ANALYZING AND SELECTING TASKS FOR MATHEMATICS TEACHING: A HEURISTIC

Pedro Gómez and María José González

In planning units and lessons every day, teachers face the problem of designing a sequence of activities to promote learning. In particular, they are expected to foster the development of learning goals in their students. Based on the idea of learning path of a task, we describe a heuristic procedure to enable teachers to characterize a learning goal in terms of its cognitive requirements and to analyze and select tasks based on this characterization. We then present an example of how a group of future teachers used this heuristic in a preservice teachers training course and discuss its contributions and constraints.

In this chapter, we propose partial answers to one of the main concerns in John Mason's opus: "Teachers set tasks because they believe that working on tasks will promote learning. But what learning might ensue from a given task, and how can tasks be selected to promote certain kinds of learning?" (Mason, 2004, p. 25). Mason's approach to this question is based on the ideas of behaviour, emotion and awareness, ideas that can enable the teacher to design and select tasks by helping him to organize questions to describe an individual's concept-image. We propose a complementary approach based on a heuristic procedure for analyzing the cognitive demands of school mathematics tasks and use this analysis to "psychologis[e] the subject matter" (p. 6). The purpose of the proposed procedure is to provide teachers with tools for analyzing and selecting tasks that promote students' development toward a learning goal. This type of analysis —that provides a method for conceptualizing the kinds of learning in a task— addresses one of Mason's main concerns regarding task interpretation and selection: "Teachers need to interpret tasks designed by other people and need to form views on what the intentions of the author were and what preparation the learners need if they are to benefit from undertaking the task. Teachers also need to be aware of how tasks might be interpreted by learners and know how to support the learners' activity in such a way that the experience can result in appropriate learning." (p. 8)

When a teacher undertakes the planning of a unit, he usually takes into account the curriculum design established for the topic at hand. This design might take different forms, but it generally defines a specific mathematical topic for which a set of learning goals has been established. A learning goal at this level is usually formulated as a statement like the following: "To recognize and use the graphical meaning of the parameters of the symbolic forms of the quadratic function and communicate and justify the results of its use." This statement can be considered one of several learning goals for the teaching and learning of the quadratic function topic in secondary school mathematics. It expresses the learning expectations that the teacher is supposed to have for his students concerning this aspect of the subject matter.

The process of interpreting learning goals in terms of tasks and activities is not self-evident. Two different teachers can develop different interpretations of the same learning goal. Furthermore, a teacher's interpretation of a learning goal makes sense in practice only when he specifies its cognitive complexity through the choice of tasks that are aligned with a specific group of students in a particular context. Interpreting learning goals and setting up students' activities to achieve them thus requires knowledge and competencies that are not necessarily part of the teacher's knowledge base.

In this chapter, we propose a heuristic for analyzing and selecting tasks that can promote students' achievement of a given learning goal for a mathematical topic. "Since asking mathematical questions is vital, both as part of the presentation of mathematics and in the setting of tasks for students, it is appropriate to ask where these questions come from. Questions arise as pedagogic instruments both for engaging students in and assessing students' grasp of, ideas and techniques." (Mason, 2000, p. 97). Inspired by Simon's (1995) idea of hypothetical learning trajectory, we seek to describe a procedure to enable teachers to characterize a learning goal in terms of its cognitive requirements and to analyze and select tasks based on that information. The procedure emphasizes the role in unit planning of the teacher's hypotheses about students' learning and their relation to the mathematical tasks he can propose in order to achieve a specific learning goal.

In what follows, we establish the meaning we give to the notions of learning goal and capacity. We then introduce the idea of learning path as a means for expressing the teacher's hypotheses about how tasks can promote his students' learning. We use these three notions for proposing a heuristic procedure that the teacher can use for analyzing and selecting tasks that can contribute to characterizing his students' development of toward a learning goal. Finally, we discuss some of the contributions and constraints of such a proposed procedure.

