HOW MATHEMATICS IS DEFINED AND MEASURED IN PISA 2003

Jose Luis Lupiáñez

University of Granada

How mathematics is defined

Today, all adults need a solid foundation in mathematics, science and technology to meet their goals. The society requires that all people, not just those aspiring to a scientific career, be mathematically, scientifically and technologically literate.

PISA defines a form of mathematical literacy that is concerned with the capacity of students to analyse, reason and communicate effectively to pose, solve and interpret mathematical problems in a variety of situations involving quantitative, spatial, probabilistic or other mathematical concepts. To compare mathematics performance across countries, PISA 2003 defines mathematical literacy as:

"An individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen" (OCDE, 2003).

The objective of the PISA assessment is to obtain measures of the extent to which students presented with problems, mainly set in real-world situations, can activate their mathematical knowledge and competencies to solve such problems successfully.

Real-life problems and situations for which mathematical knowledge may be useful often do not present themselves in familiar forms for students. The individual must translate the problem into a form that exposes the relevance and usefulness of mathematics. In schools, the usefulness of mathematics in the real world may be given little attention, and mathematical content is often taught and assessed in ways that are removed from authentic contexts.

The intention of the PISA approach is to encourage a vision of teaching and learning mathematics that gives strong emphasis to the processes associated with confronting problems in real-world context, making these problems amenable to mathematical treatment, using the relevant mathematical knowledge to solve problems, and evaluating the solution in the original problem context. If students can learn to do these things, they will be mathematically literate.

How mathematics is measured

Student's mathematics knowledge and skills were assessed according to three dimensions relating to: the mathematical content to which different problems and questions relate; the processes that need to be activated in order to connect observed phenomena with mathematics and then to solve the respective problems; and the situations and contexts that are used as sources of stimulus materials and in which problems are posed.

Content

The PISA assessment was established around four content areas. These content areas were selected taking account of the research literature and an in-depth consensus among OECD countries. The content areas are the following:

- *Space and shape* relates to spatial and geometric phenomena and relationships, often drawing on the curricular discipline of geometry. It requires looking for similarities and differences when analysing the components of shapes and recognising shapes in different representations and different dimensions, as well as understanding the properties of objects and their relative positions.
- *Change and relationships* involves mathematical manifestations of change and functional relationships and dependency among variables. This content area relates most closely to algebra. Relationships are given a variety of different representations like symbolic, algebraic, graphic, tabular and geometric, and the translation between representations is often of key importance in dealing with situations and tasks.
- *Quantity* involves numeric phenomena as well as quantitative relationships and patterns. It relates to the understanding of relative size, the recognition of numerical patterns, and the use of numbers to represent quantities and quantifiable attributes of real-world objects. This content area involves number sense, representing numbers, understanding the meaning of operations, mental arithmetic and estimating. The most common curricular branch of mathematics with which quantitative reasoning is associated is arithmetic.
- *Uncertainty* involves probabilistic and statistical phenomena and relationships that become increasingly relevant in the information society. These phenomena are the subject of mathematical study in statistics and probability.

Together, the four content areas cover the range of mathematics 15-year-olds need as a foundation for life and for further extending their horizon in mathematics. The concepts can be related to traditional content strands such arithmetic, algebra or geometry and their detailed sub-topics that reflect historically well-established branches of mathematical thinking and that facilitate the development of a structured teaching syllabus.

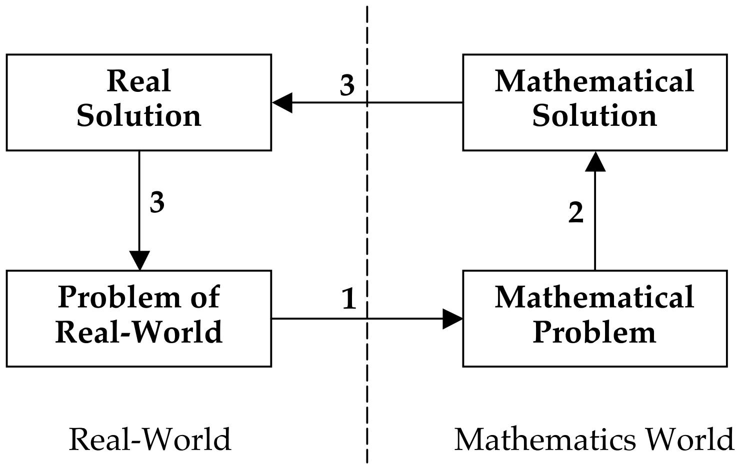
The PISA mathematics assessment sets out to compare levels of student performance in these four content areas, and the results are reported separately on each area and can be related to countries' curricular choices.

