SOME IMPLICATIONS OF WITTGENSTEIN'S IDEA OF *USE* FOR LEARNING MATHEMATICS THROUGH MATHEMATICAL MODELLING[1]

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ABSTRACT

This article reports a study which aim was to identify and characterise ways of understanding mathematics learning through mathematical modelling in a school environment from a Wittgensteinian perspective. Mathematical modelling in a school context can be understood as an approach involving real problem situations in school mathematical content. The subject to be discussed is based philosophically on the idea of meaning as the *use* we attribute to words, a concept developed by the philosopher Ludwig Wittgenstein. Based on this understanding, we analyse the theoretical definitions that Anna Sfard, following Wittgenstein, posits regarding mathematics learning in schools. We conducted a qualitative study in which we analysed the discourse of a group of students and a teacher produced during the implementation of mathematical modelling. The analysis of this discourse, based on the ideas of both Ludwing Wittgenstein and Anna Sfard, allowed us to point out that mathematics learning through mathematical modelling is characterised by the identification of similarities between the uses for which words are mobilised in the school environment and the uses of words suggested by problem situations addressed in mathematics modelling. We have also identified how the teacher guides students regarding the legitimate uses of words in mathematics classes.

Keywords: Mathematical Modelling; Mathematics Learning; Use; Grammar.

RESUMO

Este artigo tem como objetivo identificar e discutir maneiras de compreender a aprendizagem matemática que se constitui na modelagem matemática em âmbito escolar a partir de uma perspectiva wittgensteniana. Modelagem matemática no contexto escolar pode ser compreendida como a abordagem de situações-problema reais, por meio de conteúdos matemáticos escolares. A temática a ser discutida se fundamenta, filosoficamente, na ideia de significação do filósofo Ludwig Wittgenstein como sendo o uso que atribuímos às palavras. A partir desse entendimento, analisamos as definições teóricas que Anna Sfard, inspirada em Wittgentein, explicita sobre a aprendizagem matemática no âmbito escolar. Realizamos um estudo de natureza

[1] This article is a modified and expanded version of a chapter of the first author's PhD dissertation (Souza, 2012), advised by the second author.

qualitativa, no qual, analisamos os discursos de um grupo de alunos e de um professor produzidos durante a implementação da modelagem matemática. A análise desses discursos, a partir das ideias de ambos os autores, nos permitiram apontar que a aprendizagem matemática na modelagem matemática é caracterizada pela identificação das semelhanças entre os usos pelos quais as palavras são mobilizadas em âmbito escolar e os usos das palavras sugeridos pela situação-problema abordada em modelagem. Também identificamos como o professor orienta os usos legítimos das palavras nas aulas de matemática aos alunos.

Palavras-Chave: Modelagem Matemática; Aprendizagem matemática; Uso; Gramática.

1. Introduction

One of the arguments for teaching mathematics in the school environment is to enable students to *use* the mathematics that they learn in school, or a portion of it, in their daily lives, in the workplace, when using technology, and in the other sciences, among other situations (Maaβ, 2006; Blum & Ferri, 2009; Araújo, 2010; Kaiser & Schwarz, 2010). Certain authors emphasise that, in addition to these arguments, the teaching of mathematics should enable students to criticise the way mathematics is used in the various discussions that occur in society (Alrø & Skovsmose, 2002; Barbosa, 2006).

These arguments regarding the goals of mathematics teaching are explicitly present in official curriculum recommendation documents of certain countries. This is the case, for example, with the curriculum document prepared by the National Council of Teachers of Mathematics (NCTM), an institution consisting of professors and researchers in the United States. In Brazil, the National Curriculum Parameters recommend that students should use the mathematics learned in school in extracurricular situations (Brasil, 2002).

In this context, mathematical modelling[2] has shown promise for accomplishing these goals of mathematics teaching in the school context because modelling in the school environment generally refers to an approach used for problems in everyday life, the workplace or other sciences using school mathematics content.

Certain studies report the ability of students to understand the usefulness of school mathematics when they learn mathematics content in a way linked to "real situations" (Barbosa, 2006; Blum & Ferri, 2009; Araújo, 2010; Kaiser & Schwarz, 2010). Given the way mathematical concepts are approached in modelling, this study presents a possible understanding of mathematics learning based on mathematical modelling from a *point of view* guided philosophically by the ideas of Wittgenstein, particularly certain ideas contained in his book *Philosophical Investigations*[3].

[2] In certain instances, we will use only the term *modelling* in place of the expression *mathematical modelling*.

[3] In addition to Wittgenstein's *Philosophical Investigations*, we adopted the interpretations of a few of the commentators of his work, including those whose studies relate to the field of mathematics education. These commentators and their studies are Gottschalk (2004a, 2004b, 2008), Glock (1998) Vilela (2007, 2010), Jesus (2002), Moreno (2003, 2005), Miguel, Vilela and Moura (2010) and Harré and Tissaw (2005).

Next, we will present the theoretical and philosophical underpinnings of the study by focusing on the following themes: meaning, as conceived by Wittgenstein (1999), an understanding of the school mathematics system, and the learning of uses of words. From this discussion, we will discuss a *way of viewing* mathematical modelling in the school environment. Then, we will reintroduce the study objective and describe the methodological procedures and the context of data collection.

2. The idea of *use* in Wittgenstein

Wittgenstein (1999) in *Philosophical Investigations* is to point out the existence of "misunderstandings" concerning the manner in which we use words (Wittgenstein, 1999, §90, our translation)[4]. These misunderstandings, which we may also call conceptual confusions (Vilela, 2007), can be removed and diluted by our analysis of ways in which words are used in various situations in our experience.

For the philosopher, the role of philosophy is to indicate these multiple uses and begin removing these misunderstandings regarding the use of words. As such, Wittgenstein's (1999) ideas are also called "grammatical therapy" (Moreno, 2003, p. 95, our translation) or philosophical therapy (Miguel, Vilela, & Moura, 2010; Vilela, 2007).

One of the conceptual problems identified by Wittgenstein (1999) refers to what the philosopher terms referential understanding of language. This understanding of language was exemplified by Wittgenstein (1999, §1, our translation) using the following statement: "[...] every word has a meaning. This meaning is correlated with the word. It is the object for which the word stands."

According to a referential understanding of language, language has stable and autonomous foundations that are extralinguistic entities, whether ideal, mental or material. In addition, language is viewed as a mere symbolic system whose function is restricted to *expressing* or *giving voice* to these entities.