Learning Goals and Capacities

Learning goals express expectations of what the students should know and should be able to do as a result of instruction (Farrell, 1988, p. 196). These expectations can be formulated at different levels, from the national government's requirements to lesson planning. On the unit planning level, a learning goal is related to a particular mathematical topic in a specific course. Our interest centres on learning goals that involve "connected knowledge", as described by Mousley (2004), that is, the teacher's expectation that students will make connections between old and new knowledge, mathematical ideas and their representations, and academic concepts and real contexts. Although formulating learning goals is one of the most critical events in the instructional design process (Gagne, Briggs & Wager, 1994), we do not discuss the formulation of learning goals here. We will assume that the teacher has already chosen or has been assigned a specific learning goal. For instance, the Spanish education system gives the following as a learning goal for 16-year-olds studying the topic of surface areas of two and three-dimensional shapes:

> *Learning Goal (LG): To compute the area of figures given in real situations for which the data required in the formula are not directly known.*

There is more than one way in which students can make progress toward a learning goal and construct the corresponding connected knowledge (Mousley, 2004). By selecting tasks, the teacher assigns the learning goal a specific meaning and establishes how he expects students to achieve it. He expects students to use their current knowledge and to establish new connections between that knowledge and the new ideas and techniques involved in the learning goal. Hence, the teacher manages two types of expectations: those expressed in the learning goal and those related to his students' current skills, which the teacher links to the students' concrete performance. We introduce the notion of capacity to capture the second of these expectations levels.

In the context of school mathematics we use the term capacity to indicate an individual's successful performance with respect to a specific assigned task (Gómez, 2007, pp. 64-66). We say that an individual has developed a capacity when he is able to solve the tasks requiring it. For instance, we say that an individual has achieved the capacity for developing plane surfaces when he successfully solves this particular task in a variety of standard situations (developing a prism, a pyramid, a cone). Capacities are specific to individual mathematical topics and are linked to types of tasks and observable student behaviours. The teacher decides whether a given piece of knowledge is a capacity based on his knowledge of the particular group of students to which it refers. Thus, what he considers to be a capacity for one group of students might be a learning goal for a different group of students.

Table 1 shows the 11 capacities identified for learning goal LG by a group of future teachers in an optional methods course implemented at the University of Cantabria during the academic year 2006-2007. The final project for the course was to design and justify a unit for the topic, "Surface areas of two and three-dimensional shapes". This group of future teachers followed the process we are describing here to analyze and select tasks related to a learning goal. We will now use the work they did throughout the course to illustrate that process. The next section describes a procedure that the teacher can use for identifying the capacities associated with a learning goal.

Table 1
Capacities associated with learning goal LG

Id	Capacity
c_1	Recognizing the geometric elements (surfaces, segments, straight lines, etc.) to which a problem refers and drawing them.
c_2	Identifying the known and unknown data for a problem in 2D and 3D drawings.
c_3	Recognizing whether the Pythagorean theorem can be applied and knowing how to apply it.
c_4	Recognizing the properties of similar triangles and knowing how to apply them.
c_5	Knowing how to apply the properties of regular polygons.
c_6	Knowing how to apply the properties of triangles.
c_7	Knowing how to apply the properties of the circle and its circumference.
c_8	Knowing how to apply the formulas for area.
c_9	Transforming units of measurement.
c_{10}	Developing a surface in the plane.
c_{11}	Decomposing and/or completing a surface in order to compute its area.

In the context of planning, we believe that the notion of capacity plays the role of the basic procedural component for characterizing a learning goal. Developing the connected knowledge involved in a learning goal implies the appropriate coordination of a given set of capacities. These are the capacities the teacher associates with the learning goal, the basic skills that the teacher assumes his students already have and on the basis of which they can construct the new knowledge.

Learning Paths of a Task

The teacher can select tasks based on his hypotheses about how those tasks can promote his students' learning. In most cases, the teacher can describe his hypotheses in terms of sequences of capacities. We introduce the idea of the *learning path of a task* as the sequence of capacities that students might put into practice when undertaking it. For instance, for the following task,

