

Strategies Employing Prospective Primary Education Teachers in a Functional Relationship Task

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ABSTRACT

Background: The early algebra curricular proposal seeks to promote algebraic thinking from the first educational levels. Functional thinking is an approach intended to promote algebraic thinking in students. At present, various curricular programs include notions related to this thinking. However, this type of thinking has had little influence on prospective teachers, thus arising the need to investigate how they approach tasks to prepare them adequately for their teaching. **Objective:** To analyse the strategies that prospective Chilean primary education teachers use when they solve a task that implies the generalisation of a functional relationship. **Design:** We follow a qualitative methodology at an exploratory-descriptive level. **Setting and participants:** The sample included 18 prospective primary education teachers from a Chilean university. **Data collection and analysis:** We collected the data through a written test in a group of 18 prospective teachers who were in the first year of training and analysed the data. **Results:** We highlight the diversity of arithmetic and functional strategies used by prospective teachers. In addition, we highlight their trajectories to solve the task, finding that most of them begin using arithmetic strategies and end using functional strategies. **Conclusions:** This research will identify the strategies used by prospective primary education teachers and whose data could be useful for the creation of teacher education programs.

Keywords: Early algebra, Primary education, Algebraic thinking, Functional thinking.

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Estrategias que emplean futuros profesores de educación primaria en una tarea de relación funcional

RESUMEN

Contexto: La propuesta curricular *early algebra* busca promover modos de pensamiento algebraico a partir de los primeros niveles educativos. El pensamiento funcional es un enfoque por el cual se pretende promover el pensamiento algebraico en los estudiantes. En la actualidad diversos programas curriculares introducen nociones relativas a este pensamiento. Sin embargo, en el futuro profesor este tipo de pensamiento ha incidido poco, surgiendo la necesidad de indagar en la forma en que ellos abordan tareas vinculadas a él, con la finalidad de prepararlos adecuadamente para su enseñanza. **Objetivo:** Analizar las estrategias que emplean futuros profesores de educación primaria chilenos cuando resuelven una tarea que implica la generalización de una relación funcional. **Diseño:** Seguimos una metodología cualitativa de nivel exploratorio-descriptivo. **Entorno y participantes:** La muestra estuvo conformada por 18 futuros profesores de educación primaria de una universidad chilena. **Recopilación y análisis de datos:** Recogimos los datos a través de una prueba escrita en un grupo de 18 futuros profesores que cursaban el primer año de formación. Realizamos un análisis cualitativo de los datos. **Resultados:** Destacamos la diversidad de estrategias aritméticas y funcionales utilizadas por los futuros profesores. Además, destacamos las trayectorias que han efectuado estos profesores en la resolución de la tarea, donde la gran mayoría de comienza con estrategias aritméticas y finaliza utilizando estrategias funcionales. **Conclusiones:** Esta investigación permitió identificar las estrategias que emplean futuros profesores de educación primaria y cuyos datos podrían ser útiles para la creación de programas de formación del profesorado.

Palabras clave: Early algebra, Educación primaria, Pensamiento algebraico, Pensamiento funcional.

Estratégias Utilizadas por Futuros Professores do Ensino Fundamental em uma Tarefa de Relacionamento Funcional

RESUMO

Contexto: A proposta curricular da álgebra inicial visa promover formas algébricas de pensar desde os primeiros níveis de ensino. O pensamento funcional é uma abordagem pela qual se pretende promover o pensamento algébrico nos alunos. Atualmente, vários programas curriculares introduzem noções relacionadas com este pensamento. No entanto, no futuro professor, este tipo de pensamento teve pouca influência, surgindo a necessidade de investigar a forma como abordam as tarefas a ele relacionadas, a fim de prepará-los adequadamente para o ensino. **Objetivo:** Analisar as estratégias utilizadas pelos futuros professores chilenos no ensino fundamental quando

resolvem uma tarefa que implique a generalização de uma relação funcional. **Design:** Seguimos uma metodologia qualitativa a nível exploratório-descritivo. **Ambiente e participantes:** A amostra foi constituída por 18 futuros professores do ensino fundamental de uma universidade chilena. **Coleta e análise de dados:** Os dados foram coletados por meio de uma prova escrita em um grupo de 18 futuros professores que estavam no primeiro ano de formação. Fazemos uma análise qualitativa dos dados. **Resultados:** Destacamos a diversidade de estratégias aritméticas e funcionais utilizadas pelos futuros professores. Além disso, destacamos as trajetórias que esses professores percorreram na resolução da tarefa, onde a grande maioria começa com estratégias aritméticas e termina com estratégias funcionais. **Conclusões:** Esta pesquisa possibilitou identificar as estratégias utilizadas pelos futuros professores do ensino fundamental e cujos dados podem ser úteis para a elaboração de programas de formação de professores.

Palavras-chave: Álgebra inicial, Ensino fundamental, Pensamento algébrico, Pensamento funcional.

INTRODUCTION

The teaching approach where arithmetic precedes the teaching of algebra causes students to have serious difficulties in learning algebraic notions (Lins & Kaput, 2004; Molina, 2009). For this reason, in recent years, the teaching of school algebra has substantially changed the way it has been carried out in the classroom. Some of these changes focused on incorporating algebraic elements from the first educational levels based on the arithmetic present in the mathematics curriculum. This change is known as the *early algebra* curriculum proposal and is based on the algebraic character of arithmetic (Molina, 2009). Rather than manipulating symbols and their techniques, this proposal expects that students develop algebraic thinking modes from an early age to identify arithmetic patterns and structures, establish relationships and generalisations, and build increasingly sophisticated ways of representing them (Brizuela & Blanton, 2014).