Wittgenstein (1999) breaks with this referential understanding of meaning, particularly because in his process of *grammatical therapy,* he discusses that it is *in* and *through* language that words acquire meaning, thus pointing out that the meanings of words can have only *language* as a source of analysis and constitution.

For example, Wittgenstein (1999, § 57, 58) states that the meaning of the word *red* does not come from the object or from something that is red in colour. That is why these objects may not exist, or we may not have the visual experience necessary to identify these objects, however, the word red can still be *used with meaning.* In contrast, if we forget that the word is used for objects that have a particular colour, the object can exist, but the word is not used, thus losing its meaning.

Thus, Wittgenstein's (1999, §43) ideas suggest that the meaning of a word corresponds to the *use* that makes sense of it in a given language, i.e., the meanings of words must

[4] The book *Philosophical Investigations* is written through the use of aphorisms. In this article, we will quote Wittgenstein's work followed by the corresponding aphorism number.

have their *use* as analysis and identification. Certain words are used to name *objects,* but this is just one of the possible uses that can be attributed to words.

Wittgenstein (1999) called *grammar* the *meaningful* uses of a given word. Therefore, the *grammar of colours,* for example, refers to the *meaningful* uses for which we employ the words concerning *colours.* For example, in a certain grammar, saying "the colour of the eyes" may have a meaning, but talking about "the colour of the wind" and "the colour of pain" may not make sense. Thus, we conclude that *the colour of pain* is not a part of this grammar, although it may be part of another.

In addition, Wittgenstein's (1999) ideas suggest that we should understand *grammar* based on the *form of life* to which a particular use of words is bound (Glock, 1998; Jesus, 2002; Gottschalk, 2004a, 2004b, 2008; Vilela, 2007, 2010). *Form of life* in Moreno's (2003, p. 129, our translation) definition can be understood as "systems where habits, attitudes, ethics, conceptions of knowledge and decisions of will intersect".

Thus, the uses assigned to words are derived from and related to beliefs, values and conceptions of the world, among others, to which individuals who use the words in a certain way are linked; therefore, they are not an arbitrary community consensus.

These questions allow us to understand that the uses that do or do not make sense as being attributed to words are constituted by *forms of life* and in turn also constitute them. Therefore, distinct *grammars* may also be indicative of distinct *forms of life* in which these *grammars* are anchored.

The use of words within any *form of life* is not arbitrary; i.e., it is not just any use nor is it a fixed use that is delimited by criteria. This non-randomness of the use of words is characterised by the understanding that such uses are governed by *rules* (Jesus, 2002; Moreno, 2003, 2005; Gottschalk, 2004a, 2004b, 2008; Harré & Tissaw, 2005; Vilela, 2007).

Rules, in Wittgenstein (1999), have a connotation distinct from the idea of rules as fixed and immutable stipulations. The indication of the existence of rules in the uses of words is intended precisely to emphasise that when we refer to the uses of words, these uses are not random and are under the delimitations of the *form of life.*

Grammatical rules, i.e., rules related to the way words should be used such that their use is a *meaningful* one within a *form of life,* are indicators of direction. They should not be taken as fixed and absolute criteria for deciding when a particular use of a word makes sense because the rules may change, and many modifications develop into the abandonment of certain words or into expansions in their uses.

In the next section, we present understandings about school mathematics. However, we emphasise that Wittgenstein's considerations regarding mathematics were not disciplinary or typifying, and therefore, that which we shall call *school* mathematics corresponds to our definition, based on Wittgenstein's ideas.

3. A use of language: the mathematical use

In Wittgenstein's view, mathematical statements have the function of normatising, of being *standards,* i.e., *systems* for organising our experiences in the world (Glock, 1998; Jesus, 2002; Gottschalk, 2004a, 2004b, 2008; Vilela, 2007, 2010), and for providing our experiences with a certain mode of presentation.

In this sense, the statement that defines the geometric figure *rectangle* as a quadrilateral consisting of parallel sides with equal measurements should be understood as a normative statement, in Wittgenstein's (1999) view. This statement is a system that can be used to identify the geometric shapes of objects and of spaces around us. Based on this statement, we can say, for instance, that a certain wall has the shape of a rectangle.

Mathematical statements conceived as systems have the function of organising and guiding our experiences in the world based on certain aspects provided by the statements. They do not dictate how a thing is; rather, being normative implies that mathematical statements indicate how a thing "should be" in case I adopt them (Vilela, 2007, p. 153, our translation).

Therefore, the way we organise our experiences does not refute or invalidate mathematical statements (Glock, 1998; Jesus, 2002; Gottschalk, 2004a, 2004b, 2008; Vilela, 2007, 2010) because mathematical statements are the *standards of correction* themselves and thus are not open to correction, based on the situations in which the statements were taken as the *standard.* If we identify, for instance, that the wall analysed does not have parallel sides with equal measurements, this does not invalidate the geometric mathematical statement about rectangular figures.

Mathematical statements are refuted based on their internal consistency, but they are not totally independent of empirical experience, given that a change in certain facts can make the choice of certain statements impractical or inapplicable (Glock, 1998).

Other statements aside from school mathematical statements may be adopted as models of organisation of our experiences in the world. Vilela's (2007) study of various mathematics (street mathematics, peasants' mathematics, farmers' mathematics, among others) can be understood as the confirmation of the existence of a variety of mathematical systems that can be used to organise the experience of people in the world.

Given this diversity, Sfard (2008) presents a set of theoretical concepts and definitions related to the learning of what may be called school mathematics. The author's ideas are inspired by Wittgenstein's *Philosophical Investigations*, although we reiterate that Wittgenstein did not typify his understanding of mathematics.

4. Learning the *use* of words

Wittgenstein's (1999) theory of language leads us to understand the term *learning* as *the learning of the uses of words.* The learning of school mathematics then may be understood as learning the *system of school mathematics or the uses of words that make up this system.* We define this school mathematics system as a ruled set of uses of language related to a *form of life* that is approached in the school context in a disciplinary manner.

In this sense, we can say that Sfard's (2007, 2008) studies present insights into the dynamics of the learning of the school mathematics system in the classroom. Sfard redefines many classical terms that are unique to the field of psychology of mathematics education, such as abstraction, concept, acquisition, mathematical objects and mathematics. Inspired by Wittgenstein, the author defines the latter as mathematical discourse.

Regarding the learning of school mathematics discourse or, in our terms, the school mathematics system, Sfard (2007, 2008) points out that students may assign uses to words that are unique to everyday usage and have different rules when compared to the school mathematics system.

