Generalizations of third and fifth graders within a functional approach to early algebra

Abstract

We describe 24 third (8-9 years old) and 24 fifth (10-11 years old) graders' generalization working with the same problem involving a function. Generalizing and representing functional relationships are considered key elements in a functional approach to early algebra. Focusing on functional relationships can provide insights into how students work with two or more covarying quantities rather than isolated computations, and focusing on representations can help to identify the type of representations useful to them. The goals of this study are to: (1) describe the functional relationships evidenced in students' responses, and (2) describe the representations that the students use. In addressing these research objectives, we describe student responses drawn from a Classroom Teaching Experiment in each grade. We analyzed students' written responses to different questions designed to generalize the relationships in a problem that involves the function $y=2x+6$. Our findings illustrate that 11 third graders and 19 fifth graders provide evidence of functional relationships in their responses. Three third graders and all fifth graders generalized the relationship. We conclude that these differences may be due to the students' previous classroom mathematical experiences, since students in higher grades would be more likely to focus on the relationships between variables, whereas third-graders would focus on the details of arithmetic computations. In addition, we find that natural language is the main vehicle used to generalize in both grades. Unlike third graders, fifth graders perceive general rules from the numerical calculation and express these generalizations even when not explicitly requested to do so.

Keywords

early algebra; functional thinking; functional relationships; generalization; representations;

Introduction

Research in school algebra demonstrates the importance of students' ability to generalize in different grades. Although definitions of generalization vary in the field of mathematics education (e.g., Dienes, 1961; Dubinsky & Harel, 1992; Kaput, 2008; Mason, 1996; Radford, 2003; Sfard, 1991), a growing research effort seeks to demonstrate the importance of elementary students' generalization. Multiple reasons justify including generalization in the early grades. Generalization promotes flexibility in students' mathematical thinking, allowing them to: (a) set aside irrelevant information; (b) adapt, adjust, and reorganize previous experiences; (c) pay attention to ideas, skills, and properties involved in different situations; (d) make generalization a powerful tool for solving problems and understanding different mathematical situations (Carpenter & Levi, 2000; Carraher & Schliemann, 2002, 2015; English & Warren, 1998; Warren, 2005). By considering generalization as a central element of elementary students' mathematical experiences, this paper moves away from positions that relate generalization exclusively to algebraic notation (Kieran, 1989).

Broadly speaking, we adopt the idea that algebraic thinking (also known as early algebra) is necessary starting in the elementary grades and has two essential aspects: (a) generalization; and (b) using symbols to represent generalizations, which enables problem solving, communicating, and articulating ideas (Carpenter & Levi, 2000; Kaput, 2008). Generalizing and expressing generalizations through different representations are thus core elements of the learning of algebra in the early grades (Cooper & Warren, 2011). To describe students' generalizations, we focus on a functional approach to algebraic thinking (Carraher & Schliemann, 2007). Within this approach, functions are the prime mathematical content, which: (a) serves as mathematical context for introducing algebra in the elementary grades; (b) can unify a wide range of isolated topics, such as arithmetic computations, fractions, ratio, and proportion, and formulas that relate quantities; (c) serves to connect students' daily experiences and mathematics and can enrich many arithmetic activities; and (d) improves organization of the teaching and learning of algebra (Carraher & Schliemann, 2019; Carraher, Schliemann, & Schwartz, 2008; Chazan, 2000; Dubinsky & Harel, 1992; Freudenthal, 1992; Schwartz & Yerushalmy, 1992). In this context, the questions we seek to answer are: *What* and *how* do third and fifth graders generalize relationships in a problem involving a linear function (also known as a functional thinking task)? The four main reasons motivating this study are presented below.

First, an increasing number of studies show how elementary students consider the relationships between variables in different functional thinking tasks (e.g., Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015; Carraher & Schliemann, 2007; Cooper & Warren, 2011; Morales, Cañadas, Brizuela, & Gómez, 2018). Through examples that show how these students perceive and describe functional relationships when solving problems, these studies provide evidence to describe in greater depth the types of such relationships students evidence. Focusing on in-depth description of functional relationships evidenced by students provides insights into how students work with two or more covarying quantities rather than isolated computations. Further, most studies that focus on the functional relationships that students demonstrate do so in instructional contexts (e.g., Blanton, Brizuela et al., 2015; Carraher, Martinez, & Schliemann, 2008; Radford, 2018) We focus on *what* type of functional relationships students evidence when not directly instructed in generalization of covarying quantities.

Second, representations[1] are a way to describe and analyze *how* students perceive and express the structures and relationships embedded in a given problem, which helps to structure and expand students' thinking (Brizuela & Earnest, 2008). Although algebraic notation is the traditional type of representation used in school algebra, generalizations can be represented in other ways, such as using natural language to communicate and represent algebraic concepts or ideas (Radford, 2003). Focusing on the different types of representation used by elementary students (e.g., natural language, pictorial, numerical, tabular, graphical) could help us to describe students' functional thinking and identify the types of representations useful to them.

Third, generalization is a core aspect in functional thinking and working with problems that involve functions becomes an optimal scenario for students to identify patterns and generalizations (Warren, 2005), and analyze functional behavior through different representations (Blanton, Levi, Crites, & Dougherty, 2011; Cañadas & Molina, 2016). Some authors describe young students as naturally predisposed to perceiving regularities and generalizing (Mason, 1996; Schifter, Bastable, Russell, Seyferth, & Riddle, 2008), even when they are unable to represent these processes clearly. We seek to describe students' generalization in response to different questions, including questions about specific values and generalization.

