MATHEMATICAL MODELLING - BACKGROUND AND CURRENT PROJECTS IN GERMANY

Modelización Matemática – Antecedentes y proyectos en la actualidad en Alemania

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Abstract

Mathematical modelling is an important field of research in mathematics education in Germany and a key competence in educational standards. The significance of modelling problems in schools and teacher training has increased in the past few decades with various research projects. This article aims to clarify important aspects of the current status of mathematical modelling in the German-speaking world. The background to the German modelling discussion will be set out and the current significance for modelling in German educational standards presented. Research projects on the measurement of modelling competence, the use of digital and strategic tools and research on teacher training are described. Various modelling cycles are also presented and their goals and use under various circumstances outlined. The situation in schools is set out and current and future fields of research identified.

Keywords: *mathematical modelling, Germany, research, educational practice.*

Resumen

La modelización matemática es un campo importante de investigación en educación matemática en Alemania y una competencia clave en los estándares educativos. La importancia de los problemas de modelización en las escuelas y en la formación del profesorado ha aumentado en las últimas décadas con varios proyectos de investigación. Este artículo pretende aclarar aspectos importantes sobre el estado actual de la modelización matemática en el contexto de los países de habla alemana. Se presentan los antecedentes de la discusión sobre modelización, así como su papel y relevancia actual en los estándares educativos alemanes. Se describen algunos de los proyectos de investigación más relevantes sobre la medición de la competencia de modelización, el uso de herramientas digitales y sobre la formación del profesorado. Se muestran a la vez distintos ciclos de modelización, así como sus objetivos y usos, de acuerdo con las líneas de investigación previamente citadas. Por último, se describe la situación en las escuelas y se identifican líneas futuras de investigación.

Palabras clave: *modelización matemática, Alemania, investigación, práctica educativa.*

INTRODUCTION AND BACKGROUND

The use of application-oriented tasks to learn mathematics has a long history in Germany. Mathematical modelling became particularly well known in Germany in the 1980s. Blum (1985) described many application examples which reflected a wide range of topics. It became clear that the discussion on applications and modelling was increasingly significant. One of the most well-known modelling cycles in Germany (see Figure 1) was also mentioned by Blum (1985).

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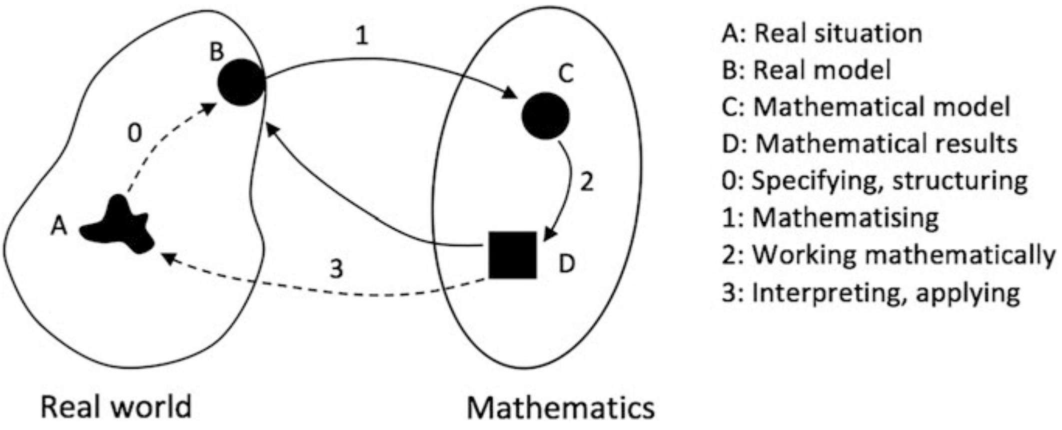

Figure 1. Modelling cycle by Blum (Blum & Kirsch, 1989, p. 134)

The founding of the German-speaking ISTRON Group by Werner Blum and Gabriele Kaiser in 1990 contributed to an increase in the intensity of the modelling discussion in Germany. The idea of ISTRON is to increase the number of applications that appear in mathematics classes. Research on modelling has also been increased significantly. Germany has also hosted important conferences on mathematical modelling (ICMI Study, ICTMA 1987 and 2009).

Different goals at various levels are pursued with the focus on application in mathematics classes. The particular opportunity of dealing with applications in mathematics classes is that interesting insights into both mathematics as a subject and in reality, are possible. We distinguish between content-based, process-based and general goals of application-oriented mathematics classes below.

The *content-based goals* of application-oriented mathematics classes can be described for two areas. Learning mathematical terms and structures (Strehl, 1979, p. 26) and knowledge of the environment are both content-based goals. In addition to the inner-mathematical goals, a further goal is enabling people to perceive and understand our world. This also corresponds to the first of Winter's three basic experiences that each student should be taught in mathematics classes (Winter, 1996). The classical content of arithmetic such as quantities and interest calculation are particularly important when it comes to achieving content-based goals. They contribute to achieving the content-based goals in both categories. Quantities, for example, are mathematical objects with generalizable structures, and working with quantities also requires people to engage with their environment. The development of concepts about quantities is a key part of working with quantities in mathematics classes. Students can only check the results of tasks for plausibility if they have the relevant concepts about certain important units. Concepts about volume, for example, could be that a pack of milk is 1 litre or that a half-filled bathtub is 100 litres.