As a result, Sfard (2008) emphasises that learning the rules associated with the uses of words unique to the school mathematics system occurs through a conflict, called "commognitive conflict[5]" (Sfard, 2008, p. 256). This conflict is settled when students change the rules that govern their use of words — the everyday usage — to rules related to the *school mathematics system.*

This delimitation by Sfard (2008), among other indications, indicates that the same word can be used in various systems anchored to various *forms of life.* Therefore, the legitimacy of the use of certain words in certain systems is a possible judgement based on the analysis of their *use*; these uses may or may not be identified as unique to the school mathematics system.

Therefore, we do not judge that a particular *use* of words should be replaced by another, as Sfard points out (2008). We understand that this practice may be a *delimitation of uses* in the sense of clarification regarding which uses are useful for purposes relating to the *forms of life* anchored by these systems.

However, mathematical learning in school predominantly corresponds to the learning of a single system, specifically the school mathematics system. This system is not the one adopted as a model in non-school environments.

Due to this uniqueness, Sfard (2008) points out that students begin their mathematics learning at school by *imitating* the uses of words used by people who already assign to the words uses that are seen as *legitimate* in the school context. Imitating such uses is to attribute to the words the same or similar uses that these people attribute. We call legitimate uses of words the uses related to the system adopted as a model in a particular social practice.

The rationale for students to *imitate* the uses assigned by the teacher or the textbook author, for example, is the fact that these uses are already historically adopted. In the words of Sfard (2008, p. 287), students likely are developing the following understanding: "if these people are talking the way they do, they must have good

[5] This neologism is a blend of the terms *communication* and *cognition*. For the author, the blending of these terms indicates that cognitive phenomena, such as the activity of thinking, are communicative phenomena. Sfard (2008) posits that thought consists of communication with oneself.

reasons. After all, they have been doing this for a long time." Thus, we can say that, for Sfard (2008), students adopt the school mathematics system the way it has been developed, as there must be reasons for this choice to be important even if they are not explicit.

Given the above, Sfard (2008) points out that the uses assigned to words by the teacher should be considered the leaders' uses*;* i.e., these are uses to which students should align themselves. However, the author points out that following rules is justified only because people have already been assigning these uses for a long period of time.

This following of rules without analysis is what justifies students often looking to the teacher to judge whether their uses are legitimate uses. Sfard (2008, p. 233) calls this situation "substantiantion". For Sfard, however, substantiation occurs only when students have not yet identified which uses of words are legitimate uses.

Sfard (2008) admits that the students' choice of assigning uses provided by teachers to words is a matter involving "power relationships" (Sfard, 2008, p. 283). The author states that this power relationship is present because of the historically established and unquestioned position assigned to the teacher as an individual whose uses assigned to words are legitimate uses.

For us, the power relationship is not only explained due to the historically attributed position of a mathematics teacher in the dynamics of learning the classroom. Rather it occurs because of the uses attributed to words that are legitimate or not in a particular social practice in which students and teacher take part.

Sfard's (2008) indication of the need for *one* leader discourse to be followed by the students is representative of the existence of power relationships between normative systems, as it is proposed that a mathematical system is a model and is approached in the school context.

These questions regarding language, mathematics and mathematics learning in the school context have substantial implications for the understanding of modelling in mathematics education.

5. Modelling: a way to organize our experiences

Mathematical modelling in the school environment from the perspective of mathematics education is an expression generally assigned to the use, development, investigation and solving of real problem situations based on formulas, models, procedures and concepts associated with the *school mathematics system.* We summarise these designations present in the literature by the term *approach of real problem situations* using the school mathematics system.

Silveira and Caldeira (2010) report that the problem situations cited in studies of such modelling are situations called reality situations that are approached *mathematically*. Based on this designation, certain authors explain that addressing these real problem situations corresponds to uniting reality and mathematics (Biembengut & Hein, 2003; Maaβ, 2006) or uniting reality and the other part of reality that is mathematics (Bean, 2007; Blum & Ferri, 2009); one may say that the situations being addressed have their basis in reality (Barbosa, 2007).

These delimitations may be indicative of the search for a definition of mathematics as belonging to a particular sphere that is distinct from the situations referred to as reality situations. Such understanding may be related to an understanding of mathematics in terms of referring, originally, to abstract and formal entities, or consisting of generalisations that are also abstractions of so-called real-world situations.

Araújo (2007) points out that this distinction between mathematics and reality is based on realistic and formalistic philosophical conceptions of mathematics. According to author, although these conceptions are distinct, they are based on the idea that mathematics has a one-to-one relationship with reality and that therefore modelling would be a tool for the *description* of real situations.

When this "one to one" identity with reality is not evident, certain researchers (Bassanezi, 2002; Biembengut & Hein, 2003; Cifuentes & Negrelli, 2011) conclude that reality is complex and therefore difficult to *translate* into mathematical terms. According to these authors, this condition indicates a need for simplification of this reality such that it may be *mathematised*.

These simplifications justify the use of hypothetical situations drawn from information about the nuances of real situations addressed by modelling. To illustrate, we quote an example by Blum and Ferri (2009) in which the curvature of the earth is not taken into account in identifying the maximum distance from which someone on a ship can see a lighthouse.

The use of the words "simplification", "translation" and "validation" of real situations in studies of modelling in mathematics education indicates an understanding of mathematics as an entity capable of representing reality.

Another possible understanding regarding the non-identity between mathematics and reality is based on an understanding of mathematics as normative as presented by Wittgenstein (1999). Based on the ideas of this philosopher, the mathematical approach to a real problem situation can be conceived as *a way to organise and address such situations.*

From now on, we will call the situations addressed in modelling "empirical situations" rather than referring to them as real. This change reflects the understanding of mathematics based on the mathematical use of the language and not in relation to the *place* of mathematics in reality.

The use of the term *empirical situations* must be understood in grammatical terms. In this case, empirical situations correspond to those adopted as a source of analysis using the school mathematics system. The mathematics system can be compared to a ruler and the empirical situation to the object to be measured. The ruler itself is not measured, as it is the standard of measurement itself. In contrast, the empirical situation takes shape from the use of this standard.

Kaiser and Schwarz (2010) identify the maximum irrigation area of a plot by correlating the shape of the plot with the geometric form of the rectangle. This example illustrates how the academic mathematics system (a normative system) was used to organise the empirical situation being studied, i.e., the optimisation of irrigation areas.