Functional thinking is an approach to *early algebra* and is considered a type of algebraic thinking (Cañadas & Molina, 2016). This thinking focuses on understanding the concept of function, or relationships between the amounts that covariate (Rico, 2006; Smith, 2008). It is used to solve problems and allows students to identify patterns and represent and generalise relationships between quantities (Cañadas et al., 2016). The reasons above are enough for this thinking to be considered a disciplinary goal in the teaching of mathematics (Rico, 2006), and for countries such as Australia, Canada, China, Chile, Korea, United States, Japan, Portugal, and Spain to incorporate it in their curricula (Merino et al., 2013; Ministerio de Educación de Chile [MINEDUC], 2012a).

The Chilean curriculum emphasises that students must be able to identify relationships between quantities and investigate how changing one amount affects the other amount (MINEDUC, 2012a).

In the teaching context, the prospective primary education teacher should be prepared to recognise the relationships established in the school curriculum regarding algebraic contents and especially those referring to functional thinking (MINEDUC, 2012b). However, algebra teaching in primary education has focused on didactic proposals centred on patterns, equations, and inequations, which indicates that functional thinking has not involved students, being a pending issue in their formation (Cañadas & Molina, 2016; Morales et al., 2018). From this perspective, the prospective primary education teachers must be involved in tasks that imply functional thinking, given their little experience in the matter (Blanton & Kaput, 2005; Morales et al., 2018). This study is pertinent because the way in which prospective teachers know mathematical content is related to how they will think about the teaching of that content (Sánchez & Llinares, 2003), especially regarding functional thinking. In this article, we aim to answer the following research questions:

- What are the characteristics of the strategies that prospective primary education teachers employ when solving a task that involves a functional relationship? and
- What resolution trajectories do they employ?

THEORETICAL ELEMENTS

Functional Thinking

Functional thinking is catalogued as algebraic thinking based on construction, description, representation, and reasoning with and about the functions and elements that constitute them (Cañadas & Molina, 2016). This type of thinking differs from arithmetic thinking because the latter is based on numerical concepts and sense of numbers, the meaning of arithmetic operations, control of basic facts of arithmetic, mental calculation and writing of arithmetic, reading and writing of verbal problems and arithmetic skills (Verschaffel & De Corte, 1996), while the former, although based on arithmetic thinking, goes much beyond, given that it develops key algebraic elements such as “variable quantities, their relationships, recursion, the correspondence between values of variables or the use of different representation systems in a problem solving context” (Cañadas & Molina, 2016, p. 212).

Blanton et al. (2015) define functional thinking as the one that implies generalising functional relationships, representing and justifying those relationships through natural, pictorial, tabular, graphic, symbolic, or algebraic language, and fluently reasoning with those generalised representations to understand and predict the behaviour of the function. A student is said to manifest functional thinking when they pay attention to two or more covariate quantities, identify the type of relationship between those quantities, and can generalise such a relationship (Confrey & Smith, 1991; Smith, 2008). The relationships that are established between varying amounts, generalisation, and systems of representation are key in functional thinking. Below, we detail those elements.

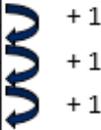
Functional relationships

In functional thinking, three relationships can be accounted for: recurrence, covariation, and correspondence. Recurrence is the most elementary, given that some value of some of the variables is implicit (Blanton & Kaput, 2005). It is defined as the relationship between the values of the same set (Johnsonbaugh, 2005). That is, from a succession of data, the recurrence relationship expresses each term of that succession according to its predecessors (Castro, 1995). For example, in a problem that relates Álvaro's and Carmen's ages, where Carmen is 5 years older than Álvaro and whose problem is modeled by the function $f(x) = x + 5$ (Morales et al., 2018), Carmen's age adds one to her previous age, as shown in the tabular representation of Figure 1. This means that there is no relationship between the values of both variables but rather a relationship between the values of one of the variables (dependent). Sometimes, this type of relationship hinders the generalisation because it is necessary to know a previous value of the dependent variable to give a satisfactory response (Morales et al., 2018).

Figure 1

Recursive Pattern.

Edad de Álvaro	Edad de Carmen
1	6
2	7
3	8
4	9

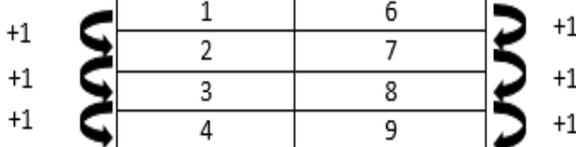


For their part, the relationship of covariation and correspondence does refer to a relationship between variables, commonly called functional relationships (Blanton et al., 2011; Confrey & Smith, 1991; Smith, 2008). The covariation relationship refers to the change of one variable and its incidence in the other variable. It refers to “a simultaneous change between two variables that occurs due to the existence of a relationship between them” (Gómez, 2016, p. 170). To identify a covariation relationship is to focus on changes between the values of the independent and dependent variables. In this way, we understand that this relationship implies observing how two values of both variables within a functional relationship vary simultaneously and coordinately (Blanton et al., 2011; Blanton & Kaput, 2005). For example, in the problem above, Carmen’s age adds one to Carmen’s age, given that Alvaro’s age increased by one, as shown in the tabular representation of Figure 2.