[1] This study focuses on external representations to distinguish them from mental or internal ones. We therefore use the term "representation" or "representations" to refer to external representations, produced with pencil and paper, that are intentional, permanent, and spatial in nature.

Fourth, this study is important in Spain because regularities and generalization form part of Spain's national curriculum. According to this curriculum, by the end of elementary school, students should be able to "describe and analyze situations of change, find patterns, regularities, and mathematical laws in numerical, geometric, and functional contexts, valuing their usefulness to make predictions" (Ministerio de Educación, Cultura y Deporte, 2014, p. 19387). However, the types of taks that are proposed at elementary level related to theses contents are not usual. We select third and fifth graders because research on functional thinking in these grades is scarce. Useful teaching implications could be derived, however, by relating our study to previous studies that focus on first graders (e.g., Morales et al., 2018).

Considering the ideas presented above and the research question—*What* and *how* do third and fifth graders generalize relationships in a problem involving a linear function?—we define two research objectives:

1. To describe the functional relationships evidenced in student's responses (*what*); and
2. To describe representations used by students (*how*).

In tackling these research objectives, we analyze evidence from a specific Classroom Teaching Experiment (CTE, hereafter) session in which students worked with a problem and answered various questions on a worksheet.

Conceptual framework

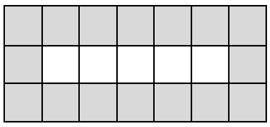
The conceptual framework that guides our study is based on Kaput's proposals (2008). For Kaput, "the heart of algebraic reasoning is comprised of complex symbolization processes that serve purposeful generalization and reasoning with generalizations" (p. 9). We focus specifically on a functional approach to early algebra, in which both generalizing relationships among covarying quantities and representing these relationships are key elements. According to Blanton and Kaput (2011), this approach (also known as functional thinking) involves the "construction and generalization of patterns and relationships, using a diversity of representations and treating generalized relationships, or functions, as the result of useful mathematical objects" (p. 6-7). Smith (2008) indicates that functional thinking is "focused on the relationship between two (or more) variables; specifically, the types of thoughts that go from specific relationships to generalizations of relationships" (p. 143). We therefore assume that functional thinking involves functional relationships, which may or may not be generalized and which may be expressed through different types of representations.

Functions and functional relationships

In this study, the function is the mathematical content that describes students' generalization. We adopt the concept of function as "a correspondence between two nonempty sets that assigns to every element in the first set (the domain) exactly one element in the second set (codomain)" (Vinner & Dreyfus, 1989, p. 357). Our focus is on linear functions, specifically the type $y=mx+b$, where m and b are constants, and variables x and y natural numbers. This type of function is deemed suitable for the age and type of work expected of elementary school students in the functional approach to early algebra (Carraher & Schliemann, 2007).

In functional thinking tasks, functions are presented through contextualized problems. The problem used in this study, the tiles problem, exemplifies such problems and, additionally, it reflects our adoption about what is a function (see Figure 1).

A school wants to replace the floor in all its corridors, where the tiles are severely damaged. Its board decides to lay white and grey tiles on all the floors. All the tiles are square and of the same size and are to be laid in the following pattern:



The school asks a company to replace the floors in all the corridors. We want you to help the masons answer some questions before they can start work.

Figure 1. The tiles problems

The tiles problem involves the function $g=2w+6$, with natural numbers as the domain and codomain. This problem involves two variables: the number of white tiles (w) and the number of grey tiles (g). For example, if we want to know the number of grey tiles to put around a number of white tiles, g is expressed in terms of w. In this case, w is the independent variable and g is the dependent variable.

In working with a functional thinking task, students have different ways to interpret and build how the dependent and independent variables relate to each other: (a) *recurrence (or recursive patterning)* describes attending to variation within one quantity (in the tiles problem, "the number of grey tiles increases by 2"); (b) *correspondence* emphasizes the relation between a corresponding pair of variables (e.g., "two times the number of white tiles plus six"); and *covariation* analyzes how two quantities covary, that is, how change in one (from a_n to a_{n+1}, for instance) produces change in the other (from $f(a_n)$ to $f(a_{n+1})$) (e.g., "when the number of white tiles increases by one, the number of grey tiles increases by two") (Confrey & Smith, 1994; Smith, 2008).

The literature on functional thinking distinguishes between (1) recursive patterns and (2) functional relationships (correspondence and covariation) and studies how to move from (1) to (2) (e.g., Blanton, Brizuela et al., 2015; Cañadas, Brizuela, & Blanton, 2016; Moss & McNab, 2011; Rivera & Becker, 2011). This distinction arises because recursive patterns center on the values of a single variable, whereas correspondence/covariation involves both variables. The literature on functional thinking reports two main trends in recursive patterns: (a) they give students difficulty when students try to focus on both variables (e.g., Carraher et al., 2008); and (b) students in the early grades of elementary school have been shown to evolve from ability to identify recurrent patterns to providing evidence of correspondence and covariation (e.g., Cañadas et al., 2016).

Generalization and representations

We argue that generalizing from a functional approach to early algebra involves attending, perceiving, and expressing how one quantity varies with respect to the other *in general* (Blanton, 2008, 2017). Representations also become the means by which students can organize and express the relationships identified in order to understand, analyze, explain, predict, and justify the way in which the variables are related. From our perspective, elementary students can express generalization using different types of representation: (a) natural language, (b) pictorial, (c) numerical, (d) algebraic notation, (e) tabular, and (f) graphical (Carraher et al., 2008). To illustrate examples of students' generalization through different types of representations, some authors have used the problem that includes a number of tables (*t*, independent variable) and the number of people who can sit at them (*p*, dependent variable) (e.g., Blanton, Isler-Baykal, Stroud, Stephens, Knuth et al., 2019; Carraher & Schliemann, 2007; Merino, Cañadas, & Molina, 2013; Moss, Beatty, Barkin, & Shillolo, 2008). Figure 2 presents two ways of introducing this problem.