Thus, from a wittgensteinian *point of view*, we understand modelling as the use of the mathematical system to organise our experiences in the world. The literature regarding modelling is replete with examples of the use of the school and/or academic mathematics system primarily for the purpose of such organisation.

Kaiser and Schwarz (2010, p. 53) also argue that it is important for modelling activities to address "authentic situations", i.e., those that students can experience in non-school environments. Similarly, Ferruzi and Almeida (2009, p. 4, our translation) emphasise that problem situations in modelling should be configured as a "simulated context", i.e., a situation likely to occur in various contexts in which students may find themselves, such as workplaces or their homes. However, Schwarkoph (2007) reports observations by students regarding the inauthenticity of problem situations regarding their legitimacy in non-school contexts. According to Schwarkoph (2007), students report that when experiencing the problems presented in the school environment, they would use other means to solve them.

The study by Schwarzkoph (2007) indicates that there are situations where the approach is specific to the school mathematics system. The author notes, for example, that in a modelling problem situation consisting of calculating the distance between two cities based on information displayed on a signpost giving the distances in kilometres from a given point to the cities, students stated that if they needed to calculate the distance between two cities, they would not base it on the information on the signpost, and they would ignore the approach called for in the school context.

Problem situations based on non-school settings may have their own system as an organisation *model*. In this case, words can have- their own grammar distinct from that of the school mathematics system. Therefore, analysing or developing these situations based on the school mathematics system may lend them a non-*authentic* character.

However, there are problem situations involving non-school contexts that may be modelled in the school mathematics system because its legitimacy in the school context is also reflected in its legitimacy in non-school contexts.

These understandings lead us to reflect on the nature of problem situations used in modelling and of the normative system chosen to organise such situations in terms of legitimacy and power relationships involved in this choice.

In the following sections, we describe the methods used to develop the discourse used as the source of analysis for the ideas outlined throughout the present article.

6. Research method and procedures for eliciting and recording classroom discourse

The study we performed is characteristic of qualitative studies (Bogdan & Bicklen, 1994; Miles & Huberman, 1994; Denzin & Lincol, 2005) in that we aimed to understand the learning of uses that are attributed to words when developing modelling' task in the school environment.

Because we conceive of learning as the learning of the *uses* of words, the speech, writings and gestures of the students and teacher were recorded as data for our analysis. We opted to refer to these as oral, written and gestural styles of discourse. We studied these discourses via *observation. Observation* is a procedure for obtaining data whose main characteristic is that it allows for an understanding of the subject being studied at the time the data are generated (Miles & Huberman, 1994; Angrosino, 2005).

The observations were recorded on video and in notes taken in the field. The video recordings were a record of the discourses of a group of students and a teacher during a lesson in which the students and the teacher worked on a modelling task. The modelling task involved a government housing program named *Minha Casa, Minha Vida* (My House, My Life) and two associated problem situations. *Minha Casa, Minha Vida* is a program of the Brazilian government, established in May 2009, that involving public financing of housing construction.

7. Spatial context of the recording of classroom discourse

The classroom discourse occurred in a state public school. The particular school provides education exclusively to youth and adults. In Brazil, the youth and adult education refers to basic education for persons over 18 years old who did have acess to school education during childhood.

This school was the *locus* of data recording for this study, as it was the school where Marcus taught. Marcus, at the time of data recording, participated with the authors in weekly meetings of a group called the Collaborative Group of Mathematical Modelling. The students chosen by Marcus to solve problem situations proposed in the modelling task were a group in their first semester of secondary education. The names of the teacher and students have been changed to preserve their anonymity.

8. Discourse analysis procedures

Based on suggestions by Charmaz (2006), we analysed the video recordings of the classroom discourse. The first step of the analysis consisted of transcribing the discourse in its entirety. We then selected for analysis only those portions that included uses of words that were important in terms of the school mathematics system. These portions were assigned designations based on the critical words being used at the time. This process is equivalent to what Charmaz (2006) calls the process of creating initial codes.

In this initial analysis, we observed that the words *average*, *sets*, *intervals* and *percentage* were frequently used by the teacher in solving the problem situation. However, of these four words, only the words *average* and *sets* were used in the present analysis. After making this selection, we developed analytical understandings of mathematics learning through modelling by focusing on the uses of these words. We call this understanding *preliminary descriptive analysis*, the objective of which is to perform an analysis simultaneous with noting descriptions of the performance on the modelling task.

The decision to develop this analysis was based on guidelines presented by Charmaz (2006), who argues that a more descriptive analysis of a phenomenon, without focusing on theoretical concepts, has the greatest potential to produce ideas that oppose such concepts, rather than only confirming them.

We have reservations regarding this understanding by Chamaz (2006) because we did not analyse data without theoretical biases, as theoretical biases were the guidelines of the present study. However, in the preliminary descriptive analysis, we sought elements for refuting and/or confirming these biases. We devoted, then, a specific section of this paper to a theoretical analysis of issues raised in the preliminary analysis.

When preparing the transcripts, we used conversational markers based on those of Silva (2002). The markers include the slash (/), which indicates a break; and the ellipsis (...), which indicates hesitation or pauses. Brackets ([]) were used for the insertion of comments by us to clarify certain discussions.

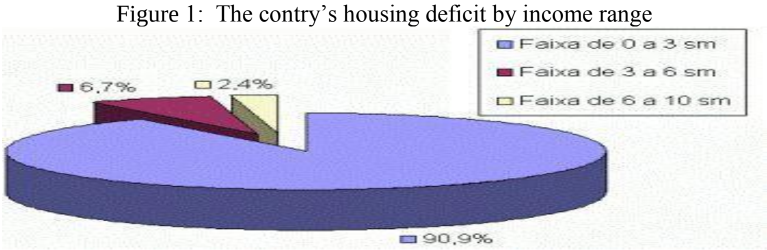
9. The modelling task and the preliminary descriptive analysis of its performance