> *t_1: Given a right triangle ABC with cathetus of length 8 and 6 units, respectively, we draw a parallel line to the minor cathetus, obtaining a new*

triangle APQ. The hypotenuse and the minor cathetus of triangle APQ are of length 5 and 3 units, respectively. Compute the area of the two triangles,

the teacher might expect his students to identify the geometric elements to which the problem refers, draw those elements, identify the known and unknown data, apply the formulae of area to ABC, apply the Pythagorean Theorem to APQ and apply the formulae for area to this triangle. That is, he might decide that a learning path of the task could be the following sequence of capacities (see Table 1):

$$c_1 \rightarrow c_2 \rightarrow c_8 \rightarrow c_3 \rightarrow c_8$$

A learning path of a task describes the sequence of capacities that the teacher expects his students to execute when facing the task. The teacher might expect his students to bring more than one learning path into play for a given task. For instance, if he considers that his students know how to relate the areas of similar figures, then he might decide that his students might also put into play the following learning path when solving task t_1:

$$c_1 \rightarrow c_2 \rightarrow c_8 \rightarrow c_4$$

The teacher can use the notion of learning path for analyzing and selecting tasks that can contribute to his students' development of the learning goal. In what follows we suggest a procedure that can give the teacher clues on how to do so.

Analyzing and Selecting Tasks

We assume that a learning goal like LG has been assigned, for which the teacher wishes to select a sequence of tasks. In a first interpretation of the learning goal, the teacher can identify and delimit the mathematical content to which it refers. The teacher can produce a preliminary list of capacities related to the learning goal by performing a subject matter analysis of the topic at hand. Subject matter analysis is a procedure for analyzing a school mathematics topic in terms of the concepts and procedures involved in the topic, the multiple ways in which the topic can be represented, the phenomena that are related to the topic, and the relationships among these elements. The end result of the subject matter analysis can be expressed in concept maps that depict the variety of meanings that the topic acquires in school mathematics (Gómez, 2007, pp. 36-56). Subject matter analysis resembles "conceptual analysis" (Thompson, 2008) in the sense that it can be used in "building models of what students actually know at some specific time and what they comprehend in specific situations" (p. 1-46). It thus provides information for identifying a list of capacities that might be associated with a given learning goal. However, these capacities must be linked to the teacher's expectations and hypotheses concerning the learning of a particular group of students undertaking a set of tasks.

Nowadays, teachers have access to a great variety of tasks for almost any topic (e.g., from experience, textbooks or the web). The issue is not to design new tasks, but to select or adapt those available to achieve the tasks most appropriate for the purpose at hand. The teacher thus starts the second phase of the process with initial sets of tasks T_i and capacities C_i. From T_i, he then chooses a task that he considers relevant and produces its learning path on the basis of the capacities in C_i. This learning path is a way of using his knowledge of his students to describe how he thinks that they would undertake the given task. In this process, the teacher might realize that the set C_i must be enlarged with new capacities or that some capacities in C_i must be formulated differently. For instance, when considering a task like the following,

t_2: We want to line the lampshade of table lamp in the form of a truncated cone. The small and large circumferences measure 30 and 60 centimetres, respectively. The height of the lampshade measures 20 centimetres. How much tissue do we need?,

the teacher might realize that his students ought to know how to perform developments in the plane or to identify situations in which the Pythagorean Theorem can be used. The teacher might have overlooked these capacities in his subject matter analysis of the topic. In this case, he can improve this analysis and enlarge the set C_i of capacities associated with the learning goal. Thus, task analysis in terms of learning paths can contribute to the teacher's identification of the capacities that characterize the learning goal. The teacher also analyzes a task based on the capacities he has identified. Analyzing the learning path of a task can lead the teacher to modify the task in order to include or eliminate a given capacity because he considers that it will not induce his students to bring into play the sequence of capacities he is interested in. For instance, when analyzing the task

t_3: We want to tile the classroom floor. How many 30×40-floor tiles are needed to cover it?,

he might decide not to choose this task because he thinks that the context will allow his students to solve the problem by directly counting the floor tiles (without using any formula). The teacher can then perform a cyclic process of refining the list of capacities and selecting tasks. The cycle ends when the teacher decides that he has obtained an appropriate set of tasks T_m and C_m: he expects the tasks in T_m to induce his students to establish links among capacities in C_m, thereby promoting development toward the learning goal.

The set of learning paths of a set of tasks T_m can be depicted on a graph. We illustrate the production of this graph with the work of the group of future teachers who proposed the list of capacities in Table 1 for the learning goal LG. They also proposed ten tasks related to these capacities and this learning goal. Table 2 shows two examples of the tasks they selected and their corresponding learning paths.

