THE IMPACT OF VISUALIZATION ON FUNCTIONAL REASONING: THE ABILITY TO GENERALIZE

Ana Barbosa
anabarbosa@ese.ipvc.pt

Isabel Vale
isabel.vale@ese.ipvc.pt

Instituto Politécnico de Viana do Castelo, Escola Superior de Educação

ABSTRACT

With this study we try to understand how pre-service teachers, from basic education (for 3 to 12-year-old students), solve problems involving the generalization of patterns in visual contexts, identifying: the strategies they use; the difficulties they present; the role of visualization in their reasoning; and the factors that influence their generalizations. Considering the aims of this study, we followed a qualitative methodology. The participants were 80 pre-service teachers. During the classes of a Mathematics unit course, these students solved a sequence of tasks involving pattern generalization in figurative contexts. Our results showed that students were able to use different types of generalization strategies, but also that some dimensions of the tasks (e.g. type of pattern, nature of the figures) can have impact in students' reasoning, causing, in some cases, a shift on the strategies used and the emergence of difficulties of different kind.

Keywords: Functional reasoning; Patterns; Generalization; Visualization; Learning.

RESUMO

Com este estudo pretende-se compreender de que forma alunos da formação inicial de professores, do ensino básico (alunos 3-12 anos de idade), resolvem problemas que envolvem a generalização de padrões em contextos visuais, identificando: as estratégias que usam; as dificuldades que apresentam; o papel da visualização no seu raciocínio; e os fatores que influenciam as suas generalizações. Considerando os objetivos do estudo, seguimos uma metodologia qualitativa. Os participantes foram 80 alunos da formação inicial de professores. Durante as aulas de uma disciplina de Didática da Matemática, estes alunos resolveram uma sequência de tarefas que envolviam a generalização de padrões em contextos figurativos. Os resultados mostraram que os alunos foram capazes de usar diferentes tipos de estratégias de generalização, mas também que algumas dimensões das tarefas (e.g. tipo de padrão, a natureza das figuras) podem ter impacto no raciocínio dos alunos, fazendo com que, em alguns casos, haja uma mudança nas estratégias utilizadas e a emergência de dificuldades de tipo diferente.

Palavras-chave: Raciocínio functional; Padrões; Generalização; Visualização; Aprendizagem.

1. Introduction

Reasoning plays a crucial role in school mathematics, being impossible to dissociate it from learning, since it is from this ability that students gradually acquire knowledge (Thompson, 1996). This is an evolutionary process that involves conjecturing, generalizing, investigating why, developing and evaluating arguments (Lannin, Ellis & Elliott, 2011). The focus is thus on the formulation of general statements that deepen students' understanding, allowing them to clarify what is true (or false). We can also add that other mathematical processes are themselves manifestations of reasoning, like problem solving or mathematical communication. It is impossible to solve problems or to make statements without mobilizing reasoning and both are avenues through which students develop it. Also, communication, the establishment of connections and the representations chosen by students also support their reasoning and, on the other hand, reasoning should be used in decision making associated with these processes. Polya (1981) associates mathematical activity with two types of reasoning: plausible, intuitive and creative activity, through which conjectures are generated; and deductive activity, through which mathematical knowledge is validated by using argumentative chains. School mathematics involves these two types of reasoning, starting with the identification and organization of significant facts in patterns, their use to formulate conjectures, followed by the test of these conjectures, culminating in an attempt to understand and argue why these premises work. In this sense, the identification of patterns is considered an essential component of the reasoning-proof activity that has become increasingly visible and prevalent in the mathematics curriculum, starting from the most elementary levels, across all subjects of the curriculum (e.g. ME-DGIDC, 2007; NCTM, 2000).