Goals are often set out for *application-oriented mathematics* classes in which it is not the outcome in terms of the knowledge of mathematical content that is the focus, but rather the process taken to achieve these results. Discussions and analyses of the environment using mathematical tools are also important (Spiegel & Selter, 2006, p. 74). Dealing with application orientation in mathematics classes requires general mathematical competence and problem-solving competence. The key heuristic strategies linked to problem solving such as working with analogies or working backwards can be used and promoted when working on application-oriented tasks. These goals also fit with the third of Winter's three basic experiences for a general mathematics class: mathematics classes should aim "to develop, by working on tasks, problem-solving competence that go beyond mathematics" (Winter, 1996). We can make a distinction here between two further areas: on the one hand, there are *process-based goals* which are specific to classes with an application orientation, and on the other hand, there are process-based goals that are relevant to mathematics classes as a whole. One key aim of application-oriented mathematics classes is developing *modelling competence*, in other words the ability to transfer problems from reality to mathematics in a suitable way for them to be processed and solved. The step of mathematization, in other words the finding or determination of a suitable mathematical model, is also an important goal (Fricke, 1987, p. 11 et seqq.). Modelling competence also involve content elements such as metaknowledge about modelling processes and knowledge of different mathematical models such as proportional relation. The focus in the case of modelling, however, is the potential to solve a modelling problem. So, the focus is on the process, not the result, in other words a process-based goal. However, while developing modelling competence is a goal that typically assumes a link to a real situation, this is not the case when it comes to problem-solving competence. In many cases, application-oriented tasks should be viewed as a problem, the solution of which also requires problem-solving competence; there are, however, many inner-mathematical problems, the solution of which does not come under the category of applications. This includes, for example, mathematical proofs. Problem-solving competence is therefore a process-based goal that cannot be attributed exclusively to application-oriented mathematics classes but rather is relevant to mathematics classes in general (Strehl, 1979, p. 26). Other process-based goals that can be achieved in application-oriented mathematics classes but are also relevant to mathematics classes in general and beyond is arguing, reflecting (Radatz & Schipper, 1983, p. 20 et seqq.) and the use of suitable tools such as measuring devices and digital tools.

There are also goals that are not specific to the content and processes of application-oriented mathematics classes but rather go beyond this. These goals can in some cases also be achieved in other subjects. One goal of application-oriented mathematics education is to increase motivation. Substantive problems at the start of a learning process can achieve this goal particularly well (Maier & Schubert, 1978, p. 14). Application-oriented mathematics classes can also contribute significantly to the general goals of mathematics classes due to their link to the real life. Through the everyday problems experienced in class that can be mathematically processed, the purpose of mathematics as a subject becomes clear to students. The focus on application means the students can also be better prepared for training, jobs, study and everyday life (Westermann, 2003, p. 148). The application orientation on calculation problems suggests using the associated sciences to a greater extent. Application-oriented mathematics class is a good point of reference for working with other subjects but also with mathematics itself. The implementation of cross-subject projects is also a general goal (Jahner, 1985, p. 25 et seqq.). During application-oriented mathematics classes, students also look at social problems. Mathematics has a general educational nature (Westermann, 2003, p. 148). If this happens in class, students are at least prepared to subsequently handle political, social and economic problems thanks to the real applications (Maier & Schubert, 1978, p. 15). For example, the discussion of tax models is a possible element of socially relevant content in application-oriented mathematics class. Discussing social and political problems means students will ultimately have the competence to take responsibility in society (see Blum, 1996; Kaiser-Meßmer, 1986; Greefrath, 2018).

CURRENT SITUATION IN TERMS OF MATHEMATICAL MODELLING IN GERMANY

Nowadays, application orientation is a natural component of mathematics classes and educational standards in Germany. Modelling has been included as a competence in the educational standards (KMK, 2012) and the curricula of the various federal states. The Standing Conference of the Ministers of Education and Cultural Affairs in Germany has a general strategy on educational monitoring since 2006. It aims to increase the focus on competence in the education system. The general competence of modelling plays an important role in the subject of mathematics. In addition to international student assessment studies (PISA, TIMSS), there are also national student assessment studies and comparative studies (VERA). These tests are carried out in grades three and eight of all comprehensive schools and look at the competence students have obtained at a given point in time. The aim of the comparative studies is to give teachers differentiated feedback of the requirements their students are meeting based on the educational standards. In addition, Germany has a pool of advanced level qualification examination questions that federal states can use since

2017. This is an important step in terms of improving the quality of the examination questions and gradually aligning the level of requirements in the various federal states. The questions were developed on the basis of the educational standards. Accordingly, there are elements of questions that test the modelling competence in the German advanced level qualification examination questions. In general, the examination requirements should include a balanced ratio of formal and application-oriented questions (KMK, 2012, p. 24).

RESEARCH ON MATHEMATICAL MODELLING IN GERMANY

There are many research projects on mathematical modelling in Germany. There has been a significant increase in the past ten years. In addition to this increase, there have also been considerable methodological developments. This could be one reason for the changes in terms of the type of research projects carried out on mathematical modelling in the past few decades. Research projects on modelling now more commonly use experimental control group designs and sophisticated statistical methods to analyse various research questions. Some research results on mathematical modelling are presented below by way of an example.

Discussion of modelling cycles

In Germany, different modelling cycles are used and discussed intensively. The entire modelling process is often presented in an idealised version as a modelling cycle. Idealised means that this representation itself is also a model. These models of mathematical modelling can be accentuated. The literature therefore contains various cycle representations of modelling. We are now presenting some of these modelling cycles in order of increasing complexity of the step from the situation to the model.

We use simple mathematising to describe modelling cycles in which just one step is used from the situation to the model. This step is called “modelling/mathematization” in Ortlieb (2004) (see Figure 2). Modelling is now the general term for the entire cycle process. In the past, this process was also known as model building. Mathematization, as in the case of Ortlieb, is the standard term for the step that ends with the creation of the mathematical model. A particularly clear representation of this generally recognised model of modelling comes from Schupp (1989) and can be broken down into the dimension of mathematics and the world, which is generally standard. An equal distinction is also made between problem and solution in a second dimension.

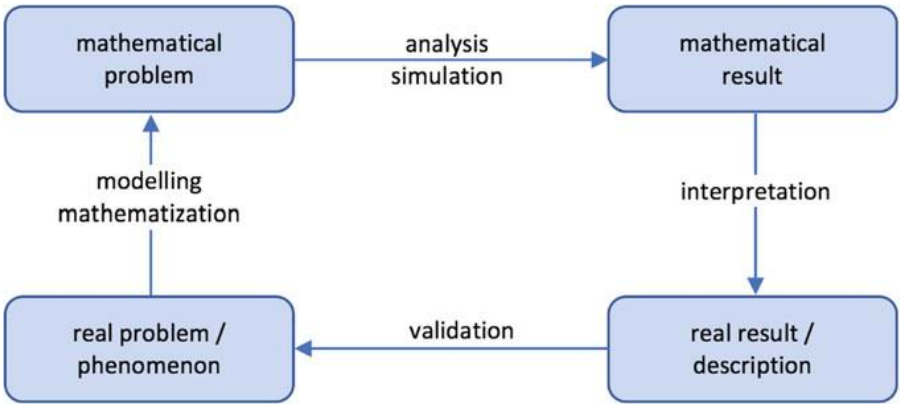

Figure 2. Modelling cycle according to Ortlieb (2004, p. 23)

In the case of the models mentioned, it is often noted that the cycle is not always complete or can be passed through several times. Büchter and Leuders (2005, p. 76) and Maaß (2015, p. 202) represent this multiple passing through of the modelling cycle as a modelling spiral. This also makes development during the modelling process clear. Experience with the problem increases after each cycle. Here, too, a distinction is made between the real and the mathematical model, but the specification of the problem is formulated as a separate step between reality and the model.