(a)

(b)

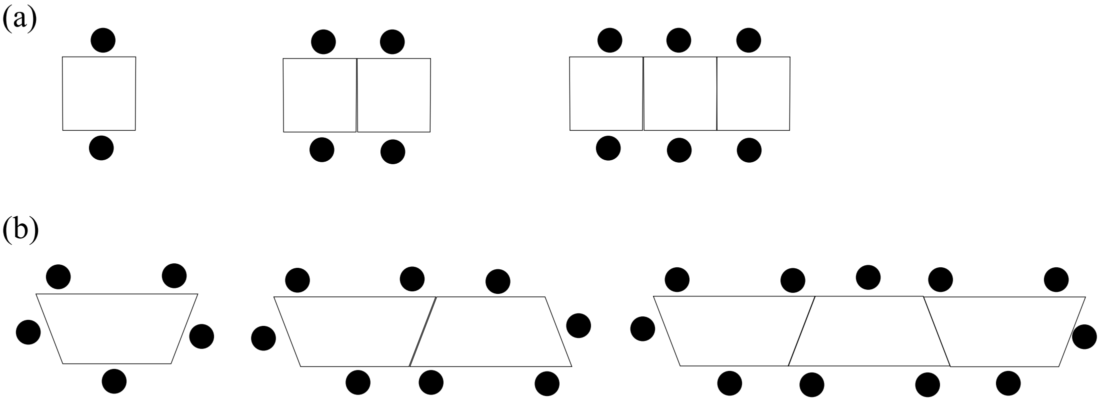


Figure 2. Tables and people problems

In studying the problem of the tables indicated in (a), which involves the function $p=2t$, Blanton et al. (2019) show that third graders used different types of representation to determine the relationship between the variables, generalized the relationship through natural language, and used algebraic notation, stating: (a) the number of people is 2 times the number of tables. If you add the number of tables twice, you get the number of people, or (b) $2\times t=p$; number of people= $2\times t$; $t+t=p$.

Different authors emphasize the role of specific representations in students' reasoning with functional thinking tasks. Some authors, for instance, note that natural language is a useful tool for expressing generalizations and consider it a crucial scaffold for the development of more symbolic representations (Radford, 2003; 2018; Stephens et al., 2017). In addition, pictorial and manipulative (or concrete materials) representations become useful tools aiding students in finding the relationships between variables. Through exploration of the relationship between "position" and "figure," elementary students can generalize the relationship between the variables successfully (Cooper & Warren, 2008; Moss & MacNab, 2011).

The study

This study is part of a broader Classroom Teaching Experiment (CTE) on functional thinking in third graders (8-9 years old) and fifth graders (10-11 years old). The study followed research design guidelines specifically established for the CTE. The CTE aims to understand teaching-learning processes when the researcher acts actively as a teacher, studying the development of ideas, tools, or models that include students, teachers, or groups (Cobb & Gravemeijer, 2008; Kelly & Lesh, 2000).

We designed a four-session CTE for each grade that posed a problem in each session. The purposes of the sessions were to explore how students generalize, considering students' work when they: (a) relate the variables involved in a functional thinking task, and (b) use different types of representations to express functional relationships. In both the third and the fifth grade classes, CTE was performed during the last term of the year. Table 1 shows the contexts and functions used in each of the four sessions for each grade; some were selected or adapted from previous studies; others were designed by the research team based on different types of linear functions.

Table 1. *Contexts and Functions presented in Each Session*

Session	Context	Function
	Third	

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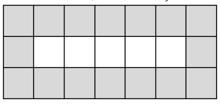
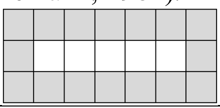
Session	Context	Function
1	María and Raúl are brother and sister. They live in La Zubia. María is the elder sibling. We know that María is 5 years older than Raúl.	$y=x+5$
2 and 3	Carlos wants to sell shirts with his school's badge so he can go on a study trip with his class. He earns 3 euros for each shirt he sells.	$y=3x$
4	A school wants to re-tile its corridors because they are in poor condition. Its administration decides to use a combination of white and grey tiles, all square and all the same size, to be laid out as in this figure (adapted from Küchemann, 1981). 	$y=2x+6$
	Fifth	
1	Carlos wants to sell shirts with his school's badge so he can go on a study trip with his class. He earns 3 euros for each shirt he sells.	$y=3x$
2	Daniel and Carla sell different shirts for their study trip. Carla gets 3 euros for each shirt. Daniel has saved 15 euros. Additionally, for each shirt he sells, he gets 2 euros.	$y=3x$ and $y=2x+15$
3	Juan has saved some money (he only has euros, no cents). His grandmother wants to reward him for a job she has given him. She offers him two deals: Deal 1. She will double his money. Deal 2. She will triple his money and then take away 7 euros. (adapted from Brizuela & Earnest, 2008).	$y=2x$ and $y=3x-7$
4	A school wants to re-tile its corridors because they are in poor condition. Its administration decides to use a combination of white and grey tiles, all square and all the same size, to be laid out as in this figure (adapted from Küchemann, 1981). 	$y=2x+6$

We chose the problems presented to students (see Table 1) according to three criteria: (a) types of functions that would be appropriate for students of these ages; (b) type of structure (additive and/or multiplicative) involved in each function; and (c) kind of values involved in each problem. We also chose contexts that were familiar and attractive to students and organized the problems from less-to-more difficulty according to results in previous studies (e.g., Blanton, Brizuela et al., 2015). Students were asked to answer several questions in connection with the problems in Table 1. These questions involved: (a) true and false items, (b) open-ended questions, and (c) create/complete function tables or Cartesian graphing.