This table shows the breakdown by mathematical **content area** of the 85 test items used in the PISA 2003 assessment, and the results of Spain:

	Number of items	**Spanish Students' Mean Performance (OECD Mean)**
Space and Shape	**20**	**476 (496)**
Change and Relationships	**22**	**481 (499)**
Quantity	**23**	**492 (501)**
Uncertainty	**20**	**489 (502)**

Process

The PISA mathematics assessment requires students to engage in a multi-step process of



"mathematisation":

Beginning with a problem situated in reality, students must organise it according to identify the most relevant mathematical concepts, and then they must transform the problem into one that is amenable to direct solution (**1**), by making simplifying assumptions, by generalising information, by imposing useful ways of represent the problem, by understanding the relationships between the language of the problem and symbolic or formal language, by finding regularities and patterns and linking it with known problems, and by identifying or imposing a suitable mathematical model.

Once the problem has been turned into a familiar mathematical form, the student's armoury of specific mathematical knowledge, concept and skills can then be applied to solve it (**2**). This might involve a simple calculation, or using symbolic, formal and technical language and operations, switching between representations, using mathematical arguments and generalising. The final step in the *mathematisation* process (**3**) involve some form of translation of the mathematical result into a solution that works for the original problem context, a reality check of the completeness and applicability of the solution, a reflection on the outcomes and communication of the results, which may involve explanation and justification or proof.

Various competencies are required for such *mathematisation* to be employed:

- *Thinking and Reasoning*
- *Argumentation*
- *Communication*
- *Modelling*
- *Problem posing and solving*
- *Representation*, and
- *Using symbolic, formal and technical language and operations*

Generally, these competencies operate together, but PISA mathematics tasks were often constructed to call particularly on one or more of there. The cognitive activities that the mentioned competencies encompass were organised within three *competency clusters*. This grouping allows discussing the way in which different competencies are invoked in response to the different kinds and levels of cognitive demands imposed by different mathematical problems.

The three competency clusters are *reproduction*, *connections* and *reflection*.

- The **reproduction cluster** is called into play in those items that are relatively familiar, and that essentially require the reproduction of practised knowledge: recognition of equivalents, performance of routine procedures, application of standard algorithms and technical skills,…
- The **connections cluster** builds on reproduction to solve problems that are not simply routine, but that still involve somewhat familiar setting or extend and develop beyond the familiar to a relatively minor degree: making links between different representations, or linking different aspects of the problem in order to obtain a solution.

- The **reflection cluster** builds further on the connections cluster. These competencies are required in tasks that demand some insight and reflection on the part of student. The problems addressed using the competencies in this cluster involve more elements than others, and additional demands typically arise for students to generalise and to explain or justify their results.

This table shows the breakdown by **competency cluster** of the 85 test items used in the PISA 2003 assessment:

	Number of items
Reproduction	**26**
Connections	**40**
Reflection	**19**

Situation

As in PISA 2000, the PISA 2003 mathematics tasks are set in a range of context, relating to day-to-day activities, school and work situations, the wider community, and scientific or explicitly mathematical problems. The tasks were grouped into four sorts of situations: *personal*, *educational* or *occupational*, *public* and *scientific*:

- **Personal situations** directly relate to students' personal day-to-day activities. These have at their core the way in which a mathematical problem immediately affects the individual and the way the individual perceives the context of the problem.
- **Educational or occupational** situations appear in a student's life at school, or in a work setting. These have at their core the way in which the school or work setting might require a student or employee to confront some particular problem that requires a mathematical solution.
- **Public situations relating to the local and broader community** require students to observe some aspect of their broader surroundings. These are generally situations located in the community that have at their core the way in which students understand relationships among elements of their surroundings.
- **Scientific situations** are more abstract and might involve understanding a technological process, theoretical situation or explicitly mathematical problem. In this category are included relatively abstract mathematical situations with which students are frequently confronted in school, consisting entirely of explicit mathematical elements and where no attempt is made to place the problem in some broader context.

These four situation types vary in two important respects. The first is in terms of the distance between the student and the situation. Personal situations are closest to students, and educational and occupational situations involve some implications for the individual through their daily activities. Public situations involve a more removed observation of external events in the community, and scientific situations tend to be the most abstract and therefore involve the greatest separation between the student and the situation. PISA assumes that students need to be able to handle range situations, both close to and distant from their immediate lives.

