O trabalho dos estudantes da escola média com variável algébrica: uma comparação entre a Itália e o México

Middle school students' work with algebraic variable: a comparison between Italy and Mexico

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Resumo

O desempenho de alunos que trabalham com variável algébrica tem sido amplamente estudado, com diferentes populações, usando diferentes ferramentas. No entanto, há poucos estudos comparativos envolvendo países diferentes e utilizando a mesma ferramenta. Com o objetivo de detectar diferenças e semelhanças no desempenho com a variável algébrica (incógnita, número genérico, variáveis em relação funcional) aplicamos o mesmo questionário para 405 estudantes italianos e mexicanos da escola média. As respostas dos alunos foram analisadas utilizando a ferramenta teórico-metodológica chamada Modelo 3UV. Identificamos pontos fortes e fracos, alguns semelhantes, outros diferentes. Os resultados mostram que muitas interpretações, erros e dificuldades manifestadas no trabalho com a variável algébrica são evitáveis, uma vez que são o produto da abordagem da álgebra elementar realizada em cada país.

Palavras-chave: *variável algébrica; Modelo 3UV; estudo comparativo.*

Abstract

Students' performances with algebraic variable has been widely studied with different populations using different tools. However, there are few comparative studies involving different countries and using the same tool. To detect differences and similarities in students' performance with algebraic variable (unknown, general number, variables in functional relationship) we administered the same questionnaire to 405 Italian and Mexican middle school students. The theoretical-methodological tool named 3UV Model was used to analyze students' responses. We identified their strengths and weaknesses, some similar, other different. The results show that many misinterpretations, errors and difficulties with algebraic variable are the product of the school approach to elementary algebra used in each country, therefor avoidable.

Keywords: *algebraic variable; 3UV Model; comparative study.*

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Introduction

The development of mathematical skills such as the ability to abstract and generalize, to interpret and mathematically model real-life situations is nowadays an essential part of young students' preparation in order to continue college studies and/or enter productive life. A school mathematics topic that promotes the development of these capabilities is algebra, and a key algebraic concept fundamental for modelling the many relationships that occur in natural and artificial worlds is variable, conceived as a multifaceted concept that include different uses as signaled by Usiskin (1988). Since decades researchers have systematically stressed student's difficulties to interpret the different roles of variable, being these a root of many problems students have with algebra and other branches of mathematics (URSINI; TRIGUEROS (2001, 2010); KNUTH et al. (2005); AKGÜN; ÖZDEMIR (2006); TAHIR; CAVANAGH; MITCHELMORE (2009); PEDERSEN (2015); WARREN; TRIGUEROS; URSINI (2016)). A low proficiency in working with algebraic variable restricts their capability to understand and use mathematical models to describe and predict different situations. Their difficulty to recognize the different uses of variable and to flexibly move between them obscures the richness derived from the relationships between them and limits students' understanding of algebra (KIERAN (2006); MALISANI; SPAGNOLO (2009)).

Researchers agree that the multifaceted character of variable is a source of difficulties for students even with years of schooling (BILLS (2001); URSINI; TRIGUEROS (2006); BIEHLER; KEMPEN (2013)). In almost any problem situation involving variable its multifaceted character is present, therefor the necessity to distinguish one use from the others, shift fluently between them and, in consequence, perform the appropriate actions required in each case. Students' difficulties with algebraic variable have been studied locally, in different nations, using a diversity of tools, and focusing frequently on one specific use of it (for example, unknown, or general number, or variables in functional relationship) and a variety of misinterpretations, errors and difficulties with each use of variable have been detected. However, only few international comparative studies have focused on students' performance with variable using the same tool to detect similarities and differences in their achievements and difficulties with this concept. For example, López; Moreno; Souza (2010) compared Brazilian and Mexican baccalaureate (15-17 years old) students' performance, and Álvarez; Gòmez-Chacòn; Ursini (2015) focused on similarities and differences between Mexican and Spanish 9th year students (14-15 years

old) and 11th year students (16-17 years old) when working with variable. The information produced by these kind of studies may help participating countries better understand their own education systems, as was signaled by Robitaille; Robeck (1996), by establishing their strengths and weaknesses in comparison to those of other countries, providing so elements to revisit their curricula to promote the necessary adjustments to improve the teaching and learning of this fundamental algebraic concept.

Aiming to contribute in this direction, in this study Italian and Mexican 1st year (respectively 11-12 and 12-13 years old) and 3rd year (respectively 13-14 and 14-15 years old) middle school students' performance with algebraic variable are compared. Italian and Mexican mathematics middle school curricula contain similar topics concerning elementary algebra but, these are differently distributed during school cycles and the different uses of variable are not equally stressed. For example, in Mexico students approach the three uses of variable from 1st year of middle school, while in Italy the study of algebra is mainly concentrated in 3rd grade and topics involving unknowns and general numbers are privileged. The goal of this study was twofold: to identify similarities and differences between the studied groups; and to analyze the understanding of the algebraic variable reached by 3rd grade students at the end of the middle school cycle which represents, for most of them, especially in Mexico, the end of their school experience.