In each session, a research team worked with the students. The team was composed of the teacher-researcher who led the sessions and two researchers who video recorded the sessions and helped to answer students' questions when they were working on the problem.

Data selection

This paper reports the findings from the last CTE session in each group: the tiles problem. In both grades, the students worked in the first three sessions with problems involving two types of functions, $y=x+a$ and $y=ax$. In the fourth session, the problem involved exclusively the type $y=ax+b$ (in the fifth grade group, the second and third sessions included this type of function and $y=ax$). Two main issues motivated our focus on the last session. Firstly, the functional thinking task was the same for both grades, providing the opportunity to explore students' generalization from different grades. Secondly, in this last session, the students were more used to working with functional thinking tasks.

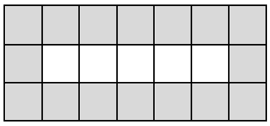
Participants

Twenty-four third graders (8-9 years old) and twenty-four fifth graders (10-11 years old) enrolled in a school in southern Spain participated in the CTE. The school was intentionally chosen because of its interest in collaborating in the research study. The families' socioeconomic level was medium-high. The students had different levels of achievement, and the school had only one third-grade and one fifth-grade classroom. As Algebra is not a content introduced at elementary level in Spain, students did not have previous algebra knowledge. Prior to the CTE, these students had not worked with problems involving generalization and/or functions. Among the arithmetic contents that could influence students' work, third graders had been taught to add and subtract, and to count one by one, two by two, five by five, and ten by ten. They had also been introduced to multiplication as repeated addition but had not yet explored multiplication tables or multiplication properties. The fifth graders had worked with the four arithmetic operations using natural, whole, and rational numbers.

Instruments and data collection

The fourth CTE session (the one we describe here) was divided into three stages. First, the teacher-researcher introduced the problem, whose underlying function was y=2x+6 (see Figure 3), and various questions. The teacher-researcher led a discussion on the context of the tiles problem, asking some questions to assess the students' understanding of the problem, and gave some instructions.

A school wants to replace the floor in all its corridors, where the tiles are severely damaged. Its board decides to lay white and grey tiles on all the floors. All the tiles are square and of the same size and are to be laid in the following pattern:



The school asks a company to replace the floors in all the corridors. We want you to help the masons answer some questions before they can start work.
Q1. How many grey tiles do they need if a corridor has five white tiles?
Q2. Some corridors are longer than others. Therefore, the masons need different numbers of tiles for each corridor. How many grey tiles do they need for a corridor with eight white tiles?
Q3. How many grey tiles do they need for a corridor with 10 white tiles?
Q4. How many grey tiles do they need for a corridor with 100 white tiles?
Q5. The masons always lay the white tiles first. How can they know how many grey tiles tiles they need if they've already laid the white tiles?

Figure 3. The tile problem

The students were then given a worksheet on which they worked individually. The questions were designed based on the inductive reasoning model described by Cañadas and Castro (2007), which involves questions that include specific values (Q1-Q4) and generalization (Q5). The students' answers to the worksheet are the data analyzed in this paper.

Lastly, one teacher-researcher led a classroom discussion about the responses to some of the questions on the worksheet. The sole difference between the two groups was that the third graders were given white and grey square papers with which to work if they wanted to.

Data and categories of analysis

We analyze each student's responses to the worksheet, considering two categories that emerge from theoretical perspectives derived from previous studies: (a) *functional relationships*, identifying the relationships underlying students' responses to the tiles problem questions (Confrey & Smith, 1994; Smith, 2008); and (b) *representations*, describing how students express the functional relationships (Carraher et al., 2008). The first category corresponds to the first research objective (to describe the functional relationships evidenced in student's responses), and the second category to the second research objective (to describe representations used by students). To illustrate, Table 2 shows examples of how we identified functional relationships and representations in a fifth grader's answers.

Table 2. *Examples of Functional Relationships underlying a Student's Responses and Representations used by the Student*

Question (Q)	Student's responses	Functional Relationship	Representation
Q2 (8 white tiles)	16 grey tiles are needed. For each white tile, there are 2 greys except for the ones on the sides— 6, or all of the white ones ×2+6 on the sides.	Correspondence	Natural language Numerical
Q2 (10 white tiles)	There are 22 grey tiles according to the previous process.	Correspondence	Natural language
Q5 (generalization)	Multiplying the whites by two plus 6 from the sides $x\times2+6=x$	Correspondence	Natural language Algebraic notation

As Table 2 shows, the student was able to provide evidence of functional relationships when asked for specific values or when generalizing. Given the problem type, students were deemed to have expressed a functional relationship when: (a) a regularity was identified in at least two of the first three questions (Q1-Q3); (b) a functional relationship was identified in Q4 and the preceding questions; or (c) when a functional relationship was identified in the answer to Q5. These criteria were adopted to ensure that functional relationships would not be identified based on the answer to a single question that might have been found arithmetically. Further, when describing students' work, we were not interested in whether their answers were correct or incorrect. Our focus was primarily on the paths they considered to relate variables and/or express general relationships.

Table 3 presents the categories, subcategories, and codes used in each student's responses.