One of the most well-known modelling cycles in Germany was described by Blum (1985, see Figure 1). Here, an intermediate step is added for the creation of the mathematical model, with simplification into the reality, the real model, being viewed as another separate step. This model was developed together with Kaiser-Meßmer (1986) and is used by many authors. Maaß (2006) and Kaiser and Stender (2013) also add the interpreted solution as an intermediate step between the mathematical results or the mathematical solution and the real situation or reality (see Figure 3). The intermediate step clarifies the different processes of "interpreting" and "validating" in the second half of the modelling cycle.

A more comprehensive model of modelling by Blum and Leiß (2007) was created from a cognitive perspective (see Figure 4). The model created by Blum (1985) was expanded to include the situation model. The creation of the mathematical model is addressed in greater detail and the process of the individual creating the model is set out in greater detail. The situation model describes the mental representation of the situation by the individual. The model by Fischer and Malle (1985) also describes the step from the situation to the mathematical model in detail. The addition of data acquisition is interesting here and plays a role in many open modelling or application tasks. For example, a lot of information needs to be determined by estimation when carrying out Fermi tasks.

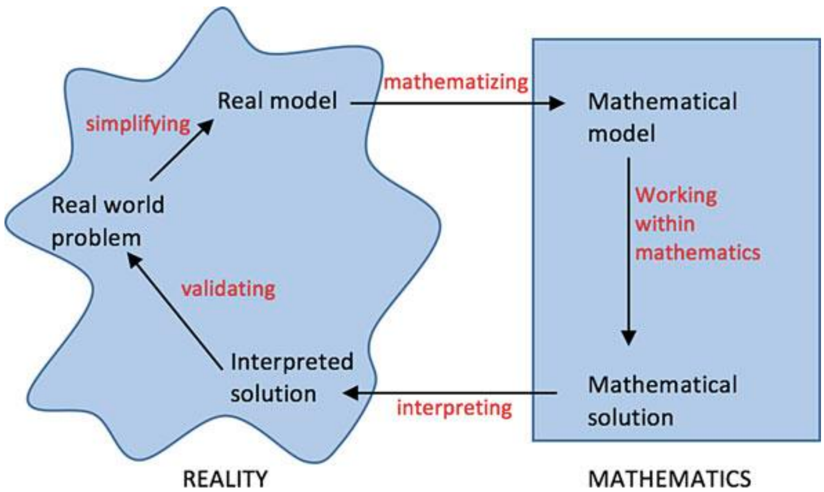

Figure 3. Modelling cycle according to Maaß (2006, p. 115)

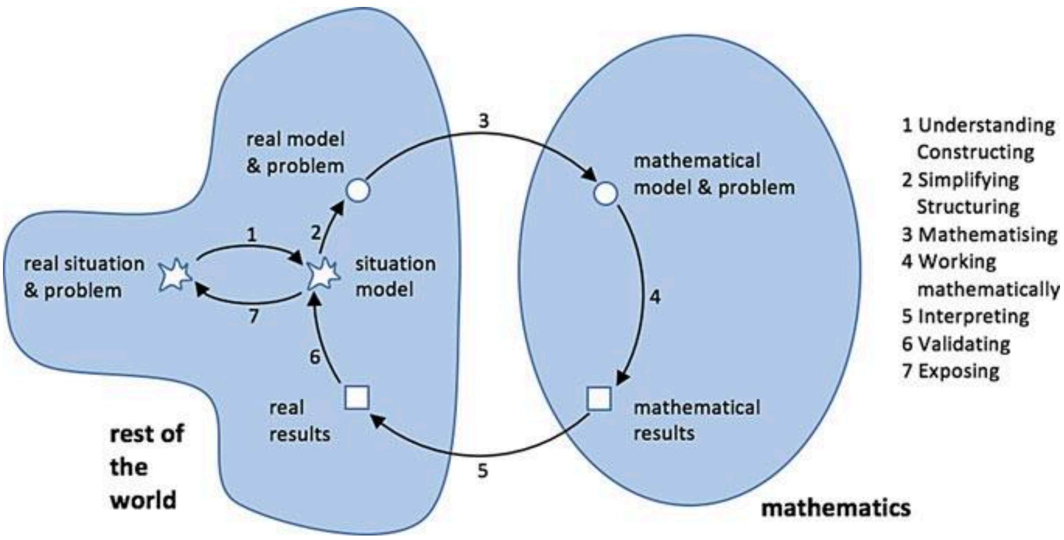

Figure 4. Modelling cycle according to Blum and Leiß (2007, p. 225)

The different models of modelling shown have different points of focus depending on their target group, the subject matter of the research and the research interest. In particular, a distinction must be made between normative and descriptive models of modelling. A certain model could be used to describe the activities carried out by school students during an empirical investigation. Very complex models such as those set out in Figure 4 are suitable for this. A cycle such as that set out in Figure 2 could also be used in a normative manner as support for students when processing modelling tasks in a classroom environment (Greefrath & Vorhölter, 2016).

Measurement of modelling competence

The assessment of modelling competence generally depends on the underlying concept of a competence. Modelling competence not only involve the ability to model but also a willingness to address problems with mathematical aspects from reality using mathematical modelling (Kaiser, 2007, p. 110). For this, a written test was developed that confronts students with a selection of situations that can be processed using mathematical methods (Hankeln, Adamek, & Greefrath, 2019).