Theoretical framework

The uses of algebraic variable that most frequently appear in elementary school algebra are *unknown*, *general number*, *variables in functional relationship*. Strongly concerned with students' learning of this multifaceted concept, Ursini; Trigueros (2001) suggest that the understanding of algebraic variable can be described in terms of students' performance with each one of these three uses. This perspective together with an analysis of the historical development of the concept of variable led them to develop a theoretical-methodological tool, named 3UV Model (summarized in Table 1), that associates to each use of variable a list of aspects that characterize its understanding and capability to work with it in different situations and at different levels of complexity. The 3UV Model has proved to be useful for designing diagnostic tools (TRIGUEROS; URSINI (2003); LÓPEZ, MORENO; SOUZA (2010)), analyzing the use of variable in textbooks and problems of different complexity (BENITEZ (2004); BELTRAME; BIANCHINI (2010)), analyzing students' and teachers' performance with variable at different school

levels (JUAREZ (2011); BOCCA; CANTER; SAYAGO (2012); ESCALANTE VEGA; CUESTA BORGES (2012); POMMER (2015); URSINI, TRIGUEROS; LOZANO (2000)), designing teaching strategies (MONTES (2003); LÓPEZ; LÓPEZ (2011)).

In this study the 3UV Model was employed to identify the uses of variable and the corresponding aspects involved in a series of items, to analyze students' responses, to establish similarities and differences between the studied groups, and to determine their strengths and weaknesses with each use of variable.

Table 1: The 3UV Model

Working with variable as unknown requires the capability to:
U1 - Recognize and identify in a problem situation the presence of something unknown that can be determined by considering the restrictions of the problem;
U2 - Interpret the symbols that appear in an equation, as representing specific values that can be determined by considering the given restrictions;
U3 - Substitute to the variable the value or values that make the equation a true statement;
U4 - Determine the unknown quantity that appears in equations or problems by performing the required algebraic and/or arithmetic operations;
U5 - Symbolize the unknown quantities identified in a specific situation and use them to pose equations.
Working with variable as general number requires the capability to:
G1 - Recognize patterns; perceive rules and methods in numeric sequences and in families of problems;
G2 - Interpret a symbol as representing a general, indeterminate entity that can assume any value;
G3 - Deduce general rules and general methods by distinguishing the invariant aspects from the variable ones in sequences and families of problems;
G4 - Manipulate (simplify, develop) general expressions;
G5 - Symbolize general statements, rules or methods.
Working with variables in functional relationship requires the capability to:
F1 - Recognize the correspondence between related variables independently of the representation used (tables, graphs, verbal problems or analytic expressions);
F2 - Determine the values of the dependent variable given the value of the independent one;
F3 - Determine the values of the independent variable given the value of the dependent one;
F4 - Recognize the joint variation of the variables involved in a relation independently of the representation used (tables, graphs, analytic expressions);
F5 - Determine the range of variation of one variable given the domain of the other one;
F6 - Symbolize a functional relation based on the analysis of the data of a problem.

Source: Drafted by the author.

Method

Data were collected in a northeastern city of Italy, Trieste, and in the capital city of Mexico, Ciudad de Mexico. Trieste is part of a region that performed in PISA 2012 mathematics' assessments higher than the OECD mean, Ciudad de Mexico performed lower than the OECD mean but, in both cities, mathematics middle school students' performance is considered good when compared with the average level of the respective country.

In the study participated 96 Italian and 99 Mexican 1st grade (respectively 11-12 and 12-13 years old) and 118 Italian and 92 Mexican 3rd grade (respectively 13-14 and 14-15 years old) middle school students. To test their performance with algebraic variable, the Trigueros; Ursini (2003) 65 items questionnaire designed to test students' understanding of the different uses of variable was adapted to 47 open-ended items, each item focusing on a specific use of variable: 8 items[2] test aspects related to variable as *unknown*; 16 items[3] test aspects concerning *general number*; 23 items[4] test aspects characterizing *variables in functional relationship*. Most items include more than one of the aspects that, following the 3UV Model, characterize the different uses of variable. To answer an item, it is necessary to work with each of the aspects involved and shift from one to the other. The questionnaire, originally in Spanish, was translated to Italian. To guarantee its clearness and understandability for students of both countries, it was reviewed by authors together with secondary school teachers, native Italian and native Mexican speakers. It was administered by researchers during students' normal school activities.

To identify similarities and differences between the studied groups' performance with algebraic variable, students' answers to the questionnaire were analyzed in several ways. First, the median values of correct, incorrect and not given answers to the whole questionnaire of each subgroup (1st and 3rd grade Italian and Mexican students) were compared. In second place, the median values of correct, incorrect and not given answers to the items grouped per use of variable were considered and compared. For both these analysis Mann-Whitney U test with two-tailed p-value was used to identify significant differences ($p \leq 0.05$). Finally, to identify 3rd grade students' understanding of algebraic variable, Italian and Mexican students' strengths and weaknesses were studied: items with 50% or more correct answers and items with less than 50% correct answers were considered and the most frequent errors committed by Italian and Mexican students were analyzed.

Results

Comparisons considering the whole questionnaire

To obtain an overview of 1st and 3rd grade Italian and Mexican students' performance

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[2] See Table 6, for an English version of these items.
[3] See Table 8, for an English version of these items.
[4] See Table 10, for an English version of these items.

with algebraic variable and identify similarities and differences Mann-Whitney *U* test was used. Table 2 shows the median values of correct, incorrect and not given answers to the whole questionnaire (47 items) and the U values obtained and used to determine significant differences (* indicates significant difference).