Table 3. *Categories used in each Student's Responses*

Category	Subcategory	Code
1. Functional relationships (Confrey & Smith, 1994; Smith, 2008)	1.1. It is possible to identify a functional relationship	1.1.1. Recursive patterns, specific values. 1.1.2. Recursive patterns, generalizing. 1.1.3 Correspondence, specific values. 1.1.4. Correspondence, generalizing. 1.1.5. Covariation, specific values. 1.1.6. Covariation, generalizing.
	1.2. It is not possible to identify a functional relationship	1.2.1. Student only answered the question. 1.2.2. Student repeated the problem wording. 1.2.3 Student furnished or alluded to pictorial representations but did not

Table 3. *Categories used in each Student's Responses*

Category	Subcategory	Code
		provide information on the relationship. 1.2.4. Student alluded to manipulative representations (white and grey paper squares) but did not provide information on the relationship. 1.2.5. Student performed arithmetic operations with no clear meaning.
	1.3. Did not answer the question	
2. Representations (Carraher et al., 2008)	2.1. Natural language 2.2. Pictorial 2.3. Manipulative 2.4. Numerical 2.5. Algebraic notation 2.6. Tabular 2.7. Graphical	

To summarize, we analyzed each student's written responses according to the two categories: functional relationships and representations. The first author coded the students' answers. The second author then checked the codes assigned. To guarantee inter-reliability of the codifications, after first author's codification, we subjected the encodings to a calibration process that included joint coding sessions and discussion of the disagreements. This process enabled us to evaluate reliability. The reliability coefficient was greater than 90%, above the acceptable minimum (Tinsley & Brown, 2000).

Results and discussion

Broadly speaking, 13 third-graders and five fifth-graders did not provide evidence of functional relationships in their responses. This situation allows us to interpret that the differences could due to the students' previous classroom mathematical experiences; students in higher grades are more likely to focus on the relationships among quantities, whereas third-graders still focus on the details of arithmetic computations or pictorial representations introduced. Figure 4 presents sample answers given by students who did not provide evidence of functional relationships in their responses, one by a third grader (T21[2]) and one by a fifth grader (F15) to Q3 (How many grey tiles do they need for a corridor with 10 white tiles? How did you figure that out?).

[2] To respect students' anonymity, each was assigned a code consisting of the letter "T" to third graders and "F" to fifth graders, and a number from 1 to 24.

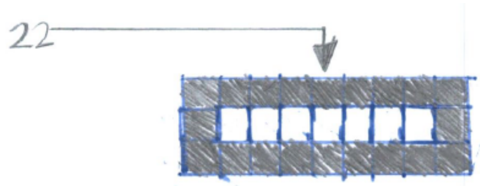
T21	F15
26. counting the squares that are in the box [white and grey paper squares].	

Figure 4. Third and fifth graders' answer to Q3

As Figure 4 shows, T21 used the white and grey paper squares to answer the question (code 1.2.4) without providing evidence of any relationships. F15, on the other hand, alluded to pictorial representations but did not provide information about the relationships (code 1.2.3).

The following presents the results, organized by grade (third or fifth) and distinguishing functional relationships demonstrated by students and the type or types of representations the students used to express the relationships identified.

Third graders

Eleven students provided evidence of functional relationships, three of whom generalized the relationship involving white and grey tiles. The following sections describe the work of these 11 students.

Functional relationships. Correspondence was the most frequent functional relationship in students' responses. Table 4 presents the question in which correspondence was identified. Shading indicates evidence of covariation in the students' answers.

Table 4. *Types of Functional Relationships demonstrated by Third Graders on Each Question*

	Questions				
Student	Q1	Q2	Q3	Q4	Q5
T03		✓	✓	✓	✓
T05	✓	✓	✓	✓	
T06	✓	✓	✓	✓	✓
T09		✓	✓	✓	✓*
T11	✓	✓		✓	✓*
T12		✓	✓	✓	
T13			✓	✓	
T14		✓	✓	✓	
T19	✓		✓		
T22	✓	✓	✓	✓	✓*
T24			✓	✓	✓

Note. T = third graders; Q = questions; * = generalization.

Broadly speaking, the number of functional relationships identified in students' responses working with specific values increased from Q1 to Q4. This idea could be explained in terms of the specific values involved in each question. For instance, answering Q4 (100 white tiles) requires finding more sophisticated ways to relate the white and grey tiles (e.g., counting or drawing would be not a useful way to find the relationship). This situation is similar to that noted by other authors (e.g., Warren, Miller, & Cooper, 2007) who argue that students' ability to identify functional relationships grows as they work with an increasing number of specific values involving two variables. Further, recursive patterning was not identified in these students, which is consistent with earlier reports (e.g., Cañadas et al., 2016; MacGregor & Stacey, 1995). This finding was foreseeable because our study posed the problem in a way intended to elicit correspondence and covariation, which relate the dependent and independent variables. We did not actually use consecutive values in the questions about the tiles problem.

Based on the types of functional relationships the students demonstrate, we identify two main trends in their responses. Some only provide evidence of functional relationships when working with specific values (T03, T05, T06, T12, T13, T14, T19, and T24), while others generalize the relationship (T09, T11, and T22). The following describes examples of both functional relationships evidenced in students' responses and generalizations of the relationship.

Eight students only provided evidence of correspondence when they worked with specific values. Figure 5 illustrates sample responses by T13 to different questions that we deem representative of this group of students.