Figure 2

Functional ratio of covariation

	Edad de Álvaro	Edad de Carmen	
+1	1	6	+1
+1	2	7	+1
+1	3	8	+1
	4	9	



Correspondence is one in which each value of the independent variable is associated with a single value of the dependent variable (Clapham, 1998). This relationship is established between the corresponding pairs of the values of both variables (Confrey & Smith, 1991; Smith, 2008). Identifying a correspondence is to focus on the pattern that allows determining a single value

of the dependent variable, given a value of the independent variable (Blanton et al., 2011). For example, in the previous problem, finding Carmen's age implies finding the pattern that determines it, in this case, adding five to Álvaro's age. In this way, any of Carmen's age can be found given Álvaro's age, just by adding five to the latter's value.

Figure 3

Functional relationship of correspondence

Edad de Álvaro	Edad de Carmen
1	6
2	7
3	8
4	9

$+5$


Generalisation

Generalisation is considered the core of algebra and an initiator of algebraic learning (Mason, 1996; Strachota, 2016).

In the context of functional thinking there are various connotations to define the generalisation, and many of them agree that the work to achieve the generalisation is important to do through particular cases. For example, Kaput (2000) indicates that the generalisation is:

deliberately extending the range of reasoning or communication beyond the case or cases under consideration, explicitly identifying and exposing similarities between cases, or increasing reasoning or communication to a level where the focus is not the cases or situation itself but the patterns, procedures, structures, and relationships throughout and between them. (p. 6)

For his part, Kruteskii (1976) alludes to the fact that generalisation is the process of moving away from the concrete situation, or the process of abstraction from what is similar and relevant in the structure of objects, relationships, or operations. Cañadas and Castro (2007), based on Polya's works, point out that one way to achieve generalisation is through the work and organisation of different particular cases, and suggest that students reach

generalisation when they can relate an identified pattern with a general rule and not only for some cases. However, Mason (1996) considers that it is also possible to reach generalisation through a single particular example or case with specific characteristics, which is known as a generic example. A generalisation can be represented by diverse representation systems.

Representation systems

In this article, we focus on external representations, which refer to “symbolic or graphic notations, specific to each notion, through which mathematical concepts and procedures are expressed, as well as their most relevant characteristics and properties” (Castro & Castro, 1997, p. 96). We considered those representations because they allow us to evidence the subjects’ productions when solving mathematical tasks (Merino et al., 2013).

In functional thinking, the systems of verbal, pictorial, tabular, graphic and symbolic representation acquire importance since it helps to understand the behaviour of the function and allows to reveal the presence of functional thinking in them (Blanton et al., 2011; Cañadas et al., 2016, Cañadas & Molina, 2016). The verbal representation system mentions natural oral or written language to express mathematical concepts (Cañadas & Figueiras, 2011). Pictorial representations allude to visual resources, such as drawings, that allow mathematical relationships to be expressed and are paramount because they are own and original representations of the subjects who solve mathematical tasks (Blanton et al., 2011; Cañadas & Figueiras, 2011). Numerical representations use numbers and operations expressed by mathematical language (Merino et al., 2013). The symbolic representation is alphanumeric, the syntax of which is described by a series of rules and procedures (Rico, 2009). This system involves symbols and signs typical of a type of mathematics that allow the precise expression of the quantities of the variables and the variables themselves in a functional relationship task. This representation system requires sophisticated mathematical thinking since it enables us to express a functional relationship (Azcárate & Deulofeu, 1990; Blanton, 2008).

Problem solving strategies

Strategies are fundamental in problem solving, as either implementation will help the subject succeed in the resolution. A strategy is understood as a “procedure or rule of action that allows us to obtain a

conclusion or answer a question using relationships and concepts, general or specific to a certain conceptual structure” (Rico, 1997, p. 31). Several authors highlight the need to investigate problem solving strategies in the functional context, given that, in that context, students manifest various difficulties in finding an appropriate strategy (e.g., Amit & Neria, 2008; Merino et al., 2013; Moss & Beatty, 2006). Considering the above definition, we assume that functional relationships can be seen as resolution strategies when someone faces a task involving covariable amounts.

BACKGROUND

Although there is a significant amount of research on the primary education teacher’s knowledge of mathematics, there is very little in the algebraic context, especially in functional thinking. Studies indicate that pre-service primary education teachers manifest different difficulties and errors when solving tasks in a functional algebra context. For example, at the international level, the TEDS-M study showed that prospective primary education teachers had difficulties in identifying an algebraic representation of three consecutive even numbers and had limited success in tasks of application of functions in geometric contexts, as they found them too challenging (Senk et al., 2012). In Chile, the national diagnostic assessment of initial teacher education shows that less than 50% of the answers of 1,323 prospective primary education teachers about patterns and successions are correct, which shows those teachers’ poor preparation for algebra-related issues (MINEDUC, 2020). From the research context, Aké (2021) found that in a functional task of the type $f(x) = 4x + 2$, 18 out of 40 primary education teachers in training solved the task correctly by establishing a functional rule, but using the verbal representation system instead of the symbolic one. While 11 of the 40 resolved it partially correct, focusing only on the particular cases of the task and not so on the general case. The rest of the prospective teachers (11) solved the task incorrectly. In this study, it is noteworthy that 37 of the 40 pre-service teachers used pictorial, numerical, and verbal representations in their answers. For their part, Polo-Blanco et al.’s (2019) study showed that prospective Spanish and Portuguese primary education teachers find it difficult to establish and generalise a correspondence relationship in a functional relationship task in a geometric context, and that they generally addressed it through covariation and recursive strategies. Wilkie (2014) reported that 30% of 105 Australian primary

education teachers manifested a low level of functional thinking in a task that involved extending a geometric pattern. Those teachers provided incorrect, inappropriate answers, and solved the task through a strategy based on the recursive pattern. For its part, only 70% of these teachers generalised the functional relationship, but 2% of them did so through a complete symbolic representation. It may be that these antecedents are the product of the inadequate algebra offered to the prospective teachers before, which is rooted in the abrupt and disconnected passage between arithmetic and algebra (Kaput, 2000). This case is observed in Rodríguez-Domingo et al. (2015), who report that secondary education students find it challenging to address algebraic tasks, especially those they translate from verbal representations into symbolic representations. Therefore, this background suggests the need to deepen the functional thinking of prospective primary education teachers in training in diverse contexts.