Table 2: Medians and U values corresponding to correct, incorrect and not given answers

	Italian students	Mexican students	
Grade	Md	Md	U
Correct answers			
1st	11	8	3288.5*
3rd	17	19	4954.5
U	3042.0*	1258.0*	
Incorrect answers			
1st	16	18	4249.5
3rd	15.5	20	3733.0*
U	5402.5	3946.5	
Not given answers			
1st	18.5	20	4336.0
3rd	10	4.5	3553.0*
U	3862.5*	1384.5*	

*p<.001

Source: Drafted by the author

The medians corresponding to correct answers show that even if 1st graders, in both countries, had very little experience with algebraic variable, Italian students performed significantly better. A significant improvement between 1st and 3rd grades was found for correct answers in both countries and, in 3rd grade, no significant difference was found any more between the two groups.

Regarding incorrect and not given answers no significant difference was found between 1st graders. But, in 3rd grade Italian students answered significantly less items than Mexican students, while these gave significantly more incorrect answers.

These results show that at the beginning of middle school cycles, Italian students' mathematics background allowed them to approach the algebraic variable significantly better than Mexican students. This difference was not found any more for 3rd graders. The knowledge acquired during the three years of middle school cycle enhanced students' performance with variable in both countries, even if, despite the significant progress most students, in both countries, had still many difficulties with this concept.

In neither of the two countries the number of incorrect answers diminished from 1st to 3rd

grade, however in both countries the number of not given answers was significantly shorter in 3rd grade suggesting that students had, on average, a better understanding of the requirements of the tasks proposed. But, significant differences between Italian and Mexican 3rd graders regarding incorrect and not given answers were found. To know some of the possible reasons for these differences a qualitative approach should be used as, for example, interviews or focus group technique, but this was not the purpose of the present study.

Comparisons per use of variable

To know how students' performance with each use of variable intervened in the results reported above, the 47 items were grouped per use of variable (*unknown*, *general number*, *variables in functional relationship*) and students' responses to each group of items were analyzed separately, using Mann-Whitney *U* test. The following tables show the medians and U values used to determine significant differences corresponding to correct (Table 3), incorrect (Table 4) and not given answers (Table 5) answers to each group of items (* indicates significant difference).

Table 3: Medians and U values of correct answers to items grouped per use of variable

Grade	Italian students Md	Mexican students Md	U
	Unknown (8 items)		
1st	1	1	2994.5*
3rd	3	3	4862.5
U	2280.5*	1046.5*	
	General number (16 items)		
1st	6	4	3508.5*
3rd	8	7	5176.5
U	3858.0*	2037.5*	
	Functional relationship (23 items)		
1st	5	3	3818.0*
3rd	7	9	4044.5*
U	3357.0*	1188.0*	

* $p < .05$

Source: Drafted by the author

Regarding correct answers (Table 3) in 1st grade Italian students performed significantly better than Mexican students with each use of variable. But in 3rd grade no significant difference was found between the two groups regarding unknown and general number, but there was a significant difference favoring Mexican students for variables in functional relationship. Comparing 1st and 3rd grades for each country a significant

improvement regarding each use of variable appeared.

Regarding incorrect answers (Table 4), in 1st grade no significant difference between Italian and Mexican students was found, independently of the use of variable. In 3rd grade significant differences between the two groups regard incorrect answers given to items involving general number or variables in functional relationship. Mexican students gave significantly more incorrect answers. Comparing 1st and 3rd grades in each country no significant difference was found for incorrect answers regarding unknown or variables in functional relationship, but in Italy 3rd graders gave significantly less incorrect answers to items involving general number than 1st graders.

Table 4: Medians and U values of incorrect answers to items grouped per use of variable

| Grade | Italian students | Mexican students | U |
	Md	Md	
	Unknown (8 items)		
1st	3	3	4496.5
3rd	3.5	4	4721.0
U	5025.5	3870.0	
	General number (16 items)		
1st	6	6	4693.5
3rd	5	6	3801.0*
U	4440.0*	4264.5	
	Functional relationship (23 items)		
1st	7.5	9	4126.5
3rd	8	10	3823.5*
U	5597.5	3869.5	

* p < .01

Source: Drafted by the author

Concerning not given answers (Table 5), the result show that in 1st grade Mexican students answered significantly less items involving general number than Italian students, showing so less familiarity with this use of variable. In 3rd grade significant differences between the two groups regard not given answers for items involving general number or variables in functional relationship. Significantly more of these items were not answered by Italian students, showing so less familiarity with these two uses of variable. Comparing 1st and 3rd grades in each country, it could be observed that median values for not given

answers diminished significantly between 1st and 3rd grade for each use of variable, suggesting that students in both countries had acquired familiarity with each one of them during middle school cycle.