Algebra is often considered an essential bridge to access higher order mathematics. However, the difficulties highlighted by many students in this area are evident. Kaput (2008) and Mason (2008) point the abrupt transition from arithmetic to algebra as one of the reasons for this failure. It is therefore essential to reflect on how this transition takes place, and how we can contribute for the development of functional reasoning in students. Pattern exploration allows the formulation and justification of generalizations and the use of these relations to make predictions, facilitating, in a more natural way, the transition to traditional Algebra, by establishing relations of functional type (Lannin, 2005; Zazkis & Liljedahl, 2002). There are also advantages in the use of visual skills in solving problems in Algebra. Generalizations based on the study of visual patterns allow students to contact with the dynamic component of the conceptual construction of mathematical objects and concepts (Rivera, 2007) and more easily assign meaning to symbols and expressions.

Teachers need to know mathematics in order to teach it well. In this sense, the learning of mathematics is strongly dependent on the teacher. Pre-service teachers need to learn both mathematics and also how to teach it. It is necessary, thus, to offer them diverse experiences to extend their scientific and didactic knowledge. In this context, we considered pertinent to understand how pre-service teachers, from basic education (3 to 12-year-old children), solve problems involving the generalization of patterns in visual contexts, by defining the following research questions: (1) How can we characterize the generalization strategies used?; (2) What difficulties do they present?; (3) What is the role of visualization in their reasoning?; (4) What factors influence their reasoning in formulating generalizations?

2. From patterns to generalization

Generalization plays a crucial role in the activity of any mathematician, being considered an inherent ability to mathematical thinking in general. Focusing on the educational context, it can also be said that is a key goal in the learning of mathematics. It is a means of communication, a tool of thought, which is the basis for the development of mathematical knowledge and the centre of activity in this area. The search for patterns has been associated with generalization, considering that it could naturally lead to the expression of generality (e.g. Mason, Johnston-Wilder & Graham, 2005; Orton & Orton, 1999). These types of tasks can be a powerful vehicle to understand relationships between quantities that underlie mathematical functions, thus contributing to the establishment of relations of functional type (Blanton & Kaput, 2005; Warren, 2008). On the other hand, they constitute a concrete and transparent way for students to begin to deal with the notions of generalization and abstraction, since the elementary years. It is also expected that, through this approach, students are able to more easily assign meaning to the language and symbolism used in algebra and the corresponding representational systems, such as graphs and tables.

Formalizing, the generalization of a pattern is based on the identification of a local regularity which is later extended to all terms of the sequence, serving as warranty for the construction of expressions of elements of the sequence that remain beyond the perceptual field (Radford, 2006). Stressing the importance of looking for patterns, to Kaput (1999) generalize means continuing the line of reasoning beyond the case or cases considered, identifying explicitly the regularity between cases, or raising the reasoning to a level where the focus is no longer on those cases or on the initial situation, but on the patterns, procedures, structures and relations between them. Lannin et al. (2011) also point out that to generalize is to identify common aspects to the cases studied, or extend the reasoning beyond the set in which it was originated. Although in these perspectives the authors emphasize, as main goal, the discovery of a general rule, other authors (e.g. Davydov, 1990; Mason, 1996) stress the importance of a cyclical movement between the particular and the general during the generalization process, stating that involves, on one hand, the identification of generality in a particular case, but also the identification of particular cases in the general rule.

Traditionally, the bridge between arithmetic and algebra is achieved through growth patterns. All types of patterns are necessary for the development of mathematical reasoning, but growing patterns lead, more naturally, to the discovery of a relationship between two variable quantities, thus facilitating functional reasoning (Lee & Freiman, 2006; Rivera & Becker, 2008). When exploring this type of patterns, it is requested that students find a relationship between elements of the pattern and its position and that they use this generalization to generate elements in other positions. They are thus motivated to think about growth patterns as functions instead of focusing only on the variation of the variables.

3. The role of visual patterns in discovering functional relations

The importance given to visualization in the learning of mathematics is based on the fact that it is not confined to the mere illustration of ideas, but it is also recognized as a component of reasoning (Vale, Pimentel, Cabrita, Barbosa & Fonseca, 2012).