METHODOLOGY

Type of research

This research is exploratory and descriptive. According to Hernández et al. (2010), exploratory studies are characterised by investigating a subject that is little studied and about which there are doubts, and new perspectives need to be opened. This is the case with this research. The aim is to investigate what little has been explored, such as functional thinking in prospective primary education teachers, opening perspectives for their knowledge, possible paths of education, and future research.

The subjects of the research

The research subjects were 18 prospective primary education teachers who were studying the first year of the career of basic general pedagogy with specialisation in a Chilean university at the time of data collection. The sample was intentional and covered the prospective teachers' interest in participating in this research. The research subjects had had training in algebra during their secondary education, but not so in their pedagogical studies, therefore, they were not familiar with functional relationship tasks, as presented in this study.

Information gathering tool

We used a written test to collect the information. This test consisted of prospective teachers individually answering a problem involving a functional relationship of the type $f(x) = 2x + 2$. The proposed problem was adapted from the work done by Carraher et al. (2008), whose context addressed the functional relationship between the number of tables and the number of guests. To demonstrate the prospective teachers' functional relationship between the amounts involved in the problem, we formulated several questions based on the inductive reasoning model of Cañadas and Castro (2007). In other words, by employing questions for particular cases and a general question, we sought the generalisation of who solves the task (see Table 1). Before applying the instrument, the participants accepted participating in this study by signing informed consent¹. Next, in Figure 4, we show the problem posed in the written test, and in Table 1, we present the characteristics of the questions asked.

Figure 4
People sitting around tables²

¹This study was not reviewed by a scientific ethics committee since it is not part of a research project, but rather, it is a study that is part of the authors' motivation. Therefore, we exempt the journal *Acta Scientiae* from the consequences derived from it, including comprehensive assistance and eventual compensation for any damage resulting to any of the research participants.

² It is Antonia's birthday, and her mom organises the tables and chairs in a square format for the guests so that at one table, only four guests can sit, and only six guests can sit at the other table, as seen in Figure 1. According to the above, answer the following questions, making the corresponding procedures and calculations with the corresponding justifications.

Antonia está de cumpleaños y su mamá organiza mesas de forma cuadrada y sillas para que se sienten los invitados, de tal manera que en una mesa se podrán sentar cuatro personas a su alrededor y en dos mesas juntas se podrán sentar seis personas alrededor de ellas, como se muestra en la figura 1.



De acuerdo con lo anterior responde las siguientes preguntas realizando los procedimientos y cálculos correspondientes junto a explicación correspondiente.

The questions asked to the prospective primary education teachers are shown in Table 1.

Table 1

Types and examples of questions

Question type	Sample questions
Close consecutive particular case question³.	A. If Antonia's mom has put together three tables, how many people will be able to sit around them? How do you know it?
Close non-consecutive particular case questions.	B. If Antonia's mom has put together five tables, how many people will be able to sit around them? How do you know it? C. If Antonia's mom has put together ten tables, how many people will be able to sit around them? How do you know it?

³We consider a particular case to be near or far according to the proximity of the number asked, with respect to the initial one.

Question type	Sample questions
Distant non-consecutive particular case questions.	D. If Antonia's mom has put together 50 tables, how many people will be able to sit around them? How do you know it?
General question.	E. Antonia's mom needs to find an exact way how to discover the number of guests that can be seated around any number of tables. How can Antonia's mom find the precise number of guests sitting around any number of tables?

Data analysis

To analyse the data, we defined a series of categories based on the conceptual framework, the research background, and the answers of the prospective teachers to each of the five questions asked in the task. In Table 2, we present the categories that we developed and used in this research.

Table 2

Categories of analysis

Category	Subcategory	Description
Arithmetic thinking strategy (E.1)	Direct (E.1.1)	Numerical or verbal response without justification or inadequate answer.
	Pictorial (E.1.2)	direct Answer with justification based on a visual action of the pictorial representation of the task or through the development of drawings; it grants a numerical answer.
	Proportional (E.1.3)	Numerical or verbal answer based on proportional reasoning, "if in five tables 12 guests are placed, in ten tables 24 guests are placed, twice as many".

Functional thinking strategy (E.2)	Covariational (E.2.1)	Numerical or verbal answer based on the relationship that is established between the values of both variables, “if something increases by one, the other increases by two”.
	Reducible correspondence (E.2.2)	Numerical or verbal answer based on the relationship that is established in the pairs of values of the variables so that one can apply axioms of the real numbers to reach an irreducible representation, “the number of chairs is twice the number of tables, plus six, of which we subtract two tables previously”.
	Irreducible correspondence (E.2.3).	Numerical or verbal answer based on a relationship that is established in the value pairs of the variables, “the number of chairs is twice the number of tables plus two”.
Generalisation (G)		General rule for determining the number of guests that can be placed around an undetermined number of tables”.