In constructing a test, it is necessary to decide whether to use holistic or atomistic modelling tasks (Blomhøj & Jensen, 2003). Holistic tasks require a full modelling cycle to be carried out while atomistic tasks are pre-constructed and focus on one or two steps in the modelling process. The use of holistic tasks is sensible when measuring general modelling competence. This has already been done in various studies (Kreckler, 2017; Rellensmann, Schukajlow, & Leopold, 2017; Schukajlow, Kolter, & Blum, 2015). In atomistic tasks, the students only need to process problems that require a limited range of modelling competence. These tasks cannot be used to obtain information about whether a person would generally be able to carry out a full modelling process. However, atomistic tasks can be used to measure different modelling competencies separately from one another, which is not possible with holistic tasks. There are already tests that use atomistic modelling tasks, but these summarise various competencies (Brand, 2014; Zöttl, Ufer, & Reiss, 2011). A test has therefore been constructed that records the competencies of simplifying, mathematising, interpreting and validating separately. Holistic tasks are not used due to the large number of test items that would be required. One example item for the competency of simplifying is the one proposed in the lighthouse task (see Figure 5).

The students' task is to select all of the information that is relevant to the calculation of the distance to the horizon. This multiple-choice item is thought to measure the competency of identifying relevant quantities and key variables. This is part of the definition of the competency of simplifying. Corresponding items for the other competencies were also developed. There are pre-tests and post-tests, each with two groups in a multi-matrix design. Each test booklets consists of 16 items and takes 45 minutes to complete. An evaluation of the test instrument involving 3300 completed tests was able to show that the data collected can best be described using a four-dimensional between-item model in which the various competencies are recorded as separate dimensions of a latent construct. This result shows how certain the modelling competencies in question can be empirically measured. It was also possible to conclude that the competencies of simplifying, mathematising, interpreting and validating can be understood as different components of a global modelling competence (Hankeln et al., 2019).

Promotion of modelling competence

Some research focuses on how to promote modelling competence in school using various different tools. One example of a project that took into account the investigation of the promotion of modelling competence is the LIMo project at the University of Münster (2015-2018). The aim of the project was to investigate whether modelling competence can be promoted using digital tools such as dynamic geometry software and using strategic tools such as a solution plan. In order to do this, an interventional study was carried out in spring 2016 in a quasi-experimental pre/post/follow-up design in 44 grade nine classes in German grammar schools and the development of the students' competence was measured using a previously developed modelling test with items testing the competencies. The intervention consisted of a series of four class sessions (each of 45 minutes) on modelling tasks. During the class, for example, students had to calculate the lawn area of a castle garden. A sketch of the castle garden was available for this (see Figure 6). The students initially had to discuss which green areas belonged to the castle garden and what simplifications they could make to calculate the area.

During their summer vacation, Marcus and Irina are standing on top of a lighthouse and enjoying the view. "How far is it to the horizon?" Irina asks.

Mark all of the following information that you consider to be important to calculating the distance to the horizon.

https://upload.wikimedia.org/wikipedia/commons/b/bf/Louisbourg_Lighthouse.jpg

☐	Between the lighthouse and the ocean, there are 25 m of sandy beach.	☐	The two are standing on the Atlantic coast in France.
☐	There are no clouds in the sky.	☐	The radius of the earth measures 6370 km.
☐	The lighthouse is 83 m high.	☐	The lighthouse's light shines as far as 10 km.

Figure 5. The Lighthouse Task (translated): multiple-choice item that measures competencies in simplifying a problem (Hankeln et al., 2019, p. 148)

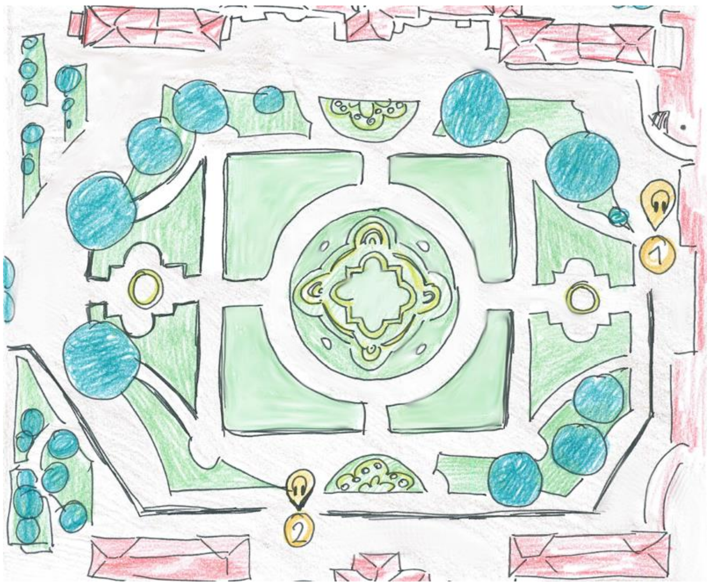

Figure 6. Sketch of the castle grounds (Hankeln, 2018, p. 152)

The 44 classes were broken down into three groups of approximately the same size. All of the groups worked on the same modelling tasks, with one modelling task being completed in each session. One group also used dynamic geometry software (GeoGebra), the second group used a five-step strategic solution plan with cognitive learning strategies in each step in the modelling process that was available on posters and worksheets for the entire investigation, and the third group used neither of the tools.

Only a short-term improvement in performance for the competencies of interpreting and validating was able to be identified for the strategic solution plan for the entire sample of consisting of both of the test groups investigated with and without a solution plan. There was a small effect of the point at which the measurement was carried out. The investigation of group membership as a factor for the development of competence showed that the solution plan has a minor effect on the development of the competency of interpreting, while no interaction effect between the test group and the time of measurement was able to be identified for the other competencies. In terms of long-term competence development, with a further measurement point defined three months after the class sessions, there was a long-term, stable increase in the competency of interpreting in the solution plan group (Adamek, 2018).

In terms of the use of the dynamic geometry software, it was assumed that it was not the modelling competencies that the students had at the first measurement point that had an impact on the effectiveness of the intervention with or without DGS but rather their competence in using the software that played a role. This assumption was confirmed in that no significant interaction effects were able to be identified between the competencies at the first measurement point and the test group. The class unit on DGS correspondingly had an equal effect on the competency development of students who were initially stronger and those who were initially weaker. The analysis of the data collected, however, showed that the factor of test group did not have a significant impact in any of the competencies taken into account. Contrary to expectations, the competencies did not differ accordingly (Hankeln, 2018).