Image 1: The modelling task assigned to students

THE DREAM OF HOMEOWNERSHIP WILL BECOME A REALITY FOR MILLIONS OF BRAZILIANS

What it is: The federal government is investing R$34 billion so that millions of Brazilians may have access to home ownership. The *Minha Casa, Minha Vida* program in a partnership with states, municipalities and the private sector enables the construction of 1 million houses for families with incomes of up to 10 times the minimum wage. The program will boost the economy, create jobs and have a positive impact on Brazilin society.
Conditions for participating in the program: One must not have been the beneficiary of government social housing programs; not own a home or receive funding in any federal unit; meet the family income requirements of the program; and make payments of 10% of income for 10 years if income is 0 to 3 times the minimum wages or make minimum payments of R$50.00, as corrected by the RR* (reference rate). Property registration shall be in the name of the female head of household. One need make no down payment or installment payments during the construction. There is no insurance for death and permanent disability (DPD) nor for physical damage to the property (PDP).
* The RR was created as part of the Collor II Plan and serves as a main Brazilian economic index; i.e., it serves as a basic reference rate of interest to be charged during the current month irrespective of the rate of inflation during the previous month. The Central Bank of Brazil will publish the reference rate (RR) monthly.
Purpose of the program: The country's housing deficit measures 7.2 million housing units. Thus, with the *Minha Casa, Minha Vida*, program it will be possible to reduce the country's housing deficit by 14%. This deficit is distributed across incomes as shown in Figure 1:

Figure 1: The contry's housing deficit by income range



The distribution of new houses among families by income brackets will be as follows:

Table I: Income ranges and numbers of houses to be distributed

Family income range	Number of houses
0 to 3 times the minimum wage (mw)	400,000
3 to 6 times the mw	400,000
6 to 10 times the mw	200,000

Based on the text above, we suggest the following questions:

1. What will be the amount of the monthly payment to be paid by the beneficiary in relation to their salary?
2. Based on Image 1, how could the houses be distributed while giving preference to people with low incomes?

Legend 1: Figure 1
Range from 0 to 3 mw - minimum wage – Faixa de 0 a 3 sm
Range from 3 to 6 mw- minimum wage – Faixa de 3 a 6 sm
Range from 6 to 10 mw- minimum wage – Faixa de 6 a 10 sm
Attencion: All commas (,) should be replaced by points (.).

The teacher (Marcus) explained the modelling task, presented additional information about the *Minha Casa, Minha Vida* program, and questioned the students regarding their personal knowledge of the program. Later, he distributed the task handout to the students and asked them to organise themselves into groups. The teacher read the entire task description and the two problem situations and then asked the class to begin solving them.

Below, we begin by presenting the transcripts of the students' discourse while they discussed how they would solve the following problem situation proposed in the task: **What will be the amount of the monthly payment to be paid by the beneficiary in relation to their salary?**

42	Ana: I don't know how to start this.
43	Nanda: We must calculate the salaries.
44	Nanda: How much is the minimum wage? Around 465 (/), no?
45	Ana: It's 465 (/), but there are some discounts.
46	Nanda: But there are also raises (/), the extra hours.
47	Ana: But it (/) doesn't figure into it, no.
48	**Nanda: So it is 465.**
49	Nanda: That amount of zeroes [Nanda refers to people who have no income] (/): We don't know if they don't earn anything (/), a person can do an odd job (/). And how much will it be per month?
53	**Josi: In relation to their salary** [reading the first question]. **Choose the salary of someone here? In relation to their salary** [Josi reads the first question again].
54	Ana: Who here works? Come on Nanda, use yours [Ana refers to Nanda's salary].
55	Ana: How much do you earn, Nanda?
56	Nanda: I earn 275.

In these opening discussions, we observed the students attempting to understand the problem in mathematical terms based on the explanation of the situation. Agreeing that the payments should be calculated *in relation to the beneficiary's salary,* the students chose to consider the salary of someone who is paid the minimum wage. Then, Josi suggested that the group adopt the salary of a member who was working as the value to substitute to begin solving the problem.

However, the students requested suggestions from the teacher because they did not envision other possible paths. The following dialog summarises the discourse that then occurred.

78	Josi: Shine a light here [an expression meaning a request for explanation] (/). It will be a multiplication (/), correct? Times 12.
79	Ana: Marcus, come here (/). Shine a light here..
80	Nanda: Through the Caixa [Nanda refers to the agency funding the *Minha Casa, Minha Vida* program], it is 300 months and also through the municipality? [Nanda quotes information given by the teacher.]
81	**Teacher: Yes (/), but in that case there (/), I *want* the following (/): there is no way for you to establish** [stipulate] **the exact amount of the payment (/), right? Because (/) it will depend greatly on the salary of each person.**
82	Nanda: No [in response to the teacher's question].
83	**Nanda: But we are basing it on the minimum wage.**
84	Teacher: Correct.
85	**Teacher: But I *want* to know the following (/): In this case (/), the program is intended for those who earn specific ranges of salaries, isn't it?**
86	Nanda, Josi, Ana: Yes
87	**Teacher: From 0 to 3, 3 to 6, 6 to 10** [the teacher reads the information in table I of the task handout]**. What are these? (/) Intervals.**
88	Students: Hmm!

89	**Teacher: Then(/) what do I *want* to know? I want to know (/) what will be the amount of a monthly payment paid by a beneficiary (/). Let’s suppose that this beneficiary earns between 0 and 3 times the minimum wage (/). What will be his (/) average payment? Because, there is no way for you to calculate it exactly (/). But what will be the average? For example (/), our friend Talia earns the minimum wage (/), Nanda earns two times, and (/) Ana earns three times (/). Are they in the same range?**
90	Students. No (/), they aren’t.
91	Teacher: Here in the program?
92	Nanda: Yes (/), from zero to three (/). They are in the same range.
93	**Teacher: So how do I find a value, an approximate average, of the amount that will be paid?**

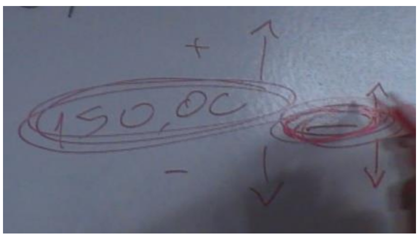
In the discourse transcribed above, we observe the professor explaining and guiding the students regarding how he *wanted* the problem to be analysed. Note the verb *to want* in all three suggestions by the teacher.

Marcus suggested that the students assume salary figures based on the ranges listed in the table presented in the handout, i.e., from zero to three, three to six and six to ten times the minimum wage, rather than a single salary, in accordance with the original definition of the problem.

Marcus’ guidance included mentioning to the students that addressing the problem would involve finding the average value. Note that the teacher assigned to the word *average* its use relative to a quantity that may not correspond exactly to one value, as we can see in his statement, “there is no way for you to calculate it *exactly*, but what will be the average?”

By identifying this use of the word *average* to the students, the teacher sought to communicate that the wage amounts should be used to find an average, and he asked the students how they might calculate this average. The students did not identify which mathematical procedures would yield a so-called average and asked again for the teacher’s presence and guidance.

