DESCRIPTION OF A PROCEDURE TO IDENTIFY STRATEGIES: THE CASE OF THE TILES PROBLEM[1]

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In this paper we present a procedure to describe strategies in problems which can be solved using inductive reasoning. This procedure is based on some aspects of the analysis of the specific subject matter, concretely on the elements, the representation systems and the transformations involved. We show an example of how we used this procedure for the tiles problem. Finally we present some results and conclusions.

The researchers related to inductive reasoning process are usually developed in problem solving context (Cañadas, Deulofeu, Figueiras, Reid, & Yevdokimov, 2007; Christou & Papageorgiou, 2007; Küchemann & Hoyles, 2005; Stacey, 1989). These investigation pay attention to the cognitive process as well as to the general strategies that students used to solve the problems proposed.

In this paper, we present part of a wider investigation (Cañadas, 2007), which is focused on the inductive reasoning process and on the specific strategies developed by students to solve problems which involved a specific mathematical subject matter. One of the methodological contributions of this research consists on a procedure to describe strategies in problem solving. We use this procedure to identify and to describe strategies of students in problems that involved linear and quadratic sequences.

This paper consists of four main parts. First, we present some theoretical and methodological aspects of our research, which are important to introduce a procedure to identify and to describe inductive strategies, which conforms the second part. Third, we show the application of such procedure for the tiles problems. Finally, we present some results and conclusions related to this problem.

THEORETICAL FRAMEWORK

Inductive reasoning

Inductive reasoning is a process that produces scientific knowledge through the discovery of general rules starting from the observation of particular cases (Neubert & Binko, 1992). Following this idea, we took as starting point the Polya´s proposal about induction (1967)[2]. We consider working on particular cases and generalization

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[2] Pólya talks about *induction* in the same sense as we refer to *inductive reasoning*. This conception is different from mathematical induction or complete induction, which refers to a formal method of proof, based more on deductive reasoning.

as two states in the process of inductive reasoning (Cañadas, 2007). One of our research objectives was to produce a systematic procedure for exploring the inductive reasoning of students in the context of problem solving.

Inductive strategies in problem solving

Problem solving is considered a highly formative activity in mathematics education. It promotes different kinds of reasoning (Rico, Castro, Castro, Coriat y Segovia, 1997), specifically inductive reasoning. Induction is a heuristic and its aim is to provide regularity and coherence to data obtained through observation (Pólya, 1967).

Strategies are the "ways of performing on mathematical tasks, which are executed in concepts and relationships representations"[3] (Rico, 1997, p. 31). We use the expression *inductive strategies* to refer to the strategies used in problems which can be solved through inductive reasoning as heuristic.

Representation systems play an important role in problem solving because they allow expressing the reasoning performed. In our research, we focused on external representation used by students in problem solving. We analyzed the way that students performed to solve written problems through the external representations.

Mathematic subject matter

Given that we choose linear and quadratic sequences as the specific subject matter, we needed to describe it to select adequate problems to propose to the students and to obtain criteria to describe students´ work on those problems. We based this subject matter description on some ideas of the subject matter analysis (Gómez, 2007). Through some aspects of this analysis, we obtained useful information about linear and quadratic sequences to elaborate a procedure to describe inductive strategies. Particularly, we focused on the elements of the sequences, the representation systems and the transformations.

The elements of sequences are the particular and general terms, and the limit. Since our interest was inductive reasoning[4], we selected particular and general terms to work on.

Since sequences are a particular kind of functions, we took into account four representation systems for functions, following Janvier (1987): Graphic, numeric, verbal and algebraic. On the one hand, particular terms can be expressed numerically, graphically or verbally. On the other hand, general terms can be expressed algebraically or verbally.

We considered three sorts of transformations:

- Transformations among different representations of the same element: *synonymous transformations* (Janvier, Girardon, & Morand, 1993).

[3] My personal translation.
[4] We consider that inductive reasoning is the process that begins with particular cases and produces a generalization from these cases. Pólya (1967) adds the idea of validation based on new particular cases to this kind of reasoning.

- Transformations among the same element inside the same sort of representation systems: *syntactic transformations* (Kaput, 1992).
- Transformations among different elements expressed in different representation systems.