The link between program-related self-efficacy, the competency of mathematization and the beliefs on the dynamic geometry software were also analysed within the group with dynamic geometry software. There is a significant correlation between program-related self-efficacy and the beliefs on the software. Students who felt more confident about their competence with the tool rated the software more positively and vice versa. It was also possible to show that the program-related self-efficacy was a significant predictor for mathematization competency in the post-test, even if we controlled for the pre-test. Students with a higher program-related self-efficacy improved their mathematization competency more than students with a lower self-efficacy, albeit with a small effect size ($\beta = 0.16$) (Greefrath, Hertleif, & Siller, 2018).

Competence to teach mathematical modelling

Teachers' professional competence can be described using various models based on Shulman (1986) in which the core areas of competence of teachers are described. Based on the model used in the COACTIV study (Baumert & Kunter, 2013) and a theoretically derived dimensions of competence by Borromeo-Ferri and Blum (2009), a model was developed specifically for the teaching of mathematical modelling.

Certain aspects and areas of competence were selected from the COACTIV model (Baumert & Kunter, 2013) with a focus on the teaching of mathematical modelling. In the field of professional knowledge, pedagogical content knowledge is characterised by specific content with regard to the teaching of mathematical modelling. The beliefs and self-efficacy can also be specified in terms of mathematical modelling. Pedagogical content knowledge was broken down into four areas of competence taking into account the dimensions of competence by Borromeo-Ferri and Blum (2009). These include knowledge of interventions, modelling processes, modelling tasks and goals of modelling. Diagnostic competence concerning knowledge of modelling processes, for example, consists of the ability to identify modelling phases and the ability to recognise difficulties in the modelling process. A quantitative test instrument on the teaching of mathematical modelling was developed on the basis of the structural model. The test consists of two parts: in the first part, modelling-specific pedagogical content knowledge is recorded in a performance test. In order to do this, a total of 70 dichotomous test items were operationalised in a multiple choice and combined single choice format. The items in the fields of knowledge about modelling processes and knowledge about interventions relate to modelling tasks that are supplemented with text vignettes on specific solution processes of students (example item, see Figure 7). In a second part of the questionnaire, beliefs and self-efficacy with regard to mathematical modelling are recorded in five scales. Abbreviated scales on constructivist and transmissive beliefs about teaching and learning in mathematics based on Staub and Stern (2002) were used. The scale on the application aspect by Grigutsch, Raatz and Törner (1998) was adapted to the content of mathematical modelling. A scale representing the use of mathematical modelling in the classroom was developed. A newly developed instrument which focuses on self-efficacy in the perception and rating of performance heterogeneity was used to determine the expectations of self-efficacy. The full test has been published (Klock & Wess, 2019).

Example item text vignette: container (grade 8)

Containers are used to store construction materials or collect construction waste on many construction sites. These containers have a specific shape to facilitate loading and unloading. How much sand is in the container shown?

STUDENT 1: It contains exactly 7,160,000 cubic metres of sand. Can that be right?

STUDENT 2: It could well be right, you calculated it with your calculator.

STUDENT 1: Well yes. Then it's right.

STUDENT 3: It’s definitely right. I can imagine it.

Which phase of the solution process are the students mainly in? Please mark accordingly.

Mathematising □

Working mathematically □

Interpreting □

Validating □

Diagnose the students’ problem with working through the task in this situation. Please mark accordingly.

The students...

...have problems making assumptions. □

...are not checking their solution sufficiently for plausibility. □

...are drawing an incorrect conclusion from their mathematical result. □

...are using an unsuitable mathematical model. □

Figure 7. Example item text vignette (Klock & Wess, 2019, p. 22)

When piloting the test instrument, data from 156 teacher training students (66.9% female) at various universities in Germany were collected. At the point at which the data were collected, the students were either at the end of their Bachelor's degree (12.7%) or doing a Master's degree (87.3%). The results of the pilot study show that the structural model of the teaching of mathematical modelling in this form was able to be empirically confirmed. Only the scale of transmissive beliefs about the learning and teaching of mathematics showed no significant charge or explained variation. Here further research is needed (Klock, Wess, Greefrath, & Siller, 2019).

Modelling in teacher training

The implementation of teaching and learning laboratories enables the inclusion of practical elements in teacher training studies at an earlier stage. An important goal of teaching and learning laboratories is the professionalization of future teachers through reflection on the teaching and learning process (Putnam & Borko, 2000). The teaching and learning laboratory MiRA$^+$ specialising in modelling was developed at the University of Münster. It is integrated into the training for grammar school teachers and consists of a lecture series with 12 seminar sessions and additional blended learning formats in the field of the design of modelling tasks. The lecture series consists of a theory-based preparatory phase, a practical phase and a reflection phase. The key element in terms of content of all phases consists of modelling processes and sensitisation to and potential-oriented handling of heterogeneity.

The preparatory phase of the seminar looks at selected backgrounds of mathematical modelling (modelling cycle, modelling competencies) and the own working on a modelling task (Figure 8). Individual promotion is discussed in connection with a productive way of handling heterogeneity. Based on this, criteria for suitable modelling tasks are then created and tasks of this type are developed by the teacher training students as part of a blended learning format with various feedback cycles for use in the practical phase are developed. Criteria and indicators on specific individual processes of modelling are then created to monitor and diagnose the students learning processes in the teaching and learning laboratory sessions in this way. The development of modelling tasks and the creation of a suitable catalogue of criteria which deals intensively with the diagnostic individual processes forms the basis for the promotion of modelling-specific diagnostic and task-based competence.

In the practical phase, a team of three teacher training students (Master of Education) supports a small group of Grade 9 students with the processing of the modelling tasks they have created during the 90-minute project sessions. The teams monitor the competencies of mathematical modelling in a targeted manner and record these in the previously created monitoring sheet. The Grade 9 students work on content that enhances the curriculum in motivating project contexts. This interlacing of theory and practice in the context of diagnostic actions and tasks represents the practical promotion of modelling-specific diagnostic and task-based competence.

