Difficulties and Errors in Number Reasoning Development.

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Abstract *The detailed study of difficulties and errors in young learner comprehension is a relevant and productive research field in Mathematics Education. Studies in the field are numerous although somewhat too varied. The present paper is suggesting methodological perspectives and principles applying to the field of research; we also show an example with school work. The use of figurate numbers as a representation system gives richer conceptual values, boosts visual reasoning and facilitates learner understanding.*

Introduction

In school mathematics, the wording of questions and problems for students is normally very clearly defined. They always have a correct answer, and any other alternative is readily perceived as inadequate. Correct, incorrect or no answer are the three possible categories. So, whenever there is an incorrect answer to a mathematical problem, it is normally said to be an erroneous answer, and the solution an error in relation to the problem to be solved (Radatz; 1980). Errors are a real part of students production and have always been studied by mathematics education researchers (Buswell & Judd 1925; Brownell 1941; Kilpatrick 1992). Errors constitute objective information found in learning and teaching mathematical processes, and they point out to shortcomings and even failures in the learning process (Brousseau, Davis & Werner 1986).

Error analysis and diagnosis have been under the influence of the predominant ideas in pedagogy and psychology, and have likewise been determined by specific curricular design in different educational systems. The pejorative connotation of the word "error" has undoubtely played a negative role in these studies. On the other hand, some other epistemological positions have positively valued errors in knowledge acquisition (Popper 1979; Bachelard 1988; Lakatos 1978).

Rational Criticism has pointed out that there is no ultimate source of knowledge and has acknowledged errors as a constituent part in knowledge acquisition. Criticism is necessary, and specially the critical search for errors to overcome our epistemological deficiencies is paramount. Consecuently, errors may positively contribute to learning processes. Besides, errors appear within conceptual frameworks based on already existing and consistent knowledge, and do not come up by mere chance. Therefore, it seems wise for instructional theories to replace the tendency to blame students for their errors by a theory which anticipates and includes these in learning processes. Finally, all instructional processes potentially generate errors for different reasons.

A young learner may realize from his/her own errors properties of a given concept of which he/she was not previously aware. *An error made* is a sign of the learner's incomplete knowledge and enables fellow students and teachers to complete this knowledge. In some other instances, it is the student him/herself who incorporates this knowledge. The positive effect of errors in learning processes is the consequence of the epistemological reflection (Bouvier 1987; Nesher 1987).

Fields of research on mathematics learning errors.

We will be considering recent research on mathematics learning errors under four general groups.

A) ***Error analysis and their causes. Taxonomies and error classifications*** (Radatz 1979). Research in the field bases in a psychological or psychopedagogical theory which provides for a explicative framework. Error analysis also offers an adequate area to increase the empirical content of the theory (Davis 1984; Moshovitz-Hadar, Zavslaski & Imbar 1987; Hiebert & Carpenter 1992). We are including in this group epistemological and purely mathematics theorical research that aims at establishing structural causes of errors originating from mathematical knowledge alone (González 1993); studies on obstacles are an example (Brousseau 1989; Sierpinska 1990).

B) ***Curricular design and mathematics learning errors.***

In this second group we are including studies considering errors as relevant in the design and development of the mathematics curriculum (Booth 1984; Hart 1984; Kerslake 1986). Diagnosis teaching is included in this group: research is being specifically carried out in error anticipation, detection and correction strategies (Bell 1986; Brekke 1991). Other studies perceive errors as a starting point for mathematical content development (Borasi 1986, 1987). Finally, studies on testing and the implications of errors in pupil assessment are also listed here.

C) ***Errors and mathematics teacher training.***

These are studies on the role played by observation, analysis, interpretation and treatment of learner's errors in teacher training. Research on the field has been quite recent and somewhat restricted (Graeber, Tirosh & Glover 1989; Tirosh & Graeber 1989, 1990).