Table 5: Medians and U values of not given answers to items grouped per use of variable

Grade	Italian students	Mexican students	
	Md	Md	U
Unknown (8 items)			
1st	4	4	4249.5
3rd	1	1	5089.5
U	2598.0*	1650.0*	
General number (16 items)			
1st	4	6	3627.0*
3rd	2	1	4176.5*
U	4697.0*	1849.0*	
Functional relationship (23 items)			
1st	10	10	4652.0
3rd	7	3	3099.0*
U	4083.0*	1641.0*	

* $p < .05$

Source: Drafted by the author

Summarizing, data presented in this section show that in both countries 3rd graders have reached a better understanding of each use of variable than 1st graders, even if the median values obtained for correct answers (Table 3) indicate that they have still difficulties to work with this algebraic concept. However, some differences between the two countries emerged. Mexican 3rd graders performed significantly better than Italian students when working with variables in functional relationship (Table 3) even if, they gave as well significantly more incorrect answers to these kind of items (Table 4). In 3rd grade Mexican students answered as well significantly more items (correctly or not) than Italian students when general numbers or variables in functional relationship were involved (Table 5).

These outcomes suggest that the approach used in Mexico, where the three uses of variable were equally emphasized, could have helped students not only to surmount the significant differences with Italian students detected in 1st grade and improve significantly their work with each use of variable, but as well obtain significantly better results when working with variables in functional relationship. To acquire the capability to work, even

if in an incipient way, with the three uses of variable from the beginning of the study of elementary algebra, and shift from one use to the other when required by a task, might be a first step to help students develop an understanding of variable as a multifaceted concept.

Third grade students' understanding of algebraic variable

Both in Italy and in Mexico 3rd grade of middle school cycle is a milestone. Afterwards, the high school cycle begins but most young people, particularly in Mexico, do not continue studying and try to enter the world of work. So, it should be important for the education system to make most students acquire, during the middle school cycle, some of the basic mathematical notions, fundamental to their intellectual development, such as, for example, the concept of algebraic variable.

To deepen in 3rd grade Italian and Mexican students' understanding of algebraic variable the aspects involved in each item were identified using the 3UV Model. To detect similarities and differences between the two groups, the percentages of correct, incorrect and not given answers to each item of the questionnaire were determined. Items with 50% or more correct answers were assumed to reveal students' strengths while items with less than 50% of correct answers point to their weaknesses. Additionally, to identify students' strengths and weaknesses in relation to each use of variable, their most common errors were analyzed.

Students' understanding of variable as unknown

Table 6 shows the items used to test students' understanding of unknown. They refer to interpretation of variable as unknown (items 4b, 4h, aspects involved: U2, G4); equation solving (items 3a, 3b, 3c, aspects involved: U2, U3, U4, G4); symbolization of unknowns and posing equations (items 1a, 1b, 15b, aspects involved: U1, U5, G4). All the items require manipulation (G4), one of the aspects characterizing variable as general number. Table 7 shows the percentages of correct, incorrect and not given answers to these items. Italian and Mexican students' common strengths (more than 50% of correct answers) regard their capability to solve linear equations with one appearance of unknown without a coefficient (item 3c); and to symbolize an unknown and use it to produce an equation with one-step operation (item 1a).

Their common weaknesses (less than 50% of correct answers) regard interpretation of symbolic variable as unknown in equations with multiple appearances of it (items 4b, 4h);

solution of 2nd degree equations (item 3b); symbolization of equations involving two-steps operations (items 1b and 15b).

Table 6: Items involving variable as unknown

In the following exercises do not calculate the number, but rewrite in mathematical language (write only a formula): 1a. An unknown number multiplied by 13 is equal to 127. 1b. An unknown number multiplied by the sum of the same unknown number plus 12 is equal 6.
Calculate the values that the letter can assume in the following equations: 3a. $13x + 27 - 2x = 30 + 5x$ 3b. $(x + 3)^2 = 36$ 3c. $4 + x = 2$
Write how many values the letter can assume in the following expressions: 4b. $3 + a + a = a + 10$ 4h. $4 + x^2 = x\,(x + 1)$
In the following problem write only the equation (it is not necessary to solve it): 15b. Juan is 15 years older than Santiago. The sum of both ages is 41. Which are Juan's and Santiago's ages?

Source: Drafted by the author

Table 7: Percentages of correct, incorrect and not given answers to items involving unknown

	1a		1b		3a		3b		3c		4b	
	It	Mx	It	Mx	It	Mx	It	Mx	It	Mx	It	Mx
Correct	82	91	39	45	52	14	0	2	60	67	36	24
Incorrect	14	8	52	54	38	51	79	80	26	20	31	50
Not given	3	1	9	1	10	35	21	18	14	13	32	26

	4h		15b	
	It	Mx	It	Mx
Correct	20	14	20	26
Incorrect	42	53	63	70
Not given	38	33	17	4

Source: Drafted by the author

Differences appeared in relation to item 3a and, as well, when analyzing the most frequent errors to items with less than 50% of correct answers. While most Italian students could solve a linear equation with multiple appearances of unknown (item 3a), Mexican students answers put forward their lack of manipulative skills. Regarding interpretation of unknown, items 4b and 4h, the most frequent incorrect answers given by Mexican students was: "3 values". This answer suggests the presence of "polysemy" error (namely, that different values can be assigned to the same literal symbol in an expression), already reported for Mexican students (GALLARDO; ROJANO (1988); URSINI; TRIGUEROS (2006)). This kind of error was not found for Italian students, whose most frequent