Although it is not an easy task, it is suggested that teachers integrate visual approaches in the mathematical experiences provided to students (NCTM, 2000). There are two major challenges in this situation: most students associate mathematics with the manipulation of numbers, numeric expressions and algorithms, which can contribute to the devaluation of visualization; on the other hand, teachers should take into consideration that there are many ways of seeing (Duval, 1998). Visual features can be grasped in two ways: perceptually and discursively. The perceptual apprehension of figures occurs when these are seen as a whole, as a single object. Discursive apprehension implies the identification of the spatial arrangement of the elements that make up the figure, either individually or in relation to each other, as a configuration of objects that are related through an invariant attribute or property.

Tasks that involve the study of patterns can be proposed in various visual and non-visual contexts, and give rise to different approaches. However, the literature indicates that the use of a visual aid in presenting problems involving the search for patterns can lead to the application of different approaches to achieve generalization, either of visual or non-visual nature (e.g. Barbosa, Vale & Palhares, 2012; Stacey, 1989; Swafford & Langrall, 2000). Also, visual patterns may contribute to generate different rules that enhance: connections between arithmetic and geometric relationships; assigning meaning to the formulated rules; the need to formulate and validate conjectures. Thus, working with functional relationships through visual growth patterns can raise the attribution of meaning to the operations that transform the independent variable on the dependent variable. Usually there are different ways of expressing the relationship between two variables in such tasks, which makes them privileged contexts to discuss multiple strategies and generalization rules, as well as to exploit equivalent expressions, which contributes to a more flexible reasoning (Barbosa, 2011). In this sense, visual patterns may be a facilitating context to functional reasoning, promoting different ways of seeing, of generalizing and, therefore, promoting better justifications (Lannin, Barker & Townsend, 2006; Becker & Rivera, 2005).

In the context of visual patterns, students that are able to analyse figures discursively, can do it in different ways. On one hand, they can identify disjoint sets of elements that are combined to build the initial figure by using a constructive generalization (Rivera & Becker, 2008). On the other hand, they can observe the existence of overlapping subsets, by counting certain elements, which are subsequently subtracted, more than once, which means that the generalization is formulated in a deconstructive way (Rivera & Becker, 2008). Several studies have concluded that students tend to use more frequently constructive generalizations than deconstructive ones (e.g. Barbosa, 2011; Rivera & Becker, 2008), since the latter category involves a higher cognitive level regarding visualization.

3.1. Generalization strategies used with visual patterns

The generalization of a pattern involves the use of a strategy, a mode of action, however there is a great diversity of approaches that enable students to generalize. Several studies have been carried out in order to understand and categorize the strategies evidenced by students of different levels, as they solve pattern problems in different contexts. The analysis of the categories proposed by some researchers (e.g. Lannin et al., 2006; Orton & Orton, 1999; Rivera & Becker, 2008; Stacey, 1989) led to the construction of the categorization presented in Table 1 (Barbosa, 2010; Barbosa, Vale & Palhares, 2012).

Table 1
Generalization strategies applied to visual patterns

Strategy	Nature	Description
Counting (C)	Visual	Drawing a figure and counting its elements.
Whole-object (no adjustment) (WO_1)	Non-visual	Considering a term of the sequence as unit and using multiples of that unit.
Whole-object w/ visual adjustment (WO_2)	Visual	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on the context of the problem.
Whole-object w/ numeric adjustment (WO_3)	Non-visual	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on numeric properties.
Recursive (R_1)	Non-visual	Extending the sequence using the common difference, building on previous terms (numeric relations).
Recursive (R_2)	Visual	Extending the sequence using the common difference, building on previous terms (features of the figures).
Difference rate (no adjustment) (D_1)	Non-visual	Using the common difference as a multiplying factor without proceeding to a final adjustment.
Difference rate w/ adjustment (D_2)	Visual	Using the common difference as a multiplying factor and proceeding to an adjustment of the result.
Explicit (E_1)	Non-visual	Discovering a numerical rule that allows the immediate calculation of any output value given the correspondent input value.
Explicit (E_2)	Visual	Discovering a rule, based on the context of the problem, that allows the immediate calculation of any output value given the correspondent input value.
Guess and Check (GC)	Non-visual	Guessing a rule by trying multiple input values to check its' validity.