Table 2
Examples of tasks and their corresponding learning paths

Example 1
A goat is grazing in a hexagonal area that is inscribed in a circumference with a radius of 10 meters. The goat is tethered to one vertex of the hexagon. How much land can the goat graze on if the rope with which it is tethered measures the radius of the circumference?
$c_1 \rightarrow c_5 \rightarrow c_2 \rightarrow c_{11} \rightarrow c_7 \rightarrow c_8$
Example 2
Compute the amount of tissue paper needed for constructing a kite formed of two sticks of length 75 and 50 centimetres, so that the short stick crosses the long one at 25 centimetres from one of its ends.
$c_1 \rightarrow c_2 \rightarrow c_8 \rightarrow c_9$

At this stage of their work, the group of future teachers produced the graph shown in Figure 1 for the learning paths of the ten tasks they selected. The graph shows the links between the capacities involved in those learning paths. The number of times that a link appears in the learning paths is indicated in parentheses. If there is no number, the corresponding link is brought into play in only one learning path. The capacities that represent the beginning of a learning path are indicated by a circle, those that represent the end of a learning path with a square. The number in square brackets indicates the number of learning paths for which this is the case.

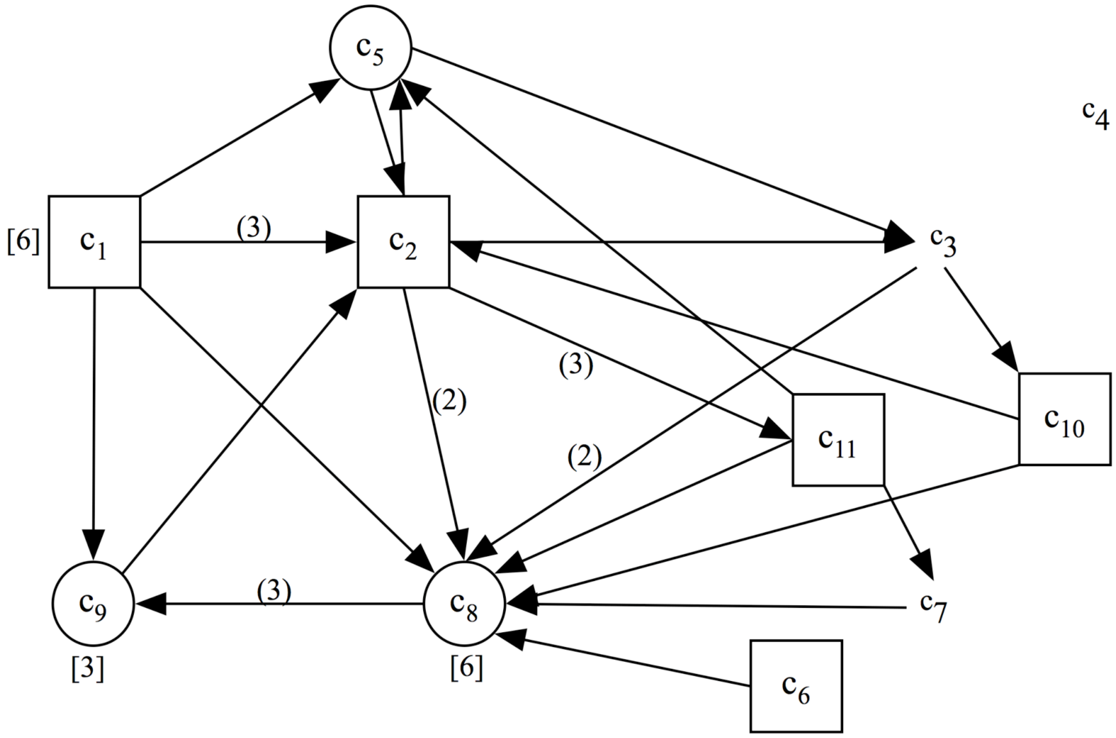

Figure 1. Graph of learning paths for the tasks selected related to LG

Deciding whether the set T_m is appropriate for the learning goal involves a final step: a global analysis of the tasks in T_m. This analysis might produce information that cannot be obtained in the individual analysis of each task. For instance, when considering the complete set of learning paths that he expects his students to bring into play when solving the tasks in T_m, the teacher might realize that some links between capacities would seldom be used, whereas others would be executed frequently; that many tasks have the same starting or ending point; or that some capacities are never brought into play. This global analysis of the learning paths of the tasks in T_m is crucial. It shows the teacher how he is interpreting the learning goal in cognitive terms. It uses his students' cognitive performance to describe how he expects to promote development of the learning goal. It shows which cognitive requirements he is stressing in the tasks in T_m and which he is not. On the basis of this analysis, the teacher might decide to reconsider his choice of tasks in T_m and of capacities in C_m as a way of characterizing the learning goal. He might rule out some tasks in T_m and select new ones or even refine his choice of capacities in C_m. We illustrate this analysis using the graph in Figure 1.

The graph shows that capacity c_4 is not brought into play by any of the learning paths of the selected tasks, that most learning paths begin with capacity c_1 and end with capacity c_8, and that the links $c_1 \rightarrow c_2$, $c_2 \rightarrow c_{11}$, and $c_8 \rightarrow c_9$ are given greater emphasis. This type of analysis can provide the teacher with information about his interpretation of the learning goal. In the case of this example, it might show him the importance he gives to those three links. This information might induce him to modify the list of capacities he has chosen (e.g., eliminating capacity c_4 in the example above) and the set of tasks he has selected (reducing the emphasis on some links and increasing emphasis on others). The process of capacity identification and task selection ends when, at a given point in this cycle, the teacher considers that the resulting graph provides a good representation of his interpretation of the learning goal in terms of final sets of tasks (T_f) and capacities (C_f), that is, when he considers that the tasks selected are the best possible choice for promoting his students' learning. He can take this decision on the basis of the cognitive requirements of the tasks: the sequence of capacities that he expects his students to bring into play when undertaking the tasks.