101	**Teacher: The reference rate (/): that’s why it comes to 50 reais (/). In this case here (/), I’m not asking you to include the RR** [reference rate] **(/). I am (/) asking you to find more or less (/) What can you do? Calculate an average (/). How is an average calculated among the three of you? How is an average calculated?**
102	**Teacher: I’ll add the three salaries and do what?**
103	**Nanda: Multiply by 10%?**
104	**Teacher: No** [the teacher responds to Nanda’s suggestion]. **Let’s suppose that 10% of Lana’s salary is 46.50 (/). How much are two salaries? Ninety three, and three salaries are 135.00 (/). Then (/), I’ll add the 46 (/) with 93 (/) with 135 (/) and divide it by how much?** [The teacher indicates the value using three fingers.]
105	Nanda: By three.
106	Ana: Three.
107	**Teacher: I’ll find an average (/). An average of roughly how much will it be (/)? Then (/) what is this average? It will be a payment that may be more or less (/). The payment may be more or may be less** [teacher draws two arrows with opposite directions] **(/), but the average will always be this** [pointing to a point between the drawn arrows] **(/). Then, you will have this average (/). Let’s suppose that the average is 150.00** [teacher writes the value 150 between the arrows] **(/). Then the payment may be more than 150**



Legend: All commas should be replaced by points

	or less than 150 (/). But how much will the average vary? [The teacher points up and down around the value 150.]	
108	Nanda: One hundred fifty [Nanda responds to the teacher's question].	
109	Teacher: One hundred fifty.	
110	Teacher: Therefore (/), the suggestion I give you (/) is to try to find (/) [a value] in relation to each range (/) because you won't be able to stipulate a beneficiary (/). Let's suppose you work at Caixa [a real estate lending agency] (/). Any person may appear there to do this, correct? Then (/) let's go (/) try to discuss a little some of the ideas we've discussed.	
111	**Teacher: You will not be able to calculate a salary and a half (/) with two and a half (/). You only calculate what? The salary (/), the incomes 1, 2 and 3, and calculate the average (/). Next, 3, 4, 5, 6, and then calculate the average (/) and so on.**	
112	**Nanda: So let's do as he said (/). Let's calculate a salary (/) with two and with three times the minimum wage.**	

We observe in the previous transcripts that the use of the word average was meant to mean "to find something approximate." Marcus then stated that this approximate value of the salary-based payments would be found by performing calculations related to the average. The verb *to calculate* is used as an indication that the students need to develop procedures associated with the school mathematics system that may prove legitimate for finding the average.

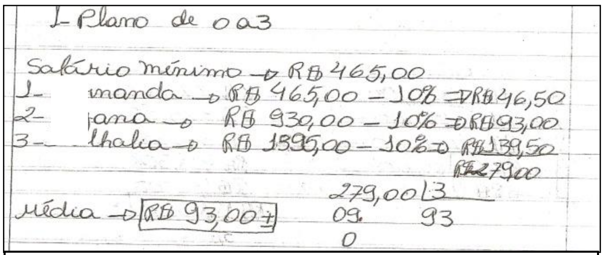
Given the lack of student responses to his questioning regarding methods for finding the average, Marcus provided more direct suggestions, such as this one: "I will add the values and do what?" Nanda replied by suggesting, "I will multiply the ten percent?"

However, Nanda's response did not correspond to the mathematical procedures for finding the average based on the school mathematics system. Thus, the teacher indicated that the students should add the values corresponding to ten percent of one, two and three times the minimum wage and then divide this amount by three.

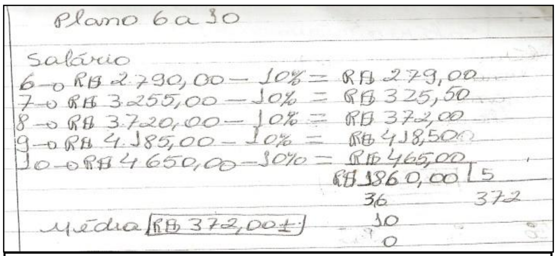
Marcus indicated the calculation that the students should first perform to obtain the average (46.50+93.00+135.00); in addition, he explained the meaning of the average value to be found, indicating that "it will be approximately the amount of the payment." Marcus also described the procedure for adding the values of ten percent of three, four, five and six times the minimum wage to find the average based on salary ranges of three to six times the minimum wage.

Nanda explained to the other members of the group that they should work on the problem "as the teacher has said." Next, we will present the procedures developed by the students in relation to the teacher's instructions with regard to the discovery of the average of the salary ranges from zero to three and three to six times the minimum wage.

Image 2: Mathematical discourse' students **Image 3: Mathematical discourse' students**



Legend 2: Plan from 0 to 3 (Plano de 0 a 3)
Minimum wage (Salário Mínimo) – R$ 465.00
Average (Média)
Attencion: All commas should be replaced by points.

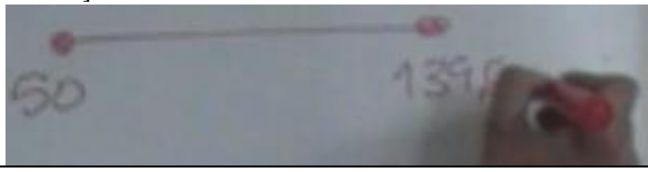


Legend 3: Plan from 6 to 10 (Plano de 6 a 10)
Minimum (Salário)
Average (Média)
Attencion: All commas should be replaced by points.

We can identify, in the notes written by the students, that the use of the word *average* presented by the teacher means "a value that represents more or less" the monthly payment. The fact that all of these values are associated with salary ranges was reflected in the mathematical procedures by placing the symbol (±) next to the calculated average value.

After completing the necessary procedures for calculating the average, the students requested the teacher's presence to assess the legitimacy of their calculations. The teacher reacted positively but suggested that he wanted the results structured in a particular way. Next, we present the discussions related to this structure.

