics Education - The 14th ICMI Study* (pp. 193-200). Boston, USA: Springer.

Kaiser, G., Blum, W., Borromeo-Ferri, R., & Stillman, G. (2011). Trends in Teaching and Learning of Mathematical Modelling – Preface. In G. Kaiser, W. Blum, R. Borromeo-Ferri, & G. Stillman (Eds.), *Trends in Teaching and Learning of Mathematical Modelling (ICTMA 14)* (pp. 1-5). Dordrecht, The Netherlands: Springer.

Kaiser, G., Carreira, S., Lingefjärd, T., & Wake, G. (2011). Introduction to the papers of WG 6: Applications and modelling. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 925-926). Rzeszów, Poland: University of Rzeszów and ERME.

Kaiser, G., & Sriraman, B. (2006). A global survey of international perspectives on modelling in mathematics education. *ZDM*, *38*(3), 302–310.

Lesh, R. A., & Doerr, H. M. (Eds.). (2003). *Beyond Constructivism: Models and Modelling Perspective on Mathematics Problem Solving, Learning, and Teaching.* Mahwah, USA: Lawrence Erlbaum.

Lesh, R., & English, L. (2005). Trends in the evolution of models and modelling perspectives on mathematical learning and problem solving. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, Vol. 1* (pp. 192-196). Melbourne, Australia: PME.

Lesh, R., & Fennewald, T. (2013). Introduction to Part I. Modeling: What is it? Why do it? In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modelling Students' Mathematical Modelling Competencies (ICTMA 13)* (pp. 5–10). New York, USA: Springer.

Lingefjärd, T., Carreira, S., Kaiser, G., & Wake, G. (2013). Introduction to the papers and posters of WG6: Applications and Modelling. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.), *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* (pp. 926-929). Ankara, Turkey: Middle East Technical University and ERME.

OECD (2013). *PISA 2012 Assessment and Analytical Framework: Mathematics, Reading, Science, Problem Solving and Financial Literacy*. Paris, France: OECD Publishing.

OECD (2019). *PISA 2018 Assessment and Analytical Framework*. Paris, France: OECD Publishing.

Pollak, H. (2007). Mathematical Modelling – A conversation with Henry Pollak. In W. Blum, P. L. Galbraith, H-W. Henn, & M. Niss (Eds.), *Modelling and Applications in Mathematics Education - The 14th ICMI Study* (pp. 109-120). Boston, USA: Springer.

Pollak, H. O. (2015). The place of mathematical modelling in the system of mathematics education: Perspective and prospect. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical Modelling in Education Research and Practice: Cultural, Social and Cognitive Influences* (pp. 265-276). Cham, Switzerland: Springer.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM*, *40*(2), 165-178.

Saeki, A., & Matsuzaki, A. (2013). Dual modelling cycle framework for responding to the diversities of modellers. In G. A. Stillman, G. Kaiser, W. Blum, & J. P. Brown (Eds.), *Teaching Mathematical Modelling: Connecting to Research and Practice* (pp. 89–99). Dordrecht, The Netherlands: Springer.

Schwarzkopf, R. (2007). Elementary modelling in mathematics lessons: The interplay between "real-world" knowledge and "mathematical structures". In W. Blum, P. L. Galbraith, H-W. Henn, & M. Niss (Eds.), *Modelling and Applications in Mathematics Education - The 14th ICMI Study* (pp. 209-216). Boston, USA: Springer.

Stillman, G. A. (2019). State of the Art on Modelling in Mathematics Education – Lines of Inquiry. In G. Stillman & J. P. Brown (Eds.), *Lines of Inquiry in Mathematical Modelling Research in Education (ICME-13 Monographs)* (pp. 1-20). Cham, Switzerland: Springer.

Stillman, G. A., Blum, W., & Kaiser, G. (2017). Crossing boundaries in mathematical modelling and applications educational research and practice. In G. A. Stillman, W. Blum, & G. Kaiser (Eds.), *Mathematical Modelling and Applications: Crossing and Researching Boundaries in Mathematics Education* (pp. 1-22). Cham, Switzerland: Springer.

Stillman, G., & Brown, J. P. (Eds). (2019). *Lines of Inquiry in Mathematical Modelling Research in Education (ICME-13 Monographs)*. Cham, Switzerland: Springer.

Wake, G., Foster, C., & Swan, M. (2015). Understanding issues in teaching mathematical modelling: Lessons from lesson study. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (CERME9)* (pp. 937-943). Prague, Czech Republic: Charles University in Prague and ERME.