During the reflection phase, the project sessions are first discussed in the form of written reflection discussions so people can benefit from the experiences of other seminar participants. Cross-task, theory-based group reflections on the respective areas of focus of the monitoring are carried out taking into account the heterogeneity aspects of the learning groups monitored in particular. The teacher training students added to their diagnostic assessments with feedback from their colleagues. The knowledge obtained is then used to professionalise the participants' own teaching activities and to evaluate the modelling tasks they created. The teacher training students also reflect on and where necessary adapt the modelling tasks in light of the criteria for good modelling tasks drawn up in the preparatory phase. The experience and knowledge gained are summarised in a reflection report.

Figure 8. Hot air balloon task: "How many litres of air are in this hot air balloon?"

As part of a study, an investigation was carried out to determine the extent to which aspects of the modelling-specific diagnostic and task-based competence can be promoted among future teachers in the mathematical teaching and learning laboratory MiRA⁺. Data were collected from 96 teacher training students using a pencil and paper test in the pre-post design (Klock et al., 2019). In addition to the experimental group at the University of Münster ($N = 35$) and the comparison group at the University of Koblenz-Landau ($N = 43$) where they used predefined modelling tasks, a baseline group in Münster ($N = 18$) was also recorded to control the test repetition effect. It was shown that the experimental group showed significant improvements with a major effect in the three aspects of development, analysis and multiple solutions while the comparison group from Koblenz only showed significant improvements with a moderate effect in the aspect of the analysis of modelling tasks and the baseline group no significant changes. In terms of the aspect of modelling-specific diagnostic competence, both the experimental group and the comparison group showed significant improvements over time with a major and moderate effect respectively in the aspects of identifying the modelling phase and difficulties in the modelling process, while the baseline group once again showed no significant changes. Accordingly, the focus of the development of modelling tasks in the teaching and learning laboratory at the University of Münster was reflected in the significant developments that go beyond all aspects of modelling-specific task-based competence while the promotion of diagnostic competence, which was deemed to be equal in both locations, was demonstrated by the results of the difference analyses (Wess & Greefrath, in press).

Modelling with digital tools

When working on application-based problems in particular, a digital tool can support teachers and students. Using digital tools in mathematics class can facilitate the introduction of more complex applications and modelling into daily practice in the classroom (Henn, 2007). Digital tools are currently frequently used for application-based problems to process models with complex function terms, for example, or to decrease the calculation effort. Digital tools can be used in different ways in class for the application and modelling.

One of these possible uses is experimenting or discovering (see Hischer, 2016, p. 180). For example, dynamic geometry software or table calculations can be used to transfer a real situation into a geometric model that can be used for experimenting. A very similar activity to experimenting is simulating real situations using the digital tools. Simulations are experiments with models that aim to provide knowledge about the real system shown in the model or the model itself (Greefrath & Weigand, 2012). For example, predictions about the population of a certain type of animal in different environmental conditions are possible using a simulation. From an applied mathematics perspective, simulations using digital tools can be seen as part of a modelling cycle in which a numerical model developed from the mathematical model is tested to validate the conceptional model by comparing it to the measurement results (Sonar, 2001). It is then possible to think about the mathematical justifications for the solution that is obtained after the experiment or simulation. Digital tools can be also used for visualisation in class. For example, specific data can be presented using a computer algebra system or statistical application in a coordinate system. This can be the starting point for the development of a mathematical model. A widespread use of digital tools is the calculation of numerical or algebraic results that students would not be able to obtain without a digital tool or would not be able to obtain in a reasonable amount of time. One example of this is the calculation of optimal complex packaging problems such as milk packaging (Böer, 1993). If this problem is addressed using functions and differential calculus, you get broken rational functions in which the zero of the first derivative is difficult to determine using school mathematics. The identification of algebraic representations from specific information is one of the calculations that can be carried out using digital tools. If, for example, a functional equation is determined from the existing data, the digital medium is also used as a calculation tool. This so-called *algebraisation* is characterised by real data being input into the digital tool and an algebraic equation being obtained. Controlling is also a sensible use of digital tools in learning processes. Digital tools can support control processes, for example with the determination of functions of specific properties in different ways. If, for example, a functional equation under certain conditions is sought, the corresponding result can be controlled both by algebraic tracking of the calculations using a digital tool and by using graphical or numerical processes, regardless of whether the result was obtained with a digital tool or not. If digital media with internet access are used in mathematics classes, they can also be used to research information, for example in connection with application contexts. Digital tools can be used in mathematics class to carry out important and varied tasks. They do not, however, replace the understanding of mathematical ideas. Digital tools can be used to support this understanding, however, as they can provide assistance with the experimenting, visualisation and calculation of examples. An exploratory phase with the digital tool is an important aid to in-depth understanding of a central concept.

The various uses of digital mathematical tools are effective in different parts of the modelling cycle in application-oriented tasks. Control processes generally belong in the final stage of the cycle. The calculations are carried out using the mathematical model created, which for example is a function in the analysis. A more precise analysis shows that the digital tools can be sensibly used when modelling in all phases of the modelling cycle.

If you look at the step of calculating using digital tools in greater detail, the processing of modelling problems using a digital tool requires two translation processes. The modelling task first needs to be understood, simplified and translated into the language of mathematics. The digital tool can, however, only be used when the mathematical expressions have been translated into the language of the digital tool and a digital tool model has been developed. The results from the digital tool then have to be transformed back into the language of mathematics. Ultimately, the original problem can be solved if the mathematical results relate to the real situation. These translation processes can be set out in an expanded modelling cycle (see Figure 9) which, in addition to the real world ("rest of the world") and mathematics also takes into account the digital tool (see Savelsbergh et al., 2008; Greefrath, 2011). Current studies (Greefrath & Siller, 2017) show, however, that an integrated modelling cycle in which digital tools can be used in each step better describes actual modelling activities with digital tools (Greefrath, 2018).