The second difference is about the extent to which the mathematical nature of a situation is apparent. A few of the tasks make no reference to matters outside the mathematical world. The PISA assessment test the extent to which students can identify mathematical features of a problem when it is presented in a non-mathematical context and the extent to which they can activate their mathematical knowledge to explore and solve the problem and to make sense of the solution in the context or situation in which the problem arose.

This table shows the distribution of items by the **three dimensions** of the PISA framework for the assessment of mathematics:

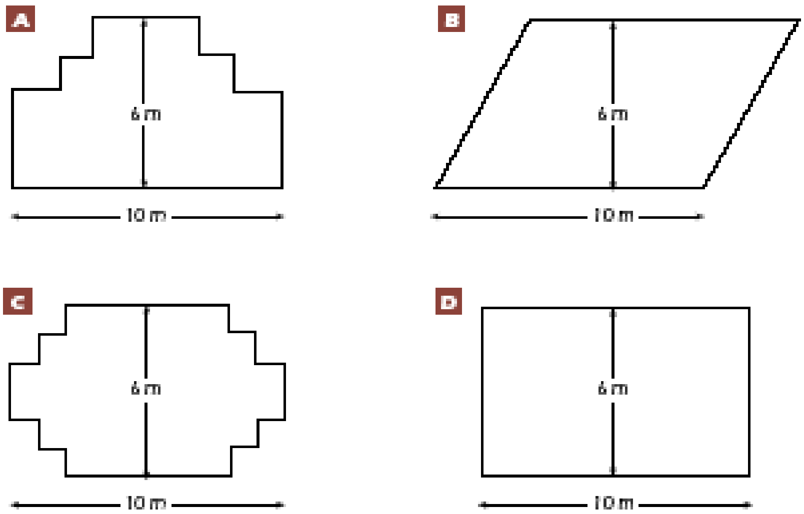
	Number of items	**Total of items by dimension**
Distribution of items by content areas		
Space and Shape	**20**	
Change and Relationships	**22**	**85**
Quantity	**23**	
Uncertainty	**20**	
Distribution of items by competency cluster		
Reproduction	**26**	
Connections	**40**	**85**
Reflection	**19**	
Distribution of items by situations or contexts		
Personal	**18**	
Educational / Occupational	**20**	**85**
Public	**29**	
Scientific	**18**	

Examples of items

Some items that can illustrate the classification above are the following:

Item 1. The Carpenter

A carpenter has 32 meters of timber and wants to make a border around a garden bed. He is considering the following designs for the garden bed:



What designs can be made with 32 meters de timber?

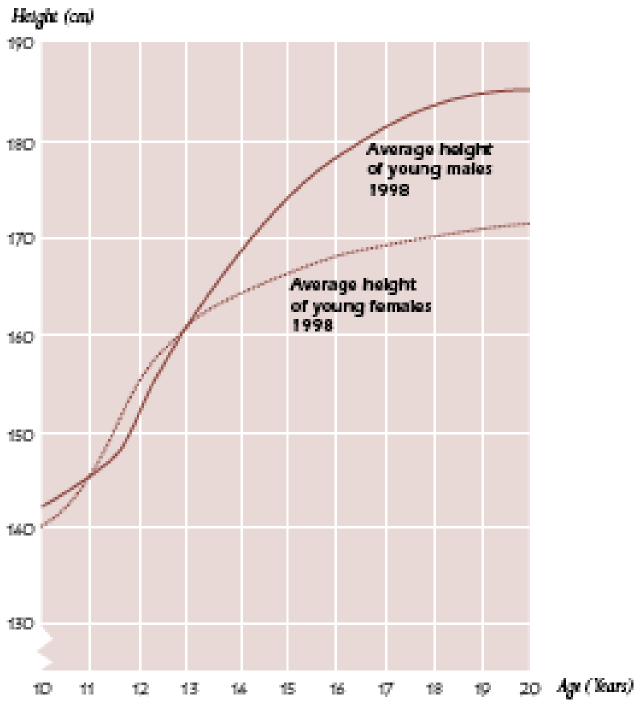
Content area: *Space and Shape*

Competency cluster: *Connections*

Situation: *Educational / Occupational*

Item 2. GROWING UP

In 1998 the average height of both young males and young females in the Netherlands is represented in this graph:



According to this graph, on average, during which period in their life are female taller than males of the same age?

Content area: *Change and Relationships*

Competency cluster: *Reproductions*

Situation: *Public*