Learning Paths of Tasks: Contributions and Constraints

"In a sense, all teaching comes down to constructing tasks for students…" (Mason, 2002, p. 105). We claim that the issue is not only to construct tasks but, from a set of tasks already available, to analyze and then select and adapt the tasks that, in the teacher's judgement, can best contribute to his students' attainment of a learning goal. Given that the teacher has a great variety of available tasks related to the learning goal, which sequence of tasks is the most appropriate to his context? The procedure we have presented provides information for making such a decision. This procedure is based on the assumption that the teacher knows the context in which instruction will take place. In particular, we assume that students' performance when solving tasks follows some patterns and that the teacher can predict them. Starting with the teacher's problem of selecting tasks that can promote his students' development of a learning goal, we have introduced the notion of learning path as a way of relating the teacher's predictions of his students' previous knowledge (the capacities), his expectations concerning the new knowledge they can construct (the learning goal), and the means with which he expects to promote such learning (the tasks). We believe that this notion and the heuristic we have suggested, enables the teacher to give a precise meaning (his meaning) to the learning goal at hand. The teacher expresses the meaning of the learning goal he assigns through the set of tasks that he believes will promote his students' attainment of it. This constitutes his prediction of how learning can evolve, based on his students' prior knowledge and the actions that he expects them to perform when solving the tasks.

In a constructivist setting, Simon and Tzur (2004) have suggested that students develop new knowledge as a consequence of iterations of activities linked to their effects. Elaborating Piaget's notion of reflective abstraction, they argue that, when students solve tasks, they create records of experience, sort and compare records, and identify patterns in those records. This mechanism provides a means for interpreting how students develop new concepts based on other, pre-existing ones. It is possible to interpret the links among capacities that are made explicit through the tasks' learning paths as the links that characterize the new connected knowledge expected in the learning goal. What is relevant when solving the tasks is the way the students might connect the capacities involved in order to arrive at a solution to the tasks and whether they become conscious of such connections. Students might develop the sequence of capacities brought into play when solving a set of tasks into a technique that can be used for solving other tasks, whether similar or not. They might do this, as Simon and Tzur suggest, by linking activities (performing sequences of capacities) and their effects (task solving). Achieving a learning goal means, among other things, becoming proficient in the recognition, selection and use of such techniques, so that what is a learning goal can now become, for another teacher and at a later time, a capacity that can contribute to the characterization of a different learning goal.