responses to these two items were: "any number"; "undetermined value", suggesting a perception of variable as a general number; and the value they obtained by solving the equations. Their answers point to a tension between acting and reflecting. In fact, those who tried to interpret the role of variable (answering "any number", "undetermined values") did not register any manipulation; in contrast, students who determined the value of the unknown registered the manipulations performed but, they did not seem to be able to reflect on the role of variable and they did not answer to the question: "how many values the letter can assume". Concerning the solution of a 2nd degree equation (item 3b). Mexican students avoided manipulation, a skill not stressed in the curriculum, and tried to solve the equation by direct inspection. Their most frequent answers were: "x = 3"; and a specific number, between 3 and -9, not justified. Even if in Italy the development of manipulative skills is stressed, these were steel incipient and some already reported misconceptions appeared (for example, $x^2 + 6x$ was assumed equivalent to $7x^2$; $3x + 3x$ to $6x^2$ and x^2 to $2x$) (e.g. Stacey, & MacGregor, 1997), however, the majority tended to manipulate the given expression.. Their most frequent answers were: "$x^2 + 9$"; "$9x^2$"; "$3x^2$", but they could not solve the equation. Regarding symbolization of equations involving two-steps operations (items 1b and 15b) most of the students, in both countries, could not consider an open algebraic expression as a mathematical object to be used to produce another expression. The most common incorrect answer to item 1b was: "$x \times x + 12 = 6$", suggesting difficulties to consider x+12 as a mathematic object that can be multiplied by x. This was further confirmed by Mexican students' answer to item 15b: "x + 15 = 41". Even if they symbolized the unknown, they could not write an equation relating it with given data. In contrast, Italian students made no attempt to pose an equation and looked for the numeric solution of the problem using arithmetic approaches.

Students' understanding of variable as general number

Items testing students' understanding of general number are presented in Table 8. They refer to recognition of figural patterns (items 7a, 7a1, 7a2, 7c, 7d, aspect involved G1); interpretation of variable as general number (items 4a, 4c, 4d, 4e, aspect involved G2); symbolization of variable as general number and its manipulation (item 1d, aspects involved G4, G5); interpretation of variable as general number and its manipulation (item 2a, aspects involved G2, G4, G5); capability to shift between the different aspects characterizing variable as general number to produce general expressions (items 5, 6, 7b, 8, 18, aspects involved G1, G2, G3, G4, G5). Table 9 shows the percentages of correct, incorrect and not given answers to these items.

Italian and Mexican students' common strengths (more than 50% of correct answers) concern recognition of simple figural patterns providing successive specific examples (items 7a, 7a1, 7a2, 7c, 7d); and symbolization of general statements with one-step operations (items 6 and item 18).

Table 8: Items involving variable as general number

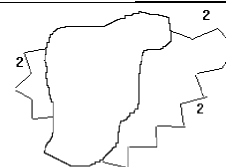| |
|---|
| In the following exercises do not calculate the number, but rewrite in mathematical language (write only a formula):
    1d. An unknown number is divided by 5 and the result is added to 7. |
| Reduce the following expression to an equivalent one:
    2a. $a + 5a - 3a =$ |
| Write how many values the letter can assume in the following expressions:
    4a. $x + 2 = 2 + x$
    4c. $x = x$
    4d. $4 + s$
    4e. $3 + a + a + a + 10$ |
| 5. To calculate the perimeter of a figure, sum the length of each one of its sides. Write the formula that expresses the perimeter of the following figure |
| 6. In the following figure, the polygon is not entirely visible. Since, we do not know how many sides the polygon has, let say it has N sides. The length of each side is 2 centimeters. Write a formula to calculate the perimeter of the polygon. |
| Observe the following figures:
    Figure 1 - Number of points: 1
    Figure 2 - Number of points: 4
    Figure 3 - Number of points: 9
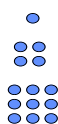
    7a.   How many points will Figure 4 have?
    7a1. Draw Figure 5 and give the number of points.
    7a2. Draw Figure 6 and give the number of points.
    7b.   Assume you can keep on drawing figures until Figure m. How many points does Figure m have?
    7c.   How many points did you add to go from Figure 1 to Figure 2?
    7d.   How many points did you add to go from Figure 2 to Figure 3? |
| 8. Observe the following identities and complete the sequence:
$1 + 2 + 3 = (3 \cdot 4)/2$
$1 + 2 + 3 + 4 = (4 \cdot 5)/2$
....
$1 + 2 + 3 + 4 + ... + n =$ |
| 18. In an m-sides polygon, the length of each side is 3 cm: what is its perimeter? |

Source: Drafted by the author

Common weaknesses regard interpretation of the role of variable in general expressions (items 4a, 4c, 4d and 4e); symbolization of general statements with two-steps operations (items 1d, 5, 7b, 8); manipulation of an expression to produce an equivalent one (2a).

The analysis of students' incorrect answers put forwards similarities and differences. Concerning symbolization (items 1d, 5, 7b, 8) a frequent common answer to item 1d was

"x/5 = y + 7", that confirms their difficulty to consider an open algebraic expression as an operable mathematical object. While the great majority could symbolize a one-step operation, writing x/5, they did not consider this expression as an operable object. They identified it with another variable (x/5 = y) but, did not perceive this expression as an equivalence, and they wrote "x/5 = y + 7" instead of x/5 + 7 or y + 7. Both groups answered item 8 writing "(5 x 6)/2" and "(8 x 9)/2" showing that they have recognized the pattern but, they were not able to symbolize the corresponding general rule. This suggests implicit contextualized knowledge (recognition of a pattern and capability to produce more specific examples), but difficulty to de-contextualize it and shift to a more abstract level of thinking where algebraic language could be used.