In this categorization several generalization strategies are identified, based on the exploration of visual patterns. Considering each case, we can distinguish differences in their nature, being either visual or non-visual strategies. In some cases the figures play an essential role in the discovery of the invariant (C, WO_2, R_2, D_2, E_2) and, in others, the work is developed in a numeric context (WO_1, WO_3, R_1, E_1, GC). It can be highlighted that different strategies may be used in solving the same problem but, depending on the characteristics of the situations presented, it is essential that students understand the strengths and limitations of each strategy, becoming more flexible in their reasoning.

4. Factors that may influence pattern generalization

There are some factors that may have a significant impact on the choice of strategies used to generalize patterns, regardless of their adequacy. The identification of obstacles to the generalization process, and the reasons that may underlie them, is essential so that the teacher can promote the development of the students' ability to generalize.

Lannin et al. (2006) identified a number of factors that can influence the use of generalization strategies, organizing them into three categories: (1) social factors resulting from interactions of the students with their peers and with the teacher, since questioning, feedback and arguments concerning the use of a particular strategy could have implications on students' thinking; (2) cognitive factors associated with mental structures that the student has developed; and (3) factors associated with the structure of the task, like the structure of the pattern (e.g. linear or nonlinear), the values assigned to the independent variable (near or far generalization, multiples of known values) or even the ability to *see*. In general, Lannin et al. (2006) concluded that, when the starting values are close (near generalization), students tend to use recursive rules, regardless of the type of pattern and the visual component of the task, although also referring that the visual analysis of the situation often leads to a different perspective on the recursive relationship, promoting the association between the proposed rule and characteristics of the context. On the other hand, students who base their reasoning only on numeric values usually have little idea about the relationship between the rule found and the context of the problem. When the starting values are multiples of known terms of the sequence, students tend to apply the whole-object strategy, with no adjustment. They also indicate that students with difficulties in seeing the pattern apply incorrectly the whole-object strategy, while those who show greater visual capabilities recognize the need to adjust the strategy, if the context does not reflect a model of direct proportionality. The use of distant values as starting point may encourage the application of the explicit strategy, although in different perspectives.

The improper application of direct proportion, particularly in the exploration of linear patterns, has been mentioned in several studies (e.g. Becker & Rivera, 2005; Lannin et al, 2006; Sasman, Olivier & Linchevski, 1999; Stacey, 1989). A thorough analysis of this phenomenon points to two situations that can explain such reasoning. On one hand the use of a strictly numerical reasoning, implying the meaningless manipulation of variables. Another factor relates to the proposed generalization for "appealing numbers" (Sasman et al., 1999, p. 5), from a multiplicative point of view. In this sense, Sasman et al. (1999) stressed the importance of the tasks contemplating also non-appealing numbers as a way to circumvent the tendency to use direct proportionality, suggesting, for example, the use of prime numbers.

The focus on numerical aspects of the pattern, even when it is presented in a visual context, is often an obstacle to generalization (Noss, Healy & Hoyles, 1997). Mason (1996) notes that there is a tendency to construct tables of values from which derives a general rule, not always correct, based on the analysis of one or two particular cases. This author suggests that opportunities should be given to students to explore different types of patterns, in which students can apply visualization and can manipulate figures, to facilitate generalization.