135	**Lana: Teacher (/), we've calculated it, and it amounted to 93 (/). It is the base (/). It can be more or less 93.**
136	Teacher: But in this case (/), what was the minimum? Did it amount to 46.50?
137	Lana: Yes.
138	Teacher: So (/), okay (/), you are not including the RR (/). Alright (/), but if the minimum is 50 reais (/), you could put 50 (/), couldn't you? But can you put that [referring to the value of 46.50]? You can (/), but you could round it to 50 (/) because wasn't that the minimum payment?
139	Lana: [Makes gestures indicating agreement with the teacher's idea.]
140	Teacher: Okay, then (/). But there is no problem (/). So that is the average [referring to the value of 93.00] (/). **Is there another way to represent it?** Did you find (/), what did you say? That the average payment by the beneficiary (/) if the salary is from zero to three (/) will be an average of 93 reais [points to the value of 93.00 recorded in Lana's notes].
141	Lana: Yes.
142	**Teacher: I can also work with these values using what?**
143	**Nanda: With the function?**
144	**Teacher: What did we see earlier?** [The teacher is referring to content covered in previous lessons.] **Wasn't it a set? From zero to three?**
145	Nanda: Yes.
146	**Teacher: Isn't it a limited set?**
147	Lana: Yes.
148	**Teacher: What is the criterion for being part of this set? One must earn from zero to three (/), correct?**
149	Lana: Yes.
150	**Teacher: So, when I have an interval like this (/), when I have a set like this (/), what can I**

	use? Interval (/). So how can I calculate that? I can find the minimum value (/) that I will have to pay and the maximum value (/) and find the interval (/). So, the person who earns from zero to three times the minimum wage will be there (/) in this interval [the teacher represents the interval using hand gestures]. **Where the minimum is what? Fifty reais, and the maximum is how much?**	
151	Lana: One hundred thirty-nine [Lana answers the teacher's question].	
152	Teacher: So the payment will be (/), whoever earns from zero to three will pay from 50 reais to 139 and?	[The teacher writes on the board the interval from 50 to 139.50.] 
153	Lana: Fifty [completes the teacher's sentence].	
154	**Teacher: So, whoever earns the minimum wage will pay what? They will be here within 50 to 139** [the teacher indicates a point within the interval written on the board] **(/). Whoever earns the minimum wage will be here** [the teacher indicates a point in the interval between 50 and 139.50] **(/). And whoever earns four times the minimum wage? No! They aren't in that interval (/). I know that (/) if I earn from zero to three (/), then what can I afford? Between 50 and 139.50** [points to the values of 50 and 139.50 written on the board] **(/). My payment will vary around this value.**	

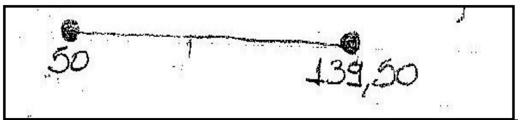
In the transcripts of the discussions presented above, we noted that the teacher asked the students to represent the calculated averages and the values of the payments based on the salary ranges by specifically asking, "Is there another way to represent it? I can also work with these values using what?"

Nanda suggested that the calculated values were mobilised by the use of the word *function*. The teacher, however, said that *function* was not a legitimate use that may be assigned to the range of values involving wage-based payments, and he instead used the word *sets*.

The teacher asked Nanda to describe the classification of income by ranges (from zero to three, from three to six and from six to ten), i.e., the very use of the word *set*. In Marcus' words "Wasn't it a set? From zero to three?" "Isn't it a limited set?" "What is the criterion for being part of this set?" "One must earn from zero to three, correct?"

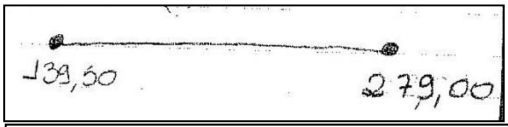
The students used the teacher's guidance to organise the information regarding income values into numerical ranges of values. Next, we will present this organisation of information by the students in terms of salary ranges of zero to three and three to six times the minimum wage.

Image 4: Mathematical discourse' students



Legend 4: Replace all commas by points

Image 5: Mathematical discourse' students



Legend 5: Replace all commas by

10. Data analysis

Based on our analysis of the transcripts presented above, we observe that the approach to the problem was configured based on the teacher's guidance. The choice of the teacher's discourse as one of *leader discourse* was received by the students as a command, which is exemplified by the following statement made by one of the students, "So let's do as he [the teacher] said."

To us, this choice is indicative of the establishment of *power relationships*, which are established by the subjects such that the students assigned to the teacher the task of judging the legitimacy of the students' discourse.

This finding may be indicative that *substantiation*, i.e., the requests by students that the teacher evaluate their discourse, is a situation that occurs due to the existence of these power relationships. Therefore, substantiation may be present in the dynamics of learning, in which such relationships are established and the occurrence of which are not simply extinguished as if the students already possessed their own criteria for such evaluations, as Sfard (2008) argues.

Among the teacher's directions for resolving the problem, we analysed his messages relating to the uses of the word *average*. We identified two types of uses for the word *average*. We refer to the first use as one of grammatical use. This grammatical use corresponds to the use of the word that *makes sense* in terms of its assignation, i.e., corresponding to a certain *form of life.*

The grammatical use of the word *average* corresponded to "finding a value more or less." In the case of this particular mathematical problem, it corresponded to finding a value among a set of payment values that is *more or less* indicative of the value of the payments to be paid by beneficiaries whose income is among the values in this set.

After identifying the uses that make sense to assign to the word, certain *grammatical uses* are linked to what we will call *procedural uses*[6]. We note that the grammatical use of the word *average* indicated the need to do something, which in this case was *to find* a value that will be *more or less the value* of the salary based payments.

In the school mathematics system, the *procedural uses* concern the performance of a set of steps, which we will call normative strategies. The normative strategies used were "to add the values of the payments and divide the result by the number of values." They may be referred to as school normative strategies because they have a symbolic structure peculiar to the school mathematics system, for example, the very structure of *45.00+93.00+139.503=93.00*); the way of presenting the result, the order of the steps

[6] This understanding can be extended to certain other words mobilized in the school mathematics system, such as *divide*. We could say that one of the grammatical uses of the word *divide* is to take parts of a quantity, while its procedural use would be the procedural calculations relative to division, usually symbolized by the sign (÷).

(first, the values are added and the result is then divided by the number of values), and the way the values are added and divided.

One of the characteristics of the school mathematics system is having a normative system whose procedural uses are marked by a symbology unique to it and predominantly configured in the preparation of written discourse. The normative uses related to the word *average* may be guided by other mathematical systems.

If, for example, we were asked about the number of pages we could write per day to complete this article, we could answer that we can write approximately five pages per day. We can verbalise our response without *writing down* how many pages we write each day or by adding up the total number of pages and then dividing by the number of days spent writing, which would correspond to the school normative strategies.
In the analysed discussions, the students did not identify which uses of the word *average* would correspond to the school mathematics system. Because of this gap, the teacher indicated the grammatical and procedural use of the word *average* that is unique to this system.