Q3 (10 white tiles)	10 + 10 = 20 + 6 = 26
Q4 (100 white tiles)	100 + 100 = 200 + 3 = 203 + 3 = 206

Figure 5. T13's answer to Q3 and Q4

T13's answers provide evidence that he focuses on the number of grey tiles when given the number of white tiles. In Q3 and Q4, he defined pairs of values (a, f(a)) for the a values in each specific value (10 and 100) and established their relationship to the number of grey tiles: 26 and 206, respectively. T13 answered both questions following the same "closed form rule" to describe a relation between quantities (Confrey & Smith, 1994), adding the number of top and bottom grey tiles to the number of tiles on the left and right sides (3+3). T13 thus provides evidence of a rule constructed to determine the unique value of any given value (x), creating a correspondence between x and y. This approach to decomposing the number of grey tiles—the same approach used in all of the questions—could indicate that T13 perceived a regularity that extended to different specific values but was not able to represent that regularity when asked directly about generalization. This result could be explained by authors who argue that young students are naturally predisposed to perceiving regularities and generalizing, even when they are unable to represent these processes clearly (Mason, 1996).

On the other hand, three students did generalize, providing evidence of correspondence. Only one of these students, T09, did so correctly. Figure 6 presents the student's answers to different questions.

Q2 (8 white tiles)	8 + 8 = 16 16 + 3 + 3 = 22
Q3 (10 white tiles)	10 + 10 + 3 + 3 = 26
Q4 (100 white tiles)	100 + 100 + 3 + 3 = 206
Q5 (general case)	*You add the number of white tiles twice and then you add 6.*

Figure 6. T09's answers to Q2, Q3, and Q4

In Q2-Q4 answers, T09 identified the same numerical regularity: the number of white tiles twice plus 3+3 (the number of grey tiles to the left and right). The student was consistent in the "way" he expressed regularity (twice the number of white tiles plus three right and three left grey tiles). This answer seems to "capture" this regularity and extend it to any number of white tiles (as Q5). T09's statement of the general rule identified is consistent with the regularity he detected from Q2-Q4. Based on this result and a similar line of inquiry, various authors describe the ability of elementary students to "generalize" regularities from arithmetic computations (Cooper & Warren, 2008; Pinto, Brizuela, & Cañadas, 2019). In some cases, like that of T09, students can express this generality for any value, while other students (e.g., the eight third-graders) do not express the general rule for any value. The other two students who generalized in Q5 did so incorrectly; their answers referred only to the six tiles needed on the right and left, i.e., the constant term in the implicit function. For instance, T11 answered Q5: "adding 6 more." She failed to see that the number of grey tiles on the top and bottom was twice the number of white tiles.

Covariation was identified in two students' answers (T03 and T12). T03, for example, provides evidence of covariation in Q3 (Figure 7).

Q2 (8 white tiles)	*20. Set them [white and grey square tiles] up and count them.*
Q3 (10 white tiles)	*22. If for 8 [referring to Q2] you need 20+2=22*

Figure 7. T03's answer to Q3

Figure 4 shows that T03 established a relationship between the number of grey tiles needed for the eight white tiles (in Q2) and the number of grey tiles given 10 white tiles (in Q3). Based on the relationship that for eight white tiles 20 grey tiles are needed, she reasoned that, if the number of white tiles increases by two, the number of grey tiles also increases by two. Her reasoning focuses on how two quantities covary and how change in one (from 8 to 10) produces change in the other (from 20 to 22).

Representations. Third graders tended to express the relationships using a numerical representation or natural language. To illustrate their representations, Table 5 shows the type of representation students used in each question. Shading indicates the student representation in which we identified covariation.

Table 5. *Types of Representations used by Students to express Functional Relationships*

	Questions				
Student	Q1	Q2	Q3	Q4	Q5
T03		M; NL	NL; N	N	NL
T05	N	N	N	N	
T06	NL	NL	N	N	NL
T09		N	N	N	NL*
T11	NL	NL; N		NL; N	NL; N*
T12		NL; N	N	N	
T13			NL	NL	
T14		NL	NL; N	N	
T19	P; N		NL; N		
T22	N	N	N	N	NL*
T24			NL; N	N	P

Note. T = third graders; Q = questions; * = generalization; NL=natural language; N=numerical; P=pictorial; M=manipulative; * = generalization.

As Table 6 shows, and considering students' responses to Q1 and Q2 (five and eight white tiles, respectively), students tend to express the relationships in the same way, numerically and/or using natural language. In their responses to Q3 and Q4 (10 and 100 white tiles, respectively), students used primarily numerical representation, employing arithmetical computations to connect both variables. In these questions, the use of arithmetical computations to connect both variables makes sense, since students were being asked about specific values. T09, T11, and T22, in contrast, used natural language to generalize the answer in Q5. All three students generalized when asked explicitly to do so. These students' use of natural language is consistent with results reported by

Radford (2018), who found that natural language is a useful vehicle for expressing general rules, as well as a useful scaffold for development of more symbolic representations. We conclude that the third graders who provided evidence of relationships between variables in specific values lacked the resources needed to express the generalization, a finding consistent with results reported by Blanton, Brizuela et al. (2015). These students were consistent in perceiving numerical regularity among variables in Q1-Q4 (their words reveal the general rule), but they could not answer when were asked for the general relationship for any number (Q5). This difference result may explain why only three students generalized.

Of the 11 third graders considered, only two used pictorial representation (T19 and T24), and only one (T03) used manipulative. Apparently, this group of students did not find either white and grey square tile papers or pictorial representation to be useful in finding the relationships between the variables in the problems. In contrast to previous studies (e.g., Cooper & Warren, 2008; Moss & MacNab, 2011), and in accordance with our findings, neither type of representation enabled students to connect the relationship between the variables to identify the covarying quantities.

Fifth graders

Nineteen of the 24 fifth graders provided evidence of a functional relationship; all of these generalized the relationship involved in the tiles problem (y=2x+6). The following describes the work of these students.

Functional representations. Correspondence was the only type of functional relationship identified in the students' responses. Table 6 shows the questions in which we identified this relationship, by individual question.