In the context of visual patterns, even when students are able to grasp figures discursively, it is necessary to take into account their complexity, a factor that can condition the establishment of generalization. Sasman et al. (1999) distinguish between transparent and non-transparent figures. In the first case, the rule which underlies the pattern is featured in a clear manner in the structure of the figures, situation that does not occur with non-transparent figures in which the rule is not easily discovered by simply observing the figures in the sequence. In these cases, it is appropriate to think of strategies that can help students identify the pattern visually and, consequently, generalize. Rivera (2007) suggests that students are encouraged to manipulate and transform the figures into simpler forms, easier to recognize, which reflects a cognitive change regarding the apprehension of the figures (Duval, 1998). Another strategy suggested by Rivera (2007) involves a symmetric counting process. Students should be able to identify symmetry in the figures presented, and subsequently focus on only one part of the figure analysed to count, applying the same action to the parts of the figure that show the same characteristics.

Some authors present suggestions that can help students overcome or minimize these difficulties. Noss et al. (1997) identified that the establishment of a connection of visual nature between the context of the problem and the corresponding symbolic representation is a determining factor in assigning meaning to rules of explicit type. Similarly, for Swafford and Langrall (2000) and Zazkis and Liljedahl (2002), asking students to analyse different values for the independent variable, testing increasing numbers, may promote the use of explicit reasoning. Stacey and MacGregor (1995) also stressed the importance of using tasks that reduce the emphasis on recursive relations, trying to get students to identify the connection between the independent and dependent variables, with the purpose of contacting with explicit relations. In summary, it’s pertinent that teachers reflect about the structure and the implementation of tasks to better promote the development of functional reasoning in students.

5. Method

Given the nature of the proposed problem and the research questions, this study follows a qualitative approach, in the form of an exploratory design (Yin, 2012). We developed a didactical experience with 80 future basic education teachers (3-12 years old students) that took place during the classes of the subject Didactics of Mathematics. The participants were on the final year of the undergraduate course. Throughout 9 hours of the mentioned unit course students solved seven tasks, focused on visual patterns, and, for each task, the following phases were accomplished: task introduction; individual resolution; and whole group discussion. The tasks had some features in common, such as: being proposed in a visual context, asking for near and far generalization, and promoting reverse thinking. The diversity was on the type of pattern (linear and non-linear) and on the nature of the figures (transparent and non-transparent).

Data was collected in a holistic, descriptive and interpretive way, including mainly classroom observations, methodological notes and written productions of the students for the tasks. To analyse data resulting from all these sources, we focused on the identification of categories that were determined by the integration of research questions and the theoretical framework. The emergence of these categories was oriented by the following themes: generalization strategies, difficulties in generalizing, role of visualization in students' reasoning, and factors that influence the formulation of generalizations. However, the interactivity between the phases of data collection and data analysis led to the consideration of new categories and the refinement of others.

6. Discussion of some results

We present some results, related to the research questions, focusing on generalization strategies and on difficulties presented by the students, trying also to understand the role of visualization and the influence of some factors in their reasoning. Rather than presenting the results task by task, we chose to reflect on certain aspects that emerged throughout the study, highlighting three different tasks: *Squares in crosses* (Annex A), that includes a linear pattern and transparent figures; *Figures with squares* (Annex B), that includes a non linear pattern and transparent figures; and *Intertwined rectangles* (Annex C), that includes a non linear pattern and non-transparent figures.

Throughout the tasks, students used a variety of generalization strategies, either of visual or non-visual nature. This fact was related to the context of the problems, since they could choose to work with figures or with numbers. The frequency of use of each strategy depended on different aspects that will be further identified and detailed in the following sections.

6.1. Near and Far generalization

When confronted with near generalization students, used one of these strategies: counting, recursive (non-visual), recursive (visual), explicit (non-visual) and explicit (visual). Analysing each one of these cases, we concluded that counting was applied by a reduced number of students that resorted to a drawing of the term of the sequence asked, in order to determine the number of elements (Figure 1).