Table 9: Percentages of correct, incorrect and not given answers to items involving general number

	1d		2a		4a		4c		4d		4e	
	It	Mx	It	Mx	It	Mx	It	Mx	It	Mx	It	Mx
Correct	29	14	23	45	31	25	25	30	20	26	23	21
Incorrect	65	83	48	46	49	51	39	48	42	45	39	50
Not given	6	3	29	9	19	24	38	22	38	29	38	29

	5		6		7a		7a1		7a2		7b	
	It	Mx	It	Mx	It	Mx	It	Mx	It	Mx	It	Mx
Correct	46	27	54	70	82	78	79	76	80	73	28	16
Incorrect	34	65	14	20	14	17	16	21	14	24	36	55
Not given	20	8	32	10	4	3	5	3	6	3	36	29

	7c		7d		8		18	
	It	Mx	It	Mx	It	Mx	It	Mx
Correct	88	90	84	89	6	3	53	58
Incorrect	5	8	9	9	60	85	15	25
Not given	7	2	7	2	34	12	31	17

Source: Drafted by the author

Differences between the studied groups regard interpretation of variable as general number (items 4a, 4c, 4d and 4e). Many Italian students interpreted the variable as an unknown (to item 4a they answered "0", after manipulating the expression), while for Mexican students the "polysemy" error was confirmed (to item 4a they answered "2 values"; "3 values"). There were as well differences regarding symbolization. For example, in Mexico the most frequent answers to item 7b were "x", "m", "13 x 13", "impossible, because m has no value", "an infinity of points", while Italian students answered "100 points", "an infinity of points". A tendency to ignore an already symbolized variable (KÜCHEMANN (1980)) and to use only numeric data was observed for Italian students whose frequent answer to item 5 was "18". Mexican students did not ignore the given variable but, their lack of manipulative skills (for example, they

considered x + x equivalent to x² and 5 + x equivalent to 5x) led them to incorrect symbolizations. Mexican students' difficulties with manipulation was further confirmed by the variety of incorrect answers they gave to item 2a. A frequent Italian students' incorrect answer to this item was "2a". This suggests that their manipulative skills were tightly linked to equation solving, in fact, the majority could solve an equation with multiple appearance of unknown (item 3a).

Students' understanding of variable in functional relationship

Items testing students' understanding of variables in functional relationship appear in Table 10. They regard correspondence (items 11a, 14b, 16b, 16c, 17a, 19c, 19d, aspects involved F1, F2 or F3); variation (items 9a, 9b, 9c, 10, 13, 16a, aspect involved F4); range of variation (items 12a, 12b, 17b, 17c 19a, 19b, aspects involved F1, F4, F5); symbolization of variables in functional relationship (items 1c, 11b, 14a, 16d, aspects involved F1, F6, G4). Table 11 shows the percentages of correct, incorrect and not given answers to these items. Once more similarities and differences were found between the studied groups.

Students' common strengths refer to their capability to work with correspondence, when data appeared in ordered numeric tabular form (item 11a and 16c) or in very simple verbal problem (14b); and to work with variation represented through very simple analytic expressions (items 9a, 9c and 10).

Common weaknesses regard correspondence when working with expressions with multiple appearance of variables (items 17a) or visually analyzing a given graph (items 19c, and 19d); joint variation (items 9b, 13, and 16a); range of variation (items 17b, 17c, 19a, and 19b); and symbolization of functional relationships (items 11b, 14a, and 16d).

Differences appeared in relation to items 1c, 12a and 12b and, as well, when analyzing the most frequent errors to items with less than 50% of correct answers. The majority of Mexican students could symbolize a simple verbally given functional relationship (item 1c), they could determine a range of variation given a very simple linear expression (items 12a, 12b) and, based on not-ordered tabular data, they identified the value of x that corresponded to the maximum value of y (item 16b). Most Italian students could not give a correct answer to these items. Their most frequent incorrect answer to item 1c was "x = 6 + x", where their tendency to interpret variable as a placeholder reappeared. To items 12a and 12b they answered writing some integer numbers belonging to the already given intervals. In relation to item 16b, their most common answer was "-20". They seemed not to be used to work with not-ordered tabular data. In fact, they considered that the value at

the bottom of the table, 400, represented the maximum value for y and, in correspondence, they identified the value of x, -20.

Table 10: Items involving variable in functional relationship

In the following exercises do not calculate the number, but rewrite in mathematical language (write only a formula):

1c. An unknown number is equal to 6 plus another unknown number.

If x + 3 = y

9a. Could x have any value?
9b. Could y have any value?
9c. Could x and y simultaneously have any value?

10. If y = 7 + x, what happens to the values of y when the value of x increases?

In order to facilitate his work, an employee has made the following table:

11a. Complete the table.
11b. Consider the table above and write a general rule. Use n to represent the number of photocopies.

Number of copies	Cost
10	2.50
15	3.75
18	4.50
	5.50
23	

Consider: y = 3 + x

12ªa. If we want the values of y to be bigger than 3 but smaller than 10, which values can x take?
12b. If the values of x are between 8 and 15, which are the values of y?

13. Observe the following expressions: n + 2 and 2·n. Which one represents a bigger value? Explain your answer.

In a market, a platform scale is used. For each kilogram the tray shifts 4 centimeters.