The proposed heuristic for task selection might offer a partial solution to the planning paradox (Ainley, Pratt & Hansen, 2006). According to the planning paradox, "if teachers plan from objectives, the tasks they set are likely to be unrewarding for the pupils and mathematically impoverished. Planning from tasks may increase pupils' engagement but their activity is likely to be unfocused and learning difficult to assess."

(p. 23) By linking a learning goal to the tasks that can promote its attainment, the idea of the learning path can enable the teacher to see planning from objectives and planning from tasks as one coherent activity. We have assumed that most teachers are constrained to plan from objectives: they are expected to promote their students' attainment of learning goals that are set in advance. Yet this does not necessarily mean that the teacher should set unrewarding tasks. Based on his knowledge and prediction of his students' learning, the teacher can select mathematically powerful tasks that promote their development of connected knowledge. We have shown that the teacher can approach planning as a process that, for a given learning goal and a particular group of students, involves a cyclic revision of capacities and tasks. Furthermore, the idea of learning path and its corresponding heuristic provides the teacher with information about the cognitive links that can be fostered by the selected tasks. In practice, when interacting with his students, he can use this information for engaging them, adapting his decisions to their actions and assessing their performance.

Mathematics teachers' practice is far more complex than the description we have given in this chapter. We have simplified teachers' practice in order to focus attention on the specific steps that shape the proposed procedure. However, the origin and formulation of some learning goals might make them inappropriate for analysis with the tools proposed. Our concern here is with learning goals that are specific to a mathematical topic, precise enough to be worked on in a few lessons, and able to be formulated in terms of connected knowledge. The procedure assumes that the teacher has sufficient knowledge of his students to be able ultimately to determine the capacities that characterize the learning goal. Nevertheless, the teacher's assumptions might not be valid. He might find that his students have not developed one or more of the capacities he assumed they had. He will then need to change his planning in order to make sure that his students have the prior knowledge needed for reaching the learning goal. Further, planning involves other activities besides task selection as presented in this chapter. For instance, tasks can be classified in several ways (e.g., introduction tasks, motivation tasks, assessment tasks). That we have not taken those activities into account does not mean that we do not consider them relevant.

We do not expect practicing teachers to perform the learning path heuristic in detail. This is activity requires more time than that usually available to the teacher. However, we believe that this procedure can be useful in teacher training. In such a setting, pre-service and in-service teachers can follow the heuristic and develop their competencies for performing it. Such an experience can then help the teacher in practice when planning lessons and units. It can enable him to interpret learning goals, give them a specific meaning adapted to his students, identify the cognitive requirements of the tasks available to him, and make explicit his expectations concerning his students' learning.

Acknowledgements

This work was partially supported by Project SEJ2005-07364/EDUC of the Spanish Ministry of Science and Technology.

References

Ainley, J., Pratt, D., & Hansen, A. (2006). Connecting engagement and focus in pedagogic task design. *British Educational Research Journal, 32*(1), 23-38.

Farrell. (1988). *Secondary mathematics instruction. An integrated approach*. Dedham: Janson.

Gagne, R. M., Briggs, L. J., & Wager, W. W. (1994). *Principles of instructional design*. New York: Holt Rinehart and Winston.

Gómez, P. (2007). *Desarrollo del conocimiento didáctico en un plan de formación inicial de profesores de matemáticas de secundaria*. Granada: Departamento de Didáctica de la Matemática, Universidad de Granada.

Mason, J. (2000). Asking mathematical questions mathematically. *International Journal of Mathematical Education in Science and Technology*, *31*(1), 97-111.

Mason, J. (2002). *Mathematics Teaching Practice: a guide for university and college lecturers*. Chichester: Horwood Publishing.

Mason, J., & Johnston-Wilder, S. (2004). *Designing and using mathematical tasks*. Milton Keynes: Open University.

Mousley, J. (2004). An aspect of mathematical understanding: The notion of "connected knowing". In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th International Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 377-384). Bergen: Bergen University College.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114-145.

Simon, M., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: an elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, 6(2), 91-104.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sepúlveda (Eds.), *Joint Meeting of the International Group for the Psychology of Mathematics Education (IGPME 32) and North American Chapter (PME-NA XXX)* (Vol. 1, pp. 31-49). Morelia: Cinvestav-UMSNH.