Figure 1. Task “Figures with squares” – Question 2

Recursive reasoning was also present in the work presented by these students, mainly the non-visual type, like we illustrate in Figure 2. In these cases, figures were converted into numbers, and students discovered the variation from one term to the next. However, some of them recognized the structural growth of the patterns, based on the observation of the figures in the sequence: “From figure to figure we add four squares in relation to the previous figure. In each extremity we add a square” (Task “Squares in crosses”, question 1).

Figure 2. Task “Squares in crosses” – Question 1

While confronted with near generalization, some students were able to deduce immediately a rule relating the two variables, using an explicit reasoning. But, we also found differences in these cases. Some students deduced the rule from features found on the figures, discovering the structure of the pattern this way: “We have a square in the middle and in each arm of the X we have the same numbers of squares as the order of the figure, so it’s 1+4x199” (Task “Squares in crosses”, question 2). Others deduced the rule based on the study of numerical relations on a table of values (Figure 3).

Figure 3. Task “Squares in crosses” – Question 2

Concerning far generalization, the following strategies were applied: difference rate with adjustment, guess and check, whole-object (no adjustment), explicit (non-visual) and explicit (visual). The two later cases have already been approached above, with the difference of now being the most used strategies. A few students, who had already identified the common difference between consecutive terms using visualization, managed to perceive that they could use that as a multiplying factor, adjusting the result based on the context of the problem: “We take figure 1 with 5 squares, thus figure 2 will have the number of squares as figure 1 plus 4 squares from the extremities. So we have 5+4x(n-1)” (Task “Squares in crosses”, question 4). The guess and check strategy was mainly use to address questions requiring reverse thinking, situation that we will analyse in the next section. The whole-object strategy with no adjustment was used in rare cases, reflecting that the majority of the student perceived that it was not adequate to solve the presented tasks, as they didn't involve direct proportion. One of the tasks where it emerged was "Intertwined rectangles", particularly to solve the second question, since it involved appealing numbers: "If 5 unit rectangles give place to 15 rectangles of any size, then 10 unit rectangles give place to 30 rectangles of any size".

6.2. Reverse thinking

Questions involving reverse thinking proved to be complex for some of these students. To overcome some of the difficulties in finding a rule, most of them recurred to guess and check (Figure 4), experimenting certain values until they reached the correct one.

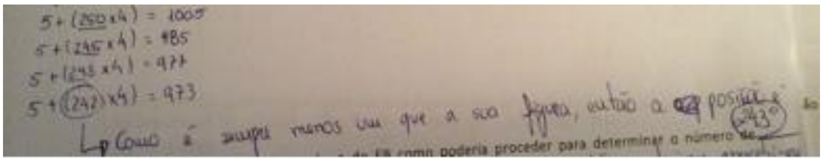

Figure 4. Task "Squares in crosses" – Question 3

In some cases the option was for the explicit strategy, either using a non-visual approach or a visual one. The first situation was illustrated by solving an equation, finding directly the order of the figure with a certain number of elements. The later situation also implied the use of reverse operations, but based on the visual structure of the pattern: "If we have 973 squares we subtract the one in the middle. Then we divide 972 by 4 because of the four arms in the X. Hence it will be the 243º X" (Task "Squares in crosses", question 3).

6.3. Transparent and Non-transparent figures

As expected, students were more successful in dealing with transparent than non-transparent figures, having sometimes trouble in deducing a rule in these problematic situations. Analysing the tasks with non-transparent figures they used strategies like counting, recursive (non-visual), explicit (non-visual) and explicit (visual), privileging the first three approaches. Concerning counting, when the order was near it may habe been considered an efficient strategy, but, as the order got far it was very difficult to apply it in a successful way, resulting in confusing diagrams (Figure 5).

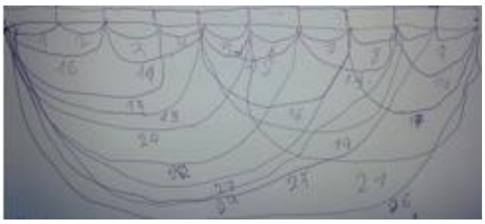

Figure 5. Task "Intertwined rectangles" – Question 2

Recursive reasoning was also an alternative to solving these tasks, but efficient only for near generalization (Figure 6). Transforming the figures into numbers made it easier for students to find relations between terms and even attempt to find a rule by manipulating those numbers, thinking about the relation between variables.