14a. Find the relation between the weight of a product and the shift of the tray.
14b. If the tray shifts 10 cm, which is the weight of the product?

Observe the table and answer the questions:

16a. Determine what happens with the value of y when the value of x increases.
16b. For which value of x, y achieves its maximum value?
16c. For which value of x, y achieves its minimum value?
16d. Write a general rule that relates variable x to variable y.

x	y
0	0
10	100
-15	225
25	625
20	400
-10	100
15	225
-20	400

Given the expression: 40 - 15x - 3y = 17y - 5x

17a. Which value of y corresponds to x = 16?
17b. Between which values must be x, if we want the value of y to be between 1 and 5?
17c. Assume that x takes values between -5 and 5, for which value of x, y reaches its maximum value?

Observe in the following graph in which way the values of y change, when the values of x increase. You can answer giving an approximate value.

19a. Between which values of x the values of y increase?
19b. Between which values of x the values of y decrease?
19c. For which value of x, y reaches a maximum value?
19d. For which value of x, y reaches a minimum value?
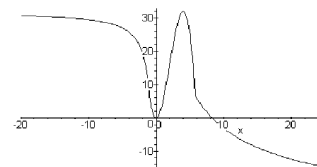
Source: Drafted by the author

Table 11: Percentages of correct, incorrect and not given answers to items involving variables in functional relationship

	1c		9a		9b		9c		10		11a	
	It	Mx	It	Mx	It	Mx	It	Mx	It	Mx	It	Mx
Correct	37	60	74	85	44	42	60	63	67	75	52	70
Incorrect	57	39	14	13	42	54	23	34	16	18	31	24
Not given	6	1	13	2	14	3	17	3	17	7	18	7

	11b		12a		12b		13		14a		14b	
	It	Mx	It	Mx	It	Mx	It	Mx	It	Mx	It	Mx
Correct	17	43	31	50	31	50	8	4	5	22	61	76
Incorrect	35	33	45	45	45	45	75	93	58	67	19	15
Not given	48	24	24	5	24	5	16	2	37	11	19	9

	16a		16b		16c		16d		17a		17b	
	It	Mx	It	Mx	It	Mx	It	Mx	It	Mx	It	Mx
Correct	1	1	42	50	59	61	33	34	5	1	1	0
Incorrect	96	83	30	33	14	23	25	33	19	49	24	47
Not given	3	16	29	17	27	16	42	34	75	50	75	53

	17c		19a		19b		19c		19d	
	It	Mx	It	Mx	It	Mx	It	Mx	It	Mx
Correct	3	4	14	13	5	2	30	40	33	42
Incorrect	15	32	43	74	52	82	25	42	21	38
Not given	81	64	42	13	45	16	45	18	46	20

Source: Drafted by the author

Both, Italian and Mexican students had great difficulties to determine a corresponding value when working with expressions with multiple appearance of variables (item 17a) that required manipulation and substitution of given values. Most students did not answer the item. Some Italian students wrote "I do not understand the question". A diversity of Mexican students' attempts put forwards their difficulties with manipulation. Items 19c and 19d required visually analyzing a given nonlinear graph to identify the values of x corresponding to maximum and minimum values of y. The majority identified the values of y but they did not relate them to the corresponding values of x, disregarding the correspondence. Difficulties to perceive joint variation from simple analytic expressions (items 9b and 13) or from not ordered tabular data (item 16a) appeared in both countries. The most frequent answer to item 9b was "no", suggesting that many students looked at the expression x + 3 = y only as an operation to perform with x in order to obtain a result, represented by y, without perceiving the joint variation of the two variables involved. To item 13 the great majority, in both countries, answered "2·n is greater because it multiplies", focusing only on the operations involved in each expression (2+n and 2·n). They applied a rule learned by hart since primary school (the product of two numbers is

greater than their sum) but incorrectly generalized. To item 16a the great majority, in both countries, answered "increases", even if some of them symbolized correctly the functional relationship "$y = x^2$" (item 16d). Once more, the focus was not on the relationship but on the result represented by y.

To determine a range of variation (items 17b, 17c, 19a and 19b) is not an easy task, it implies shifting repetitively from focusing on variation to focusing on correspondence in order to determine the endpoints of a new interval. The great majority of Italian students did not answer questions 17b and 17c, some of them wrote "I do not understand the question", or "I do not know". Neither most Mexican students answered these items, but many of them tried to determine an interval. Even if they failed, their attempts suggest that they understood the question. Students' responses to items 19a and 19b, "from 0 to 9" and "from 10 to 20", respectively, show that, in both countries, they focused only on the first and the forth quadrant and they were confused about the meaning of increase/decrease and positive/negative values.