PROCEDURE TO IDENTIFY INDUCTIVE STRATEGIES

We elaborated a procedure to identify strategies based on representation systems. Each strategy is constituted by a sequence of transformations. To identify a strategy in a specific problem response, we start from particular terms expressed in the statement of the problem and we detect the kinds of transformations performed.

In Tables 1, 2 and 3, we collect how we refer to all the possible kinds of transformations. Tables 1 and 2 contain transformations from the term in the first column to the term in the second column. For example, in Table 1, T6 refers to a transformation from a particular term represented graphically to a verbal representation of such term.

Element		Particular Term		
		Numeric	Graphic	Verbal
Particular Term	Numeric	TSN	T3	T5
	Graphic	T1	TSG	T6
	Verbal	T2	T4	TSV

Table 1: Transformations involving particular terms.

Element		General Term	
		Algebraic	Verbal
General Term	Algebraic	TSA	T8
	Verbal	T7	TSV

Table 2: Transformations involving general term.

In Table 3, C refers to a transformation from particular term to general term and CB to a transformation in the inverse sense.

Element		General Term			
		Algebraic		Verbal	
Particular Term	Numeric	C1	C1B	C4	C4B
	Graphic	C2	C2B	C5	C5B
	Verbal	C3	C3B	C6	C6B

Table 3: Transformations involving general and particular terms.

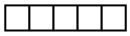
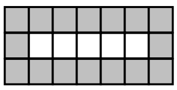
METHODOLOGICAL FRAMEWORK

We asked 359 Spanish students to work on a written questionnaire. Students belonged to years 9 and 10 in four different schools.

The questionnaire had six problems which involved linear and quadratic sequences that could be solved using inductive reasoning as a heuristic. Given that our interest was inductive reasoning, we considered problems that contained information about particular cases. One of these problems was the "tiles problem".

The tiles problem[5]

In the following lines, we present the tiles problem as it was presented in the questionnaire:

> Imagine some white squares tiles and some grey square tiles. They are all the same size. We make a row of white tiles:
>
> 
>
> We surround the white tiles by a single layer of grey tiles.
>
> 
>
> - How many grey tiles do you need to surround a row of 1320 white tiles?
>
> - Justify your answer.

DATA ANALYSIS

One example

In what folows, we will use the described procedure to identify the inductive strategy observed in one student´s response to the tiles problem. Figure 3 shows the student´s solution

1320
x 2
2640
2640 + 6 = 2646 tiles
We need the double number of white tiles, plus 3 tiles at each of both ends.

Figure 3: One solution to the tiles problem.

We observe that, first; s/he makes a transformation from graphic system of the particular term of the statement to numeric system (T1, see Table 1). After that, the student makes a transformation in this representation system (TSN, see Table 1).

[5] We present the English version of the problem we posed in our research.

Finally, s/he gets the generalization verbally (C4, see Table 3). So, s/he used the following inductive strategy: T1-TSN-C4.

Inductive strategies in the tiles problem

We applied the described procedure to identify inductive strategies for each student's response to the tiles problem. Table 4 shows such strategies, the number of students who used each of them, the elements involved, and whether they produced the generalization or not.

Inductive Strategies	Freq	Elements	Generaliz	Partial Freq
No transformations	52			52
T1	10	Particular terms	No	247
T1-T5	6			
T1-TSN	151			
T1-TSN-T5	54			
TSG-T1	2			
TSG-T1-TSN	14			
TSG-T1-TSN-T5	3			
TSG-T6	1			
T6	2			
T6-T2-TSN	4			
T1-TSN-C1-TSA	1	Particular and general terms	Yes	60
T1-C4	9			
T1-TSN-C4	36			
TSG-C1-C1B-T5	1			
TSG-T1-C4	1			
TSG-T1-TSN-C4	7			
TSG-C4-C4B-TSN	2			
T6-C3-C3B-TSN	1			
C5-C4B-TSN	2			
Total				359

Table 4. Inductive strategies in the tiles problem.