Table 6. *Correspondence evidenced by Fifth Graders in Each Question*

	Questions				
Student	Q1	Q2	Q3	Q4	Q5
F01		✓		✓	✓*
F02		✓	✓	✓	✓*
F03		✓	✓	✓	✓*
F04					✓*
F05	✓*				✓*
F06	✓*	✓*			✓*
F07		✓		✓	✓*
F08	✓*				✓*
F09					✓*
F10				✓	✓*
F11		✓		✓	✓*
F12				✓	✓*
F13		✓		✓	✓*
F14		✓	✓	✓	✓*

F17			✓	✓	✓*
F21		✓	✓	✓	✓*
F22			✓	✓	✓*
F23	✓	✓	✓	✓	✓*
F24	✓	✓	✓	✓	✓*

Note. F = fifth graders; Q = questions; * = generalization

As Table 6 shows, evidence of students' generalization was found in responses to Q1, Q2, and Q5. Sixteen students only generalized when answering Q5, and three students generalized when answering questions involving specific values and the general case (F05, F06, and F08). The following describes the work both of students who only generalized when answering Q5 and of those who did so when answering different questions.

The 16 students who generalized only when prompted (in Q5) expressed the general relationship between value pairs (correspondence). In their responses to Q1-Q4, this group of students found the relationship between white and grey tiles based on the specific demands of the questions. When explicitly asked to find the general relationship for any value, they generalized. Figure 8 presents F01's answers to Q2-Q5, which we deem representative of these students.

Q2 (8 white tiles)	*22 grey tiles are needed because if you multiply the white tiles by two and add three at the beginning and the end, you get the result.*
Q3 (10 white tiles)	*26 grey tiles are needed because if you multiply the white tiles by two and add three at the beginning and the end, you get the result.*
Q4 (100 white tiles)	*206 tiles are needed, because you multiply the tiles 100x2 plus 6 to the right and left.*
Q5 (general case)	*Multiplying the number of white tiles by 2 and adding 6 gives you the result.*

Figure 8. F01's answer to Q5

Figure 8 shows that F01 detected the regularity from her responses to Q2-Q4, using the same relationships: multiplying the number of white tiles by two and adding three to the right and three to the left. This student used the same functional relationship for 8, 10, and 100 tiles (Q2, Q3 and Q4). In all three cases, she related the value pairs (a, $f(a)$), finding the number of grey tiles needed for a=8, 10, and 100 to be 22, 26, and 206, respectively. Apparently, she generalized her responses to Q2-Q4 but did not give the general relationship because it was not requested.

Of the 16 students who generalized when prompted, 12 expressed the generalization in terms of a rule that would yield the function $y=2x+6$ in algebraic notation (e.g., F01's answer in Figure 8). The other four students generalized the rule determining the relationship between white and grey tiles incorrectly. Figure 9 shows two students' responses to Q5.

F12	F21
Multiplying the top row by 2 and the bottom row + 2.	*Well, since the grey tiles surround the white tiles (...), you multiply [the white ones] by 2 and add 2 to the right and left.*

Figure 9. F12's and F21's answers to Q5

Both students correctly identified the relationship "twice the number of white tiles" but failed to identify the constant part of the function. These students understood that the number of grey tiles remained constant regardless of the number of white tiles (although this number is incorrect).

As shown in Table 6, three students generalized in different questions on the worksheet. Figure 10 depicts F06's responses to Q1 and Q5.

Q1 (5 white tiles)	*They need 16 tiles.* *There are two grey tiles for every white tile except on the sides, where there are 5. Or all the tiles ×2 + 6 on the sides.*
Q5 (general case)	*Multiplying the number of white tiles times 2 and adding 6.* $x \times 2 + 6 = x$

Figure 10. F06's answers

F06 generalized in the question that explicitly asked for this relationship (Q5), as well as in the question that asked for the relationship for a specific value (Q1). In contrast to the third graders who generalized, these answers show that fifth graders generalize without following a worksheet designed for this purpose, in line with similar findings from other studies (e.g., Amit & Neria, 2007).

Representations. Table 7 presents the representations used by fifth graders to express the relationships in each question.

Table 7. *Types of Representation used by Students to express Functional Relationships*

	Questions				
Student	Q1	Q2	Q3	Q4	Q5
F01		NL; N		N	NL; A*
F02		NL	NL	NL	NL*
F03		NL	NL	NL	NL*
F04					NL*

F05	A*				NL*
F06	NL*	NL*			NL; A*
F07		NL		NL	NL*
F08	A*				NL*
F09					NL*
F10				N	NL*
F11		NL; N		N	NL*
F12				NL	NL*
F13		NL; N		N	NL*
F14		NL	NL	NL	NL*
F17			NL	NL	NL*
F21		NL	N	NL	NL*
F22			NL	N	NL*
F23	N	N	N	N	NL*
F24	N; NL	N	N	N	NL*

Note. F = fifth graders; Q = questions; * = generalization; A= Algebraic notation; NL=natural language; N=numerical.

Three main findings emerge from the data presented in Table 7. Firstly, and as in the case of the third graders reported here, natural language is the most frequent representation used to generalize. Figure 11 shows different students' generalizations expressed through natural language.