Difficulties with symbolization of simple functional relationships derived from tabular data (item 11b and 16d) or a simple verbal problem (item 14a) appeared in both groups. The most frequent answer to item 11b was "n = 0.25", in both countries. They correctly calculated the price of one photocopy (0.25) but they assigned this value to n instead of symbolizing the price of n photocopies. However, the majority perceived the relationship and answered correctly item 11a that required completing tabular data. The responses to item 16d illustrate different unsuccessful attempts to symbolize a functional relationship from not-ordered tabular data. In Italy a frequent answer was "y = kx", without specifying the meaning of k, while in Mexico a variety of answers witness students' difficulty to symbolize the relationship (frequent answers were "xy"; "$x^2 = x$"; "x = 10 and y = 100"; "x + 100 = y"; "x/10 = y"). In both countries some students wrote "when x increases y increases as well" showing both, a difficulty to go further from a sole perception of variability, already present even in younger children, and a tendency to disregard the negative values of x. Responses to item 14a further confirmed difficulty to symbolize. The great majority, in both countries, wrote "1kg = 4cm", a response showing attempts to use given data as support for symbolizing the functional relationship. Another frequent common answer to items 11b, 14a and 16d, was "direct proportional relationship" that describes the kind of relationship they were perceiving instead of symbolizing it.

Concluding remarks

The purpose of this comparative study was to identify middle school students' performance with algebraic variable in different countries in order to detect similarities and differences concerning their strengths, weaknesses and the most frequent error committed. This kind of information may help participating countries, and others with similar outcomes, revisit their curriculum focusing on how the teaching of elementary algebra, in particular the concept of algebraic variable, is approached and, if necessary, implement changes to help students improve their understanding of the concept of variable, a core concept in algebra.

Data were collected in two countries, Italy and Mexico. Answers given by 1st and 3rd grade middle school students to a questionnaire, designed with the purpose to detect the understanding of algebraic variable, were analyzed using the 3UV Model. This theoretical-methodological tool allowed to identify students' strengths and weaknesses with each use of variable.

The analysis of the differences found between Italian and Mexican middle school students' performances with algebraic variable, suggest a dependence on the particular approach to elementary algebra used in each country. It was observed, for example, that the experience acquired in primary school strongly influenced students' first approaches to algebraic variable. Even if the average performance of 1st grade students was in both countries very low, Italian students obtained significantly better results in the whole questionnaire and with each use of variable as well.

When 3rd graders' performance was analyzed, even if a general positive improvement was found in both countries, this was especially notable for Mexican students who performed equally well as Italian students when working with unknowns and general numbers, and significantly better than Italian students when working with variables in functional relationship. Even if this was not a longitudinal study and students tested in 1st and 3rd grades in each country were not the same students, they belonged to the same schools and have followed the same curriculum in primary school. Therefor it can be affirmed that the results obtained show that it is possible for students to surmount initial disadvantages and reach, at the end of lower-secondary school, similar or even better results than students with an initial better preparation. This suggests that students' improvements in the understanding of algebraic variable depended mainly on the subjects tackled and emphasized in middle school cycle.

A detailed analysis of 3ʳᵈ grade Italian and Mexican students' work with each use of variable put forward different strengths and weaknesses that point to two different approaches to elementary algebra: one, identified for Italian groups, where the work with unknown and general number was privileged (equation and problem solving, development of manipulative skills linked mainly to equation solving) while work with variable in functional relationship was not stressed; another, linked to Mexican groups, where the work with the three uses of variable was developed contemporarily, from the first approaches to elementary algebra, tackling the different aspects characterizing each use, although not deeply enough to avoid some difficulties when the complexity of the task increased.

The influence of different approaches is further confirmed when we observe that although both groups acquired some manipulative skills, much more pronounced for Italian students, each group tended to associate it to a specific use of variable. While Italian students obtained better results when manipulating in order to solve a linear equation with multiple appearances of unknown, Mexican students manipulated better when an equivalent expression to a given one had to be produced. Most Mexican students could solve simple tasks that required working with aspects characterizing variables in functional relationship, while the majority of Italian students had difficulties.

Students' errors, misunderstandings and misconceptions found in this comparative study have been already reported by different researchers, in different countries, but, the fact that some of them were not present in both populations or linked to the same situations, confirm that they strongly depend on the teaching approaches, therefore, they could be avoided. A clear example of this is the presence in Mexico but not in Italy of the "polysemy" error, often reported for Mexican students at different school levels. The results of this comparative study provide evidence that this error is not of epistemological but of didactical nature and that it can be avoided through the development of an alternative teaching approach.

Some similarities between the two studied groups point to actions that could be taken in the participating countries to improve students' work with algebraic variable and the understanding of this concept. For example, the results show that at the end of middle school most students in both countries were still linked to numeric calculations, to specific cases, to arithmetic approaches, being this last aspect more pronounced in Italy. The majority had serious difficulties to de-contextualize their knowledge and adopt a more general perspective, to generalize and produce general statements, and to pose simple

equations using algebraic language. That is, the great majority lacked the capability to reflect and take autonomous decisions, they have learned mainly to follow instructions. These results point to the necessity to help students develop metacognitive skills and independent decision taking, the capability to generalize and deal with abstraction. A good understanding of algebra, in particular of variable, is necessary for mathematics learning and its application to different fields, as required by the current globalized world and stressed, for example, by STEM education, but it is fundamental to help future adult citizen start developing from the first beginning of middle school the capability to analyze, generalize, think abstractly and to get used to reflect and not only to follow instructions.

The information provided in this paper can help those interested in this topic to review how this core algebraic concept is approached in their own countries and, hopefully, encourage researchers to develop more comparative studies. The results obtained can be a basis for suggesting new approaches to the teaching of elementary algebra that offer students better opportunities to gain a good understanding of its basic concepts, as the concept of variable and, through this, help them to participate in a more informed way in the contemporary world.

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