RESULTS

We identified 19 different inductive strategies in this problem, whether they generalized or not. There were 247 students that remained working on particular terms (C does not appear in the sequence of inductive strategies). On the other hand, there were 60 students who obtained the expression of the general term.

T1-TSN, T1-TSN-T5 y T1-TSN-C4 were the strategies used by most students. Observing the information in Tables 1, 2 and 3, we deduce that these students performed a transformation from the graphical representation to the numeric system (T1) and, after that, a syntactic transformation in the numerical representation (TSN).

Through the different strategies identified, we observe that students used the four possible representation systems: numeric, verbal, graphic and algebraic. Although the tendency to use the numeric representation is clear, there were 31 students who started their responses in the graphic representation (TSG). The verbal representation usually appeared at the end of the response (T5, T6, C4 or C5 at the end of the inductive strategies).

We now describe strategies of students who did not generalize and strategies of students who did, separately.

Students who did not generalize

Six of the students who answered to the problem started working on the verbal representation, as shows the transformation T6 in their strategies (T6 and T6-T2-TSN). There were 20 students who started with the graphical representation (TSG as the first term of the sequence that represent the strategy: TSG-T1, TSG-T1-TSN, TSG-T1-TSN-T5 y TSG-T6.

In general, the numeric system was the most frequent representation used by students who did not achieve the generalization (247 students).

The verbal representation was performed by 70 students, as we deduce from the frequencies of strategies that include T5 and T6. 63 of these students used this kind of representation at the end of their response, when they tried to justify their answers using particular terms.

Students who generalize

Of the 60 students who achieved the generalization, just two generalized directly from the statement (as reveals strategy C5-C4B-TSN). These students reached the generalization without any previous transformation among particular terms.

There were 55 students that generalized and had previously worked on particular term in the numeric representation (T1 precedes C1 or C4) and three students worked on particular term in the graphic representation before generalizing (TSG-C1-C1B-T5, TSG-C4-C4B-TSN). Eight of the students that generalized, combined graphical and numeric representation before expressing the general term for the sequence.

The generalization was expressed algebraically by three students. The respective strategies are T1-TSN-C1-TSA, TSG-C1-C1B-T5 and T6-C3-C3B. The remaining 57 students that get the generalization used the verbal representation to express it.

As part of the strategies of students who generalized, we paid attention to how they used the generalization. On the one hand, two of the three students who expressed the general terms algebraically, used the generalization to calculate the particular term required by the problem (students who use the strategies TSG-C1-C1B-T5 and T6-C3-C3B). The third one got the generalization in the last transformation, so s/he did not use the general term for the first task proposed in the problem. This student used the general expression as a way to justify her/his response. On the other hand, four of the 57 students that generalized verbally, used such expression to calculate the particular term that the problem asks for (students who used inductive strategies TSG-C4-C4B-TSN and C5-C4B-TSN).

CONCLUSIONS

The procedure presented in this paper allowed us to identify and describe strategies used by students in the tiles problem. The information obtained through the procedure allowed us to get conclusions related to the work on particular cases and the generalization, as part of the inductive reasoning process. Moreover, we got data about the representation systems used related to these states of inductive reasoning. In this paper we have shown some of these results for the tiles problem.

In the tiles problem, we get some conclusions related to the inductive strategies and to the inductive reasoning process. First of all, we highlight that students denote a preference for the numeric system, although the four possible representation systems are employed by different students. Another general conclusion is that most of the students remain working on particular cases.

Students show a tendency to use verbal representation at the end of their responses. This fact reveals us that they use this system in the justification of their responses. The verbal representation is also the most frequent way of expressing generalization. This is surprising if we consider that students used to express the generalization algebraically in their classrooms. The majority of the students that generalize verbally tend to do so when they try to justify their answers, and not as a way to calculate new particular terms of the sequence. Probably it could be interesting for teaching to consider this way of expressing generalization before working on it algebraically.

The generalization, both algebraic and verbal, is occasionally used to calculate the particular term required in the problem.

The procedure to identify inductive strategies can be useful for other mathematical subject matters and maybe for other cognitive processes. In the case of other mathematical subject matters, we could consider an analogous procedure based on specific elements, representation systems and transformation of such subject matter.

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