We consider as units of analysis the verbal (written), pictorial, numerical, and symbolic ones granted by each of the future teachers in each of the five questions that make up the task. First, we classify each answer of the prospective teachers according to the categories: Arithmetic thinking strategy (E.1) and Functional thinking strategy (E.2). Secondly, and once their answers have been classified, we codify them based on the subcategories that make up E.1 and E.2 (see Table 1) in such a way that we give a value to each answer. Thirdly, once the answers have been coded, we order them in Table 2 and describe the results section. The description of the prospective teachers’ answers is complemented by the notion of representation system. Finally, we

analyse the future teachers' answers to the five questions of the task, determining the trajectories of their strategies.

RESULTS

Below, we show the findings regarding the strategies employed by the prospective teachers in the five questions that make up the task. Subsequently, we describe the trajectory developed in the five questions asked. For the description, we rely on representative examples of the strategies they have employed, along with the systems of representation they have used to justify their answers.

Strategies of prospective primary education teachers

In Table 3, we present a summary of the strategies manifested by the prospective primary education teachers. The teachers are assigned the letter P and a number. For example, P10 corresponds to the prospective teacher number 10. In turn, each strategy is defined by a code that is detailed in the same table.

Table 3

Strategies of prospective primary education teachers

N. prospective teacher	Strategy by question				
	Question A	Question B	Question C	Question D	Question E
P1	E.1.2	E.1.2	E.2.3	E.2.3	E.2.3 ^G
P2	E.1.2	E.1.2	E.2.2	E.2.2	E.2.2 ^G
P3	E.1.2	E.1.1	E.1.2	E.2.3	E.2.3 ^G
P4	E.2.3	E.2.3	E.2.3	E.2.3	E.2.3 ^G
P5	E.2.3	E.2.3	E.2.3	E.2.3	E.2.3 ^G
P6	E.2.1*	E.2.1	E.1.1	E.1.1*	E.1.1*
P7	E.1.2	E.2.2	E.2.2	E.2.2	E.2.2 ^G
P8	E.1.2	E.2.1	E.1.2	E.1.3*	E.1.1*
P9	E.1.2	E.1.2	E.1.3*	E.1.1*	E.1.1*
P10	E.2.2	E.1.2	E.2.2	E.2.2	E.2.2 ^G
P11	E.2.2	E.1.1	E.1.1	E.1.1	E.1.1
P12	E.1.2	E.1.2	E.1.3*	E.1.3*	E.1.1*
P13	E.2.2	E.2.1*	E.2.1*	E.2.1*	E.1.1*

N. prospective teacher	Strategy by question				
	Question A	Question B	Question C	Question D	Question E
P14	E.1.2	E.2.1	E.2.1	E.2.2	E.2.2 ^G
P15	E.1.2	E.2.2	E.2.2	E.2.2	E.2.2 ^G
P16	E.1.2	E.2.2	E.2.2	E.2.2	E.2.2 ^G
P17	E.1.2	E.1.2	E.1.2	E.1.3*	E.1.3* ^G
P18	E.1.2	E.1.2	E.2.2	E.2.2*	E.2.3 ^G

Note: N. P= Prospective teacher's number; E.1.1=Direct; E.1.2= Pictorial direct; E.1.3=Proportional; E.2.1=Covariational; E.2.2=Reducible correspondence; E.2.3= Irreducible correspondence; * Incorrect answer; ^G=Generalisation.

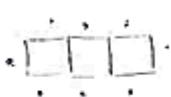
In Table 3, we show the seven strategies found in the prospective teachers' answers in the development of the task. Three of them correspond to the type of strategies focused on arithmetic thinking (E.1.1; E.1.2; E.1.3) and three others based on functional thinking (E.2.1; E.2.2; E.2.3). Below, we show results obtained in each question of the task.

Question A

In this question, related to a particular consecutive case, the most used strategy was the arithmetic thinking type, given that it was used by 12 of the 18 prospective teachers. Moreover, the direct pictorial strategy (E.1.2) was the only one of its kind. The prospective teachers that employed this strategy sometimes gave a numerical answer based on a visual action of the configuration of the task pictorial representation (tables [squares] and people around the tables [circles]). An example of this procedure is when the prospective teachers point out that on the sides of the tables that are not juxtaposed, it is possible to locate people. On other occasions, they produced a pictorial representation so that they continued the configuration of the task representation by drawing the tables (squares) and people (circles) around each of them. Some of the prospective teachers answered using the two previous actions. This case is evidenced in P8's answer (see Figure 5), where we appreciate that he does so by means of the pictorial representation, drawing the tables (juxtaposed squares) and the people around them (black circles). Attached to the above, he answered "8" numerically and justified it through a verbal representation, describing the visual action, in which he indicates that people can sit at the sides of the tables that do not juxtapose.

Figure 5

P8's direct pictorial strategy (E.1.2)

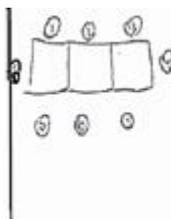


- Al rededor de ellas se podrán sentar 8 personas.
- Realizé un procedimiento mediante lo pictórico llegando al resultado, ya que por cada lado que no interseca con otra mesa cae una persona.