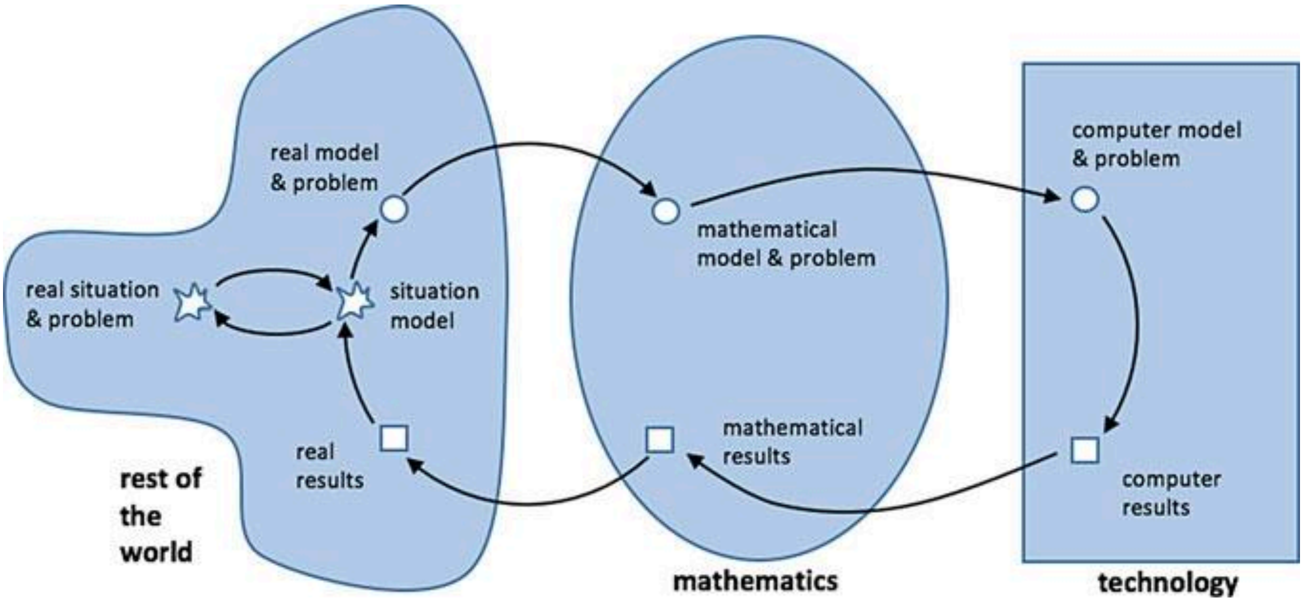

Figure 9. Possible use of digital tools in modelling cycles (Greefrath, 2011, p. 303)

MATHEMATICAL MODELLING IN SCHOOLS

Modelling has been much discussed in the German-speaking world in both mathematics education and school practice since the ISTRON Group was founded in Germany in the year 1990. There are a wide range of materials for how modelling can be taught in class (e.g. materials for reality-based mathematics classes ISTRON or material for mathematics course units MUED). Various approaches are followed to establish links to reality and modelling more strongly in schools. The possibilities that Fermi tasks and solution plans offer are presented below by way of an example. Challenges in the classroom can be also set out.

Challenges

Application-oriented tasks are not used as intensively as would be desirable in classrooms for a variety of reasons. There are, for example, organisational, personal and material-based obstacles.

Tasks on applications and modelling tasks often require longer to process than the time available in class. Extensive research or experiments sometimes need to be carried out for applications of this type. Project work is ideal for this if it appears to be organisationally possible. It is also difficult to include correspondingly extensive tasks in examinations. This in turn has an effect on the actual use in class and the students' motivation. The use of tasks from contexts outside of mathematics presents both students and teachers with new challenges. There may be personal reservations about challenging or additional activities (such as simplification and translation in the mathematical model) in mathematics classes. For teachers, too, application-oriented mathematics classes require a lot of different competencies (Blum, 2015).

Nowadays there are many materials that include links to the real world, but material and modelling tasks are not yet sufficiently well integrated into many schoolbooks that additional materials are not required. One difficulty when it comes to modelling tasks is the issue of the assessment of students' work. Maaß (2007) suggests that both the mathematical calculating and the modelling process be included in the assessment. She proposes that the development of the real model, the interpretation of the solution, the critical reflection, the documentation and the type of approach also be included in addition to the mathematical calculating.

Students may also have difficulties in many cases in application-oriented classes (Blum, 2015). Problems may occur in many places when working on modelling tasks in particular, which should be clarified on the modelling cycle (Maaß, 2004). Problems can occur when creating the real model and the mathematical model in the first steps of the idealised modelling cycle. False assumptions may be included in the model or the real situation inappropriately simplified. Students often do not have knowledge of relevant quantities, for example in terms of lengths and numbers. In the case of these difficulties, the text of the task and the representation of the problem play a particular role. A clear drawing can often have a positive impact on the understanding of the problem. Problems can also occur when transferring the real model to the mathematical model. It depends, among other things, on the available mathematical models, for example linear functions. False symbols and algorithms may be selected, or mistakes may be made in the formulas. Problems can also occur when working in the mathematical models. Students often identify the mistakes in calculation themselves in modelling tasks, but they need a suitable opportunity. The interpretation and validation of the results of the mathematical model are often not taken seriously enough. Students are often lacking control competence, particularly when it comes to considering the plausibility. Validation in particular must be addressed in greater detail (Maull & Berry, 2001).

While the difficulties mentioned can be specifically attributed to specific points in the modelling cycle, there are also difficulties that affect the entire modelling process. Students may lose their overview and stop pursuing their solution plan or not create a link to mathematics to further process their solution plan. It is also problematic when students are unable to show their work, because they do not document their work in sufficient detail. This means that their competence is very difficult to assess (Maaß, 2004). Here it is helpful if the students get a clear structure for their work.

There are various options to counteract these difficulties that may arise for students working on modelling tasks. On the one hand, there are tasks on particular competencies of modelling enable certain areas of the modelling cycle such as validation to be targeted (for example Fermi tasks), and on the other strategic solutions plans that were specifically developed for modelling tasks. Making students aware of the modelling by setting out the cycle can prevent mistakes that affect the entire modelling cycle (Greefrath, 2018).