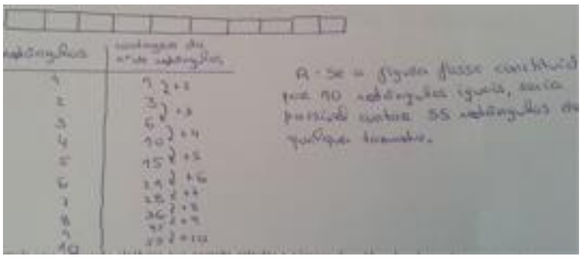

Figure 6. Task “Intertwined rectangles” – Question 2

It is relevant to state that, when working with non-transparent figures, most students were not able to reach far generalization and find a rule for the pattern, presenting difficulties related to the complexity of visual transformations or the complexity of numerical relations.

6.4. Interpreting the meaning of an expression

Some of the tasks included a question where students had to interpret the meaning of an algebraic expression in the context of the problem. Only a few students were able to do it, by giving significance to the numbers and variables: “I saw 4 sets. In each set there is one more element than the number of the figure, however, as only one of the sets has always one more element than the number of the figure, we have to subtract the 3 unit added” (Task “Squares in crosses”, question 5). In alternative, the majority of the students verified if the given expression was equivalent to the one they had found. Others used the guess and check strategy, testing the expression for a few particular cases, making generalizations based on this reasoning.

7. Concluding remarks

As shown by the results presented, tasks that involve the exploration of patterns in visual contexts promote the emergence of multiple generalization strategies, enhancing the development of a more flexible reasoning (e.g. Lannin et al., 2006; Rivera & Becker, 2008; Sasman et al, 1999). Although assigning a greater emphasis on visual representations, these tasks allow the implementation of strategies of different nature, either placing the focus on figures or transforming the given information into numbers. However, in addition to expecting that students are able to apply and adapt different strategies in the process of generalization, it is equally important to understand the advantages and limitations of each approach.

For example, the counting and recursive strategies are useful when we want to discover near terms in a sequence, but it is proven that they are difficult to apply in far generalization, as some students revealed. Counting led almost always to correct answers, but there were situations in which this strategy was not applied properly. Trying to solve far generalization questions through counting is an exhaustive process and can result in disorganized and complex representations, being an obstacle to perceiving the structure of the pattern. It was also noticeable that when counting was made based on the perceptual apprehension of the figures (Duval, 1998), it did not contribute to finding the rule, but the discursive apprehension (Duval, 1998) allowed students to identify the invariant, facilitating the development of functional reasoning. The explicit strategy was recognized as a process that allows for more expeditious generalizations, being particularly valued by students in far generalization. It was noted that in questions that had the use of reverse thinking underlying, guess and check was an efficient alternative to the explicit strategy, since most of the students showed difficulties with this type of questions. Whole-object was used in very few cases, as it was inadequate for the proposed problems. This situation relates to working strictly in purely numerical contexts that prevents students from understanding the misuse of proportional reasoning (e.g. Becker & Rivera, 2005; Lannin et al, 2006; Sasman et al, 1999; Stacey, 1989), since none of the presented patterns suited to the direct proportion model.

Depending on how students see a particular pattern, the visual approaches used can generate different rules/expressions to represent it (e.g. Rivera & Becker, 2008; Vale et al, 2012). This enables the teacher to exploit the notion of equivalence, a crucial concept in algebraic thinking, while also preventing students to conclude that everyone should converge to the same solution. By analysing how students viewed the presented patterns and the nature of generalization established, it was found that they mainly formulated constructive generalizations, being more evident for them the identification of disjoint subsets in figures (Rivera & Becker, 2008), because, in terms of visualization, this type of reasoning is more accessible than a deconstructive type. These evidences show us that it is important to develop the ability to visualize in different contexts, discussing with the students different possibilities, so they may acquire tools to solve future problems.