In this question, strategies of the functional thinking type were the least employed, given that six of the 18 prospective teachers employed it. Four of them used the irreducible correspondence strategy (E.2.2) and did so through the visual action of the task representation configuration. This helped them to determine that at the ends of each table, it is only possible to place three people, while at the centres, there are two people (one on the upper side and the other on the lower side of the table). Thus, these prospective teachers answer that around three tables, “8 people” sit together. P10’s answer reflects the above (see Figure 6), in which, on the one hand, the pictorial representation of the three tables (drawings of the tables, juxtaposed squares) and the people around the tables (circles) stands out. Attached to the above, he justified by means of verbal representation. In P10’s answer, we see indications of a regularity when he shows that, to find the number of people that can sit around the three tables, he must only add the number of guests that sit at the tables of the two ends, plus those that sit at the central tables.

Figure 6

P10's reducible correspondence strategy (E.2.2)

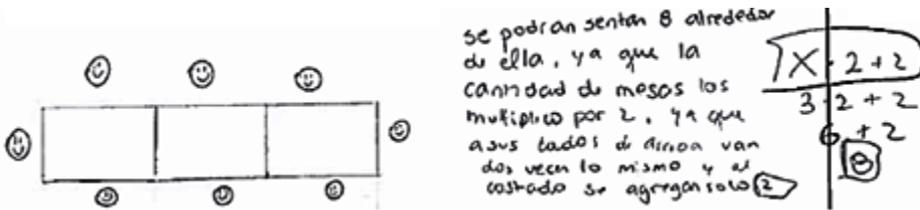


Se podrán sentar 8 personas alrededor de 3 mesas juntas, porque en cada lado de las cuadradas hay una persona exceptuando los que se unen, en este caso son 2 lados unidos, y los extremos serán 3 personas y el medio 2

Meanwhile, the irreducible correspondence strategy (E.2.3) was used by two of the six prospective teachers. They determined that the number of people that sit around three tables can be found by multiplying 3 (relative to the number of tables) by 2 (because two people are located at each table, upper side and lower side) and to the product, another 2 are added (which are the people located at the ends: circles on the left and right of the representation of the tables, see Figure 7). P4's response depicts the description above (see Figure 7). In P4's answer, the different representation systems used are highlighted. He initially used pictorial representation so that he continued setting the task by drawing (juxtaposed squares representing the tables and circles representing the people around the tables). Subsequently, he determined numerically, the number of people that can sit around the tables: "8 around them". In addition, he used verbal representation to explain the procedure followed in the strategy. He used the symbolic representation, which highlights the use of the letter "X" to represent the number of tables and which he subsequently replaced with the number 3 (number of tables considered in the question). Finally, he operates by multiplying 3 by 2 and to which product he adds 2 (people located at each end), thus obtaining eight as an answer.

Figure 7

P4's irreducible functional relationship strategy (E.2.3).



Question B

In this question, related to a particular non-consecutive case, strategies of the arithmetic thinking type (E.1) and functional thinking (E.2) were used coincidentally by nine of the 18 prospective teachers, respectively. In the first strategy, seven prospective teachers used the direct pictorial strategy (E.1.2), and two used the direct strategy (E.1.1). We emphasise that the latter two did not use a justification in their answer, for example, P11 answered: "12 people", and, although it is correct, there is no evidence of how he arrived at that answer.

Of the nine prospective teachers that used the functional thinking strategy (E.2), three used the covariational strategy (E.2.1), four used the reducible correspondence strategy (E.2.2), and two used the irreducible correspondence strategy (E.2.3). We note that the E.2.1 strategy appeared in this question. An example of this is P14's answer given that, faced with the question about how many people can sit at five tables, he answered verbally and numerically: "12 people, since by adding two more tables, it would be two people per side (i.e., four)". Therefore, we noticed that P14 focused his attention on how the change in the value of the independent variable (number of tables) affected the value of the dependent variable (number of people). In this case, the number of tables increased by two and the number of people increased by four.

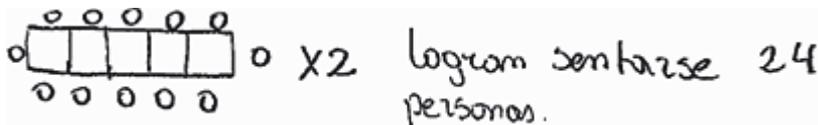
Question C

In this question, related to a particular non-consecutive case, the most used strategy was functional thinking, given that it was used by 11 of the 18 prospective teachers. Six of them used the reducible correspondence strategy (E.2.2), three used the irreducible correspondence strategy (E.2.3), and two used the covariational strategy (E.2.1).

In this question, the strategy based on arithmetic thinking was the least used, given that seven of the 18 prospective teachers used it. Three of them used the pictorial direct response (E.1.2), two used the direct response strategy (E.1.1), and two used the proportional strategy (E.1.3). In Figure 8, we highlight P12's answer, which represents the proportional strategy, although it is incorrect. P12 had previously replied that 12 people could be seated around five tables (Question B); however, when asked how many people could be seated around ten tables (question C), he replied "24 people", that is, he doubled the number of people obtained in the previous question (from 12 to 24 people). This is because the number of tables between Question B and C also doubled (from five to ten tables). We highlight that P12 answered pictorially representing five juxtaposed squares and 12 circles around the tables (in both cases shown in answer to Question B), together with the expression "x2", which represents the action of multiplying by two the number of circles (12). In this way, P12 obtained 24 people as an answer for ten tables together. While the reasoning used is proportional, his response is incorrect because he does not consider the functional relationship involved in the task.

Figure 8

P12's incorrect proportional strategy (E.1.3).