Based on this guidance, the students *found* the average value of the remaining salary ranges, making use of the normative procedural and grammatical uses assigned by the teacher. This allows us to understand that the students *imitated* the uses assigned by the teacher when we compare their discussions to what Sfard (2008) calls *imitation*.

Several of the teacher's suggestions sought to elicit from the students their own attributions of the word uses as they relate to the school mathematics system. Before explaining how to calculate the average, the teacher questioned the students regarding such a procedural use. However, Nanda's response did not match the procedural use unique to the school mathematics system. Another response by Nanda was judged by the teacher as not corresponding to the school mathematics system; this example corresponded to her understanding that the values of the payments could be seen as uses of the word *function*.

Both moments of disagreement between these uses of certain words resulted in the adoption of the teacher's suggestion and the abandonment of Nanda's suggestion after the students observed that the uses assigned by her were not the same uses unique to the school mathematics system. We did not classify this disagreement regarding the use of words as a *commognitive conflict*, which is Sfard's (2008) term. This is because Nanda suggested a procedural use of the word *average* corresponding to the completion of the multiplication procedure and not division, and this procedure was based on the school mathematics system itself. We arrived at a similar conclusion regarding the use of the word *function* by Nanda.

These observations indicate that disagreements between the uses of words result not only from the adoption of rules unique to systems distinct from the school mathematics system, as suggested by Sfard (2008). Disagreements can also occur as a result of a suggestion of a possible peculiarity within a system itself.

We suggest that, in the face of a *disagreement between the uses* assigned to words by the teacher and the students, the teacher's negative judgment alone can not explain the uniqueness of the school mathematics system regarding its legitimacy in the school environment, in addition to not showing the existence of the possibility of uses of other normative mathematical systems.

To organise the experiences of "finding the average salary" and "providing a mathematical way of working with the information related to salary ranges," the teacher and students sought to relate these experiences to *uses* of the school mathematics system.

The uses assigned to words related to the school mathematics system were adopted by the teacher to organise the experiences of "finding the average salary" and "providing a mathematical way of working with the information related to salary ranges," yet, we wonder why the use of the word *set* was adopted by the teacher in the discussion of the salary-based payments.

On this issue, we understand that when we mathematise an empirical problem and adopt the school mathematical system as a normative model, we search for similarities between the grammars that involve the use of words in the normative system and the grammar suggested by the situation we seek to organise using this system.

We believe that the teacher based his guidance on this resemblance to establish that *the values referring to payments based on salary ranges* could be seen as a *set*. In Marcus' words: "Isn't it a set?" "What is the criterion for being part of this set?" "One must earn from zero to three, correct?"

This questioning represents the teacher's desire that the students identify, in the calculations related to the payments based on salary ranges, the grammar of the word *set* in terms of the school mathematics system. In this case, the term *set* is related to the amounts of the payments to be paid by beneficiaries who earn from zero to three times the minimum wage.

The similarity between grammars can be identified by recognising the following aspect: the classification of values by salary range corresponds to a grouping of values of the same nature, i.e., the salary-based payments. This grammar can be viewed as resembling the word *set*, based on the use of the school mathematics system as a model system. We found, for example, in Dante (2004, p. 8, our translation), a *grammar* of the word *set*: a "set is any collection of elements."

We may classify the similarities identified above as *family resemblances*. This expression was used by Wittgenstein (1999, §67, our translation) to denote, among other ideas, the similarities between the grammars that involve the use of words.

However, we emphasise that without an analysis based on the school mathematics system, the classification of the amounts of the payments based on salary ranges is merely a way of organising these values. It is the normative point of view based on the school mathematics system that allows for such an identification of similarities.

Accordingly, we emphasise that the similarities between the grammar suggested by the situation and the grammar of the words within the school mathematics system are established only when a normative system is adopted. In other words, for similarities to be established, it is necessary to adopt one of the grammars as a reference. In the modelling task analysed in this article, the grammar adopted as a reference was the school mathematics system.

11. Implications for the implementation of modelling in a school context

One of the main arguments for the encouragement of modelling practices in mathematics education is to recognise the usefulness of the school mathematics system. This *utility* corresponds to the use of the school mathematics system to develop, investigate and resolve situations that students may face in their lives (Biembengut & Hein, 2003; Blum & Ferri, 2009; Kaiser & Schwarz, 2010).

Accordingly, certain studies, such as one by Kaiser and Schwarz (2010), indicate that the problem situations adopted in modelling in the classroom should simulate those found in non-school environments.

The school mathematics system is also adopted as a legitimacy model in other contexts, a fact that justifies the usefulness of this system in everyday life. However, we consider it important to include other normative mathematical systems as a way to organise and address the problem situations analysed in modelling tasks.

Without neglecting the fact that the legitimacy of the school mathematics system embeds the existence of power relationships under the legitimacy of other systems, we understand that situations experienced by subjects in the classroom or outside it require a broad spectrum of possibilities to organise them, and not only those resulting from use of the school mathematics system.

Consequently, for learning through modelling to be characterised as *helpful* in the sense that it allows students to "use and engage mathematics *in the necessities of life*" (Pisa, 2009, p. 14), the expansion and adoption of various normative mathematical systems is critical to remove the need to choose only one system.
One way to understand mathematics learning through modelling from a Wittgensteinian perspective corresponds to the learning of mathematical systems that are useful and legitimate in various *forms of life*. Thus, mathematics learning may be seen as *forms of viewing* and organising our various experiences of being in the world.

The use of the expression *form of viewing* was based on that of Wittgenstein (1999) and corresponds to the understanding that the system we adopt as normative is a *form of viewing* (Wittgenstein, 1999, §144) because, based on this system, our experiences require some form of presentation.

Thus, we believe that modelling in classroom can be understood as a pedagogical approach in which students can learn various ways of addressing and normatising authentic situations derived from existing social demands rather than non-authentic situations that are developed only for the purpose of serving as "school tasks".

We also emphasise that, in modelling, the similarities between the grammar suggested by the problem and the grammar of the use of words adopted to normatise such situations are not similarities unique to these grammars but are created from a *normative* *point of view*. We suggest that this point of view should be expanded to other areas beyond that provided by the school mathematics system.

Finally, the thematisation of mathematics learning based on modelling raises questions regarding the uses we assigned to the terms *utility*, *problem situation*, *learning*, *school* and *mathematics*, among others. The ideas discussed in this article should be seen as *guidance indicators* of a possible *form of viewing* these issues.

12. Acknowledgements

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