Reflecting on possible factors that may have influenced students' reasoning concerning generalization, we highlight aspects mainly related to the structure of the tasks: (1) All tasks included questions centred on near and far generalization. The magnitude of the values assigned to the variables influenced the type of strategies applied, and, in general, students used different strategies in addressing these situations (Lannin et al, 2006; Stacey, 1989); (2) In some tasks we presented questions promoting reverse thinking, asking the order that a given term occupied in a sequence. In these cases, many students showed difficulties and were not always able to use the inverse operations, applying alternative strategies such as guess and check; (3) Figures representing the patterns may be transparent or non-transparent, if the structure of the pattern is readily identifiable in the figure or not. The majority of students showed difficulties deducing a rule for non-transparent figures, presenting a tendency for numerical approaches and for recursive strategies. They were rarely successful in the attempt to identify a rule directly from the figures, since the apprehension was compromised (Duval, 1998; Rivera, 2007). (4) The structure of the pattern (linear or non-linear) is also a factor to consider in the choice of generalization strategies, since, for example, the recursive relationship in non-linear patterns is not as obvious as in linear patterns, implying that students seek to focus on the functional relationship.

When formulating/selecting tasks related to visual patterns, teachers should take into consideration a wide variety of factors that may influence the development of functional reasoning of students, including all these aspects in the planning of the work in the classroom. It is important to select proposals that enable the implementation of different strategies and promote a dynamic perspective of the possible approaches, so that students can understand and establish parallels between visual and non-visual strategies. The discussion about the potential and limitations of each approach can be an important contribution to develop a more flexible reasoning, fluency in communication and increase their repertoire of representations.

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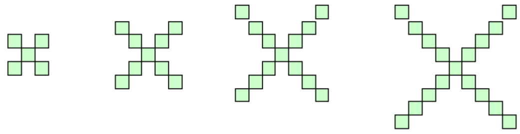
Annex A
Squares in crosses

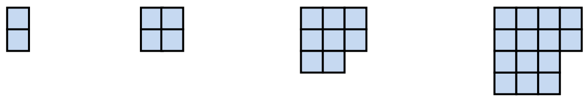
Consider the first four figures of a given sequence:



1. How many squares will the 8th figure have?
2. How many squares will the 199th figure have?
3. Is it possible to find a figure in this sequence with 973 squares? If it is possible, determine the position of that figure in the sequence.
4. Formulate a general expression for this sequence.
5. John presented the following algebraic expression to determine the number of squares of figure *n*: 4(n+1)-3
Is the expression correct?

Annex B
Squares in crosses

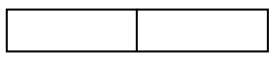
Consider the first four figures of a given sequence:



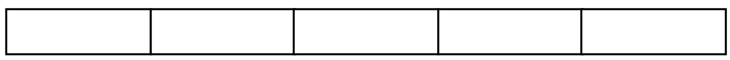
1. Draw the next figure.
2. How many squares will the 7th figure have?
3. And the 53th figure?
4. Is it possible to find a figure in this sequence with 8558 squares? If it is possible, determine the position of that figure in the sequence.
5. Formulate a general expression for this sequence.
6. Daniel presented the following algebraic expression to determine the number of squares of figure *n*: $(n-1)^2+(n-1)+2$
Is the expression correct?

Annex C
Intertwined rectangles

In this figure you can count three rectangles:



Now consider a figure constituted by five unit rectangles:



1. How many rectangles, of any size, can you count?
2. What if the figure had 10 unit rectangles? How many rectangles of any size could you count?
3. Determine an algebraic expression that represents the number of rectangles of any size in a figure with *n* unit rectangles.