Example: Fermi tasks

Fermi tasks can be characterised as specific modelling tasks (Kaiser & Stender, 2016). Fermi tasks are under-determined, open tasks with a clear end goal but unclear starting point and an unclear transformation with the focus on the data acquisition, mostly through multiple instances of guess work. They are named after the nuclear physicist and Nobel prize winner Enrico Fermi (1901-1954). He was known for his rapid solution of problems for which there were practically no data.

The classic example of a Fermi task is the determination of the number of piano tuners in Chicago. There is initially no information on this. It is possible, however, to gradually estimate the quantity by making sensible assumptions about the number of inhabitants in Chicago, the size of a household, the percentage of households with a piano, the period of time between two piano tunings, the duration of a piano tuning session and the workload of a piano tuner with an accuracy of around 100, thereby sensibly answering the question. The answer is therefore determined by making a suitable selection and sensibly estimating the intermediate data.

Other than their openness, Fermi tasks are also characterised by their link to reality and a certain accessibility. They are challenging and do not merely stimulate the person tackling them to ask further questions but also to use mathematics in the world. The term Fermi tasks is also used in a broader sense for open tasks in which the task consists of just a question or a small amount of information and where applicable a figure. When using Fermi tasks in mathematics classes, the focus is less on the calculation and more on the other steps in the modelling cycle such as simplifying and validating. Fermi tasks can teach students how to handle inaccuracy, which often does not take on a particularly significant role in mathematics classes. Fermi tasks promote estimation and working with inaccurate information in particular. Mathematising to models which are as simple as possible also plays an important role. Fermi tasks in the broader sense can also focus on researching and experimenting and on finding various approaches. Students also learn to ask questions themselves and develop heuristic strategies. They use everyday knowledge and calculate using quantities (Büchter, Herget, Leuders, & Müller, 2006; Leuders, 2001, p. 104).

Fermi tasks are used regularly in German comparative studies in Grade 8. A more precise investigation of nine Fermi tasks of this type from tests in the years 2015-2018 showed that all of the items could be assigned to the field of arithmetic and they can generally be assigned to the general idea of measurement. Mathematical modelling is felt to be the most important process as a general competence in almost all cases. The openness of the task is generally achieved (solely) through an unclear starting position. This openness is mostly based on information being missing from the task text. There are, however, three tasks that cannot be deemed to be either under-determined or over-determined. The authenticity found in almost all of the tasks is achieved by an authentic situation. In one case, however, a hypothetical problem is given (whether the routes to school of all of the students in a school could be longer than the distance from the earth to the moon), so in this example we cannot refer to an authentic context. In almost no instances can the use of mathematics be seen as authentic. None of the tasks used is currently really relevant to the students themselves. To solve most of the tasks, the students need to work along a modelling cycle. All of the competencies of modelling occur in almost all of the Fermi tasks. Only one task aims to estimate the length of a bee using a photo with a focus on only very few modelling competencies. The assumption can be made that the full modelling cycle will not be worked through but that only simplification is needed to successfully estimate the answer. Fermi tasks are a good opportunity to include authentic situations in tests and examination questions. For example, the volume of a dustbin should be determined using a photograph. In this way, test and examination questions can be more open and include contexts relevant to students. This promotes modelling competence and this aspect is then included in class development. This is possible at varying degrees of difficulty, which may be dependent on the number of quantities (like number, length, weight) used (Greefrath, 2019).

Example: strategic solution plans

One option to help students working on modelling tasks is to use strategic solution plans. These are often based on steps in the modelling process or competencies that play an important role in modelling. Solution plans can help students process modelling tasks. As part of the DISUM project, Blum (2010) developed a solution plan for students based on a simplified modelling cycle. This solution plan comprises four steps: understanding the task, creating the model, using mathematics and explaining the result. Each step is explained to students using a question and a number of explanatory points.

Blum’s solution plan is one of what are known as indirect general strategic tools as it makes reference to general technical modelling methods but in principle does not provide any specific assistance based on the content of the task. The universality of the strategic tool, however, is left out in favour of content-based information in two steps of this solution plan. The reference to equations and Pythagoras’ theorem means that the strategic tool is content-based. This solution plan can be edited for students. Its use can also be practised using sample tasks. In a study by Schukajlow, Krug and Rakoczy (2015) as part of the DISUM project, significant differences in student performance were able to be demonstrated when modelling using this solution plan with reference to the content area of “Pythagoras’ theorem”. Teaching using the solution plan proved to be a more effective form of teaching and learning. In addition to this, students in the solution plan group also have perceived stronger the use of the solution plan.

A five-step solution plan was used in the LIMO ("Lösungs-Instrumente beim Modellieren" [Solution tools in modelling]) project at the University of Münster. This solution plan comprises the steps: 1) Understanding and simplifying, 2) Mathematising, 3) Working mathematically, 4) Interpreting, and 5) Controlling. Five steps have been selected to highlight the step of validating the result and the path to a solution (Adamek, 2018).

The actual path to a solution does not always follow the same route as the respective predefined solution plan. Even if students' solutions that are fixed in writing are sometimes similar in structure and based on the solution plan, the detailed solution process varies considerably. This recalls the individual modelling routes described by Borromeo-Ferri (2018). Whether or not the solution plan plays a role in the solution process or merely impacts the outcome, the key question in the assessment is the effectiveness of these plans in providing assistance with the solution. The assumption should be made that there are students who will find it extremely difficult to use a plan of this type and others who will only need a short introduction to use the plan (Greefrath, 2018).

OUTLOOK

On the one hand, mathematical modelling is a fixed component of the current educational standards and schoolbooks in Germany. On the other hand, as in most countries, applications and modelling play a less significant role in everyday teaching and classroom activities. The empirical results presented show some areas of focus of research on modelling and application in the past few years in Germany. New test instruments provide opportunities for research and development of teaching and learning. The impact of digital tools on school practice and research projects on mathematical modelling is seen as an important task. Effective promotion of modelling competence among students and the professionalization of future teachers are currently the core elements of research. At the same time, tools and strategies are being developed and researched to help students to model problems independently and train teachers to teach mathematical modelling (Barquero, Carreira, & Kaiser, 2017; Greefrath & Vorhölter, 2016). In the future, digital media could offer new impetuses for mathematical modelling both for classes in school and for research methods at universities.

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