Question D

In this question, related to a particular non-consecutive case, the most used strategy was functional thinking, given that it was used by 12 of the 18 prospective teachers. Among this type of strategy, seven of them used the reducible correspondence strategy (E.2.2), four used the irreducible correspondence strategy (E.2.3), and only one used the covariational strategy (E.2.1).

In this question, the strategy based on arithmetic thinking was the least used, given that six of the 18 prospective teachers used it. Three of them used the direct response (E.1.1) and three used the proportional strategy (E.1.3).

Question E

Table 3 shows that 12 prospective teachers generalised the strategy used, and six did not. Six of them used the reducible correspondence strategy (E.2.1). Five used the irreducible correspondence strategy (E.2.3). Only one used the numerical thinking of proportionality strategy type (E.1.3).

The task proposed in this research sought that, through inductive reasoning, prospective teachers generalised the strategy used. Below, we show concrete examples of their answers regarding how they did so.

In Figure 9, we show P10's generalisation. His answer showed us that he followed the pattern considering that there are three people around each table at the ends, and at each table in the centre, there are two people. We observed that P10 resorts to a symbolic representation to generalise in such a way that he assigned the letter "X" to represent the number of tables. Then, although he does not explicitly say so, he symbolically represents the number of central tables with the expression "X-2". Once he did the above, he matched the representations "X-2" with "X", where both represent the number of centre tables. Finally, P10 concluded that he could find how many people can sit

around any number of tables through the symbolic representation $(X * 2) + 6$, where 6 is the total number of people placed around the tables at both ends.

Figure 9

P10's generalisation of the reducible correspondence strategy (E.2.2).

Al tener la número de mesas, cuenta la del medio contando 2, y luego los extremos. contando 3x mesa
 \downarrow
 $X - 2$ mesa de los extremos
 \downarrow
 $X \cdot 2 \rightarrow n^{\circ}$ personas que siben
 $(X \cdot 2) + 6 \rightarrow n^{\circ}$ personas que siben en las mesa de los extremos.

In Figure 10, we observe P4's answer relative to the generalisation of the irreducible correspondence strategy (E.2.3) performed. Initially, he generalised verbally, describing how to calculate the number of people around any number of tables. Subsequently, he makes use of the symbolic representation " $X * 2 + 2 = ?$ ", where "x" represents the number of tables and the expression "?" represents how many people that can sit around any number of tables.

Figure 10

P4's generalisation of the irreducible correspondence strategy (E.2.3)

\rightarrow teniendo la cantidad de mesas
 \rightarrow lo multiplica por dos, ya que a cada lado van yendo las respectivas personas acorde a la mesa
 \rightarrow y finalmente sumándole 2, ya que son las que están al costado.
 $X \cdot 2 + 2 = ?$ cantidad de persona
 \downarrow
 cantidad de mesa

In Figure 11, we observe the generalisation of the proportional strategy (E.1.3) carried out by P17. To generalise, he relied on the answers to the previous questions (C and D), given that he incorrectly answered: “110 people for 50 tables” (Question D), a result of the multiplication by 5 of the 22 people who correspond to 10 tables (Question C). Based on the above, he used concrete examples that also helped his generalisation but was always supported by a previous amount. This prospective teacher used verbal and numerical representation to express generalisation.

Figure 11

P17’s generalisation of the proportionality strategy (E.1.3)

Si son cantidades pequeñas como 3,5,7, podría dibujarlas y ver la cantidad y ya llegando a números más grandes podríamos usar la multiplicación, como vimos anteriormente teniendo el dato de las 10 mesas podríamos sacar cantidades grandes como 60 mesas($22 \times 6 = 132$), 70 mesas($22 \times 7 = 154$).

Strategies trajectory of prospective primary education teachers

From Table 3, we created Table 4, which brings the trajectories taken by the prospective teachers in the five questions of the task.

Table 4

Trajectory carried out by prospective teachers in the five questions of the task.

Trajectory	Type of strategy (and prospective teachers)
Conservation of the type of strategy	E.1 (P9-P17-P12)
	E.2 (P4-P5-P15)
Change of type of strategy	E.1 to E.2 (P1-P2-P3-P7-P14-P16-P18)
	E.2 to E.1 (P6-P11-P13)
Reiterated change of type of strategy	E.1 to E.2 to E.1 (P8)
	E.2 to E.1 to E.2 (P18)

Note. E.1= Arithmetic thinking strategy; E.2= Functional thinking strategy.

In Table 4, we see that the prospective teachers took three trajectories when they answered the five questions of the task. The first refers to the *Conservation of the type of strategy*, meaning that the prospective teachers maintained the same strategy when solving the five questions posed by the task. The second refers to the *Change of the type of strategy*, when the prospective teachers start with a specific strategy that they subsequently change. Finally, the third refers to the *Reiterated change of the type of strategy*, when the prospective teachers begin with a specific strategy, continue with a different one, and finish with the first strategy or another.

We also noticed that the Change of type of strategy trajectory was the most recurrent, given that we found it in ten of the 18 prospective teachers. We highlight that, of ten prospective teachers, seven changed from an arithmetic thinking strategy (E.1) to a strategy based on functional thinking (E.2). An example of this trajectory is the answers of P1 (see Table 1), given that in Questions A and B, he answered using strategy E.1.2 and in Questions C, D, and E, he answered using strategy E.2.3. In addition, P1 verbally generalised strategy E.2.3, since, for Question E, he answered: “the number of available tables is multiplied by 2 and I must add 2 available spaces in the header of each end”. The other three prospective teachers changed from a strategy based on functional thinking (E.2) to one based on arithmetic thinking (E.1). They initially employed a strategy based on functional thinking. However, as they progressed through the questions, they manifested a strategy based on arithmetic thinking. Such a situation is represented in P13’s (see Table 3) answers. In Questions A, B, C, and D he maintained a strategy based on functional thinking (although with incorrect answer); however, in the generalisation question (E), he employed a strategy based on arithmetic thinking, given that he answered: “We could multiply the tables by the number of people that can fit in [...]”. While P13 answered by multiplication, his answer does not show evidence of a strategy based on functional thinking.