F02	**F03**	**F06**
Multiplying the white tiles and adding 6, which are those on the left and right.	*Multiplying the white tiles by two and adding the three at the beginning and 3 at the end.*	*Multiplying the white tiles by 2 plus 6 on the right and left.* $X \times 2+6=X$

Figure 11. Examples of students' generalizations using natural language

As Figure 11 shows, F02 and F03 use only natural language to express the general relationship between the variables, while F06 uses this type of representation and algebraic notation (each of these types of representations expresses the general rule by itself). The group of students who generalized using natural language related both variables clearly: they multiplied the number of white tiles by 2 and added 6 (e.g., F02's answer) or 3 + 3 (F03's answer), which corresponds to the constant number of grey tiles. This group's level of sophistication in generalization is similar to that of the third graders, except that the fifth graders explicitly referred to "multiplication" whereas the third graders referred to repeated addition. It seems that fifth graders' prior knowledge of multiplication could help them to find the general relationship.

Secondly, four students use algebraic notation spontaneously: F01, F05, F06, and F08. Even though this type of representation has not yet been introduced in class, everything seems to indicate that they have been instructed externally to use such representation. For instance, two of the students who generalized answering question that involve specific value, used algebraic notation. We present F05's answer to Q1 (five white tiles) in Figure 12.

Q1 (5 white tiles)	*They need 16 tiles.* *I found the answer with this formula: (x+2)+6=16* *x = number of white tiles*

Figure 12. F05's answer to Q1

F05's answer shows that she can relate the pairs of values (number of white and grey tiles) for any number of white tiles. We thus find that fifth graders' use of algebraic notation is related directly to the presence of spontaneous generalization (in terms of Pinto & Cañadas, 2018): students who generalized in this way used algebraic notation.

Conclusions

This paper sheds light on what and how intermediate and upper-level elementary school students generalize when answering a functional thinking task. Various initiatives currently promote the idea of incorporating functional thinking in the early grades, where generalization is a crucial element. Because the adoption of algebraic ideas in elementary school is relatively recent, however, many issues relating to their incorporation remain to be considered. More specifically, our findings illuminate the idea that incorporating functional thinking in the elementary grades: (a) could encourage students to develop strategies of inquiry (Yerushalmy, 2000); (b) provide a useful context to promote students' generalization, representation, justification, and reasoning with relations and quantities (Blanton et al., 2011); (c) be a useful tool for solving problems (Warren & Cooper, 2005); and (d) prepare students for more powerful mathematics in later years (Blanton & Kaput, 2005).

An increasing number of studies report elementary school students' generalization with problems involving different types of linear functions (e.g., Blanton, Brizuela et al., 2015; Carraher, Martinez, & Schliemann, 2008; Cooper & Warren, 2011; Morales et al., 2018). These studies illuminate how students relate, express, and generalize relation among variables. Our study's originality lies in providing an in-depth way to describe students' work, without prior instruction: (a) relating variables and expressing these relationships rather than performing isolated computation; and (b) answering questions involving specific values and generalization. Our study also contributes to deepening knowledge, and providing detailed evidence, of how the mathematical representations introduced throughout elementary school are useful to relate variables based on the specific requirements of the task. Previous research has identified criteria for analyzing elementary school students' generalization of functional thinking problems. Carraher et al. (2008), for instance, emphasize: (a) the form of the underlying mathematical function, which could relate both variables involved in the task or only one of them; (b) the variables mentioned; (c) the types of arithmetic computation used; (d) the use or otherwise of symbolic-algebraic notation; and (e) the meanings of the components of the written expression. Our study adds three more: (a) the relationship between variables identified by students; (b) the variety of representations used in addition to algebraic notation; and (c) the type of question in which students generalized (spontaneous or prompted generalization, in the terms of Pinto & Cañadas, 2018).

As described above, students in both groups were not used to working with situations that involve generalizing. As in reports by other authors (e.g., Carraher et al., 2008), however, these students' prior mathematical experiences seem to have influenced how they attend to and relate covarying quantities, and how they perceive general rules. These findings could explain why most fifth grade students generalized, while the third graders did not. The third-grade students tended to focus on specific values (which involved the numbers 5, 8, and 10). As the arithmetical computations enabled the third graders to answer the first questions we asked them, these students did not see the need to find the relationship between the variables mentioned. In addition, our findings help us to demonstrate that incorporating functions in the elementary school classroom "can enrich many arithmetic activities by prompting students to make generalizations and relate the tasks to abstract ideas and concepts" (Carraher & Schliemann, 2019, p. 12). Introducing this content during elementary school could also reduce possible difficulties post-elementary students encounter when they work with functions (Cañadas & Molina, 2016; Stephens, Ellis, Blanton, & Brizuela, 2017).

The tiles problem was specifically designed to promote functional thinking in elementary students. We stress the importance of problem design for three reasons. First, it enables students to use different procedures when answering the questions involving specific values until they are able to generalize. Second, the introduction of pictorial or manipulative representations in students' work with functional thinking tasks could help students to understand the dynamic relationships between variables and thus serve as a first step to start students thinking about covarying quantities. Finally, the tiles problem helps researchers identify the elements of generalization and functional thinking that can be deduced from elementary school students' spontaneous replies. Such deduction should lead to useful conclusions for teaching, as these elements are present in a number of countries' Mathematics curriculums (Ministerio de Educación, Cultura y Ciencia, 2014). The findings discussed support students' ability to define a general rule, albeit incorrectly on occasion. Some of the difficulties students faced when trying to establish a rule for the relationship between variables stem from a mistaken notion of the variables' interchangeability. While acknowledging that generalization is not simple and requires time (Dienes, 1961), we consider that learning sequences must be designed to guide students to a general rule that is valid for different specific values, while working with different types of mathematical representation.

The analysis of the students' written responses could be one limitation of this study, as the worksheets may not capture all of the students' ideas. Yet these situations open a new perspective that calls for future research to investigate in greater depth the means students have to relate variables and express these relationships. Interviews could be a useful way to achieve this goal.

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