In turn, the Conservation of the type of strategy trajectory was the second most recurrent, given that we evidenced it in six of the 18 prospective teachers. We highlight that three of these prospective teachers retained the type of arithmetic thinking strategy (E.1), and another three retained the functional thinking strategy (E.2).

Finally, we observed that the Reiterated change of the type of strategy trajectory was the least frequent, given that we evidenced it in two prospective teachers (P8 and P18). P8 begins by using an arithmetic thinking strategy (E.1). Then, he resorts to a strategy based on functional thinking (E.2) and then returns to the strategy with which he began (E.1). For his part, P18 begins with a strategy based on functional thinking (E.2), to continue with a strategy based on arithmetic thinking (E.1) and finishes the task by resuming the strategy used initially.

CONCLUSIONS

In this study, we have highlighted the variety of strategies that prospective primary education teachers use to solve a task that involves a functional relationship. Initially, the predominant type of strategy was arithmetic thinking, where the direct response (E.1.1) and the direct pictorial response (E.1.2) were the most used, which reveals the difficulty that this task presents for prospective teachers to establish a functional relationship spontaneously. However, as the questions progressed, most prospective teachers changed their strategies to functional thinking. This may have happened because, by asking for distant cases (Question D), it implies a change to a more effective strategy, focused on a functional relationship. An example of the above is the answers of P3 (see Table 3) as, in Questions A, B, and C, he employed the arithmetic thinking strategy, while in Questions D and E, he employed the functional thinking strategy, which he generalised. In this way, we underscore the importance of the design of the task carried out in this study, in which we base ourselves on the inductive reasoning model of Cañadas and Castro (2007), which suggests that particular cases are intended for generalisation.

In some exceptional cases, prospective teachers that managed to employ the functional thinking strategy from the beginning. P4 (see Table 3) is one of those cases. He began with the functional thinking strategy, maintaining it for the other questions, generalising in a symbolic way. This may be due to this prospective teacher's previous education or his mathematical skills, which allowed him to solve this type of task.

The representation systems helped us describe the answers of the prospective professors participating in this research, although we have not made an exhaustive analysis. However, we can mention that some of the prospective teachers who generalised did so verbally (e.g. P1), while others did

so symbolically (e.g. P4, see Figure 7). This finding contrasts those found by Aké (2021), for which prospective primary education teachers only represented a functional relationship verbally. The representation systems used by prospective primary education teachers will be analysed in future studies.

We emphasise that functional-type strategies were the most used by prospective teachers. Here, the strategy of reducible and irreducible correspondence predominated instead of covariation and was the only one to be generalised, which may indicate that it is more accessible to prospective teachers. It is relevant to mention that, in this study, among the strategies used by prospective teachers, the recurrence relationship was not found, as seen in other studies (Polo-Blanco et al., 2019; Wilkie, 2014). This result may be due to the context that frames the proposed task, which is familiar and close to future teachers, rather than the geometric contexts and function tables presented in previous research tasks. In this sense, a possible way of research is opened to focus on how the task context contributes to promoting functional thinking strategies for prospective primary education teachers.

We found a wide range of prospective teachers' trajectories in the five questions of the task. On the one hand, three prospective teachers kept the Conservation of the type of strategy, which highlights the maintenance of functional thinking strategy. Regarding Change of type of strategy, the majority (seven) initially employed a numerical thinking strategy and subsequently switched to a functional thinking strategy, indicating a modification in their reasoning to approach the task from a functional perspective. On the other hand, some teachers took the trajectory of the Reiterated change of the type of strategy, when one of them (e.g., P8) initially employed the arithmetic thinking strategy, to subsequently change to functional thinking, ending with the strategy used at the beginning. The above accounts for the variability of reasoning on the part of a subject who solves a task with these characteristics. We believe it is important to continue investigating those aspects that make the prospective teachers change strategies.

Recent research shows that students from early educational ages can solve tasks involving algebraic notions, such as those relating to functional thinking (e.g., Blanton et al., 2015; Cañadas et al., 2016; Morales et al., 2016; Pinto & Cañadas, 2018). For this reason, the prospective primary education teacher must be prepared to promote the teaching of algebra through tasks and activities that favour the promotion of functional strategies in students. In this way, we agree with Kieran (2017) on the need to provide opportunities for prospective teachers to promote their algebraic thinking and connect it with the

primary education curriculum. This poses a great challenge to the teacher education: to promote training programs aimed at the prospective primary education teachers to develop professionally so that they can generate significant changes in their practices, which could have a favourable impact on the students' learning (Blanton & Kaput, 2005). For this purpose, the introduction of tasks presented in this study may be appropriate as a way to promote functional thinking in prospective teachers.

AUTHORSHIP CONTRIBUTION STATEMENT

R.M.M. and J.P. conceived this research. Both researchers collected the data and actively participated in the development of the theory, methodology, organization and analysis of the data, and discussion of results.

DATA AVAILABILITY STATEMENT

Data supporting the results of this research will be made available by the corresponding author R.M.M, upon reasonable request.

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