D) ***Implementation of an error analysis techniique.***

We are mentioning some purely technical studies (Mulhern 1989). The right/wrong character of learner's production supports the systematic use of statistical procedures. A considerable number of psychometrical studies deal with errors in learning. Technique in analysing error cause hypothesis are likewise listed here (Resnick, Nesher, Leonard, Magone, Omanson & Polet 1989)

Error analysis and Number Thinking

Psychometrical studies have been devoted to errors and difficulties in Arithmetic ever since the first studies in the field, although the four groups listed above are not restriced to error analysis of numerical knowledge. Four groupings are usually employed to organize different error analysis methods in:

1.- Count the number of wrong answers to a variety of problems; Similar to psychometrical method, the diagnosis value is a limited one.

2.- Analysis of error type: the normal procedure is to classify different error types, and then analyse how much they deviate from the right answer. Inferences about factors causing the deviations follow.

3.- Analysis of error pattern: such an analysis unveils systematic errors that normally result from incorrect comprehension. When tasks change, an error pattern becomes apparent thus showing relevant clues on the strategies employed.

4.- Task sequences leading to error production: an error pattern is considered, hypothesis on possible causes are drawn up, and a task is designed which should lead to similar errors being produced.

Systematic errors are the outcome of wrong premises. This is termed as defective comprehension. It is not an easy task to follow learners' reasoning. Very often, a defective comprehension leads to consistent and systematic errors. However, most case studies do not explain the causes nor offer alternatives to overcome them. They too often are limited to frequency and error classifications.

A more general analysis can be reached by close observation of the learners' deepest representation levels, by a closer scrutiny of the evolution of meanings that are at the base of a child´s performance. A whole range of errors can be explained at the deepest level of a defective comprehension. The criterion to interprete and solve errors is here termed defective comprehension.

Number sequence comprehension.

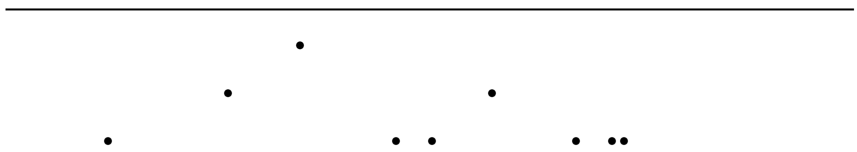
Here follows a working session carried out in a group of 36 eighth year E.G.B. students (13-14 year olds) during the academic year 92-93.

The session was in fact a normal classroom activity, and it was the fourth in a row dealing with representation of number sequence. In previous sessions, students had work upon number representation with dot configurations, had compared them and concluded that some shared an identical representation pattern. They had studied several arithmetical development of a given number and had expressed them in a dot configuration. Some time was devoted to translating dot configurations into arithmetical development. Lineal and squared number sequences had also been worked on.

The geometric representation of the first terms of a sequence was given. Students were used to giving the number translation of every single one of the terms and the arithmetical development of the sequence. They also had a grasp of the meaninng of the nth expression in the representation. A task including all these values was given to the students (Castro, 1994).

Task: Here follows the first three numbers of a dot sequence.

1st 2nd 3rd



* Draw the next number.

* Draw the shape the nth place would have.

* Write the representing number under each shape.

* Write out the development underneath each number.

* Now give me the name of these numbers.

Description of the working session in the class.

The teacher presents the task on the board. Simultaneously, she has the following conversation with the students:

TQ.- What is the 4th term like?

SA.- 4 vertical and 4 horizontal dots. One common.

TQ.- And the 7th?

SA.- The same. But it has 7 dots.

TQ.- And what about the nth?

SA.- The same. With n dots.

Representations are drawn on board following the same order.

I.- now, we write the numbers under the representations.

The teacher writes 1,3,5,7,... in the third row.

TQ.- What do we write in the nth place?

SA.- "n"

TQ.- Not just that. It wouldn't be correct.

Some students suggest 2n -1; some others n + (n-1). Some others believe both are the same.

TA.- If the two expressions are the same, please tell me which one you think it is more appropiate to represent the dot number of the shape in the nth place. Answer individually.

SA.- I think it is n + (n-1). We just agreeded it was 2 + (2-1) for number 3.

OI.- Some students strongly say *they do not agree because that was the number development* . For that reason, they do not think it to be the best representation. they believe "2n - 1" is a better one, as *it is pretty clear that the vertical and the horizontal lines has the same number of dots, and you have to drop one dot because they share it.*

SA.- I think it is n, *because what we have to represent now is the number and not its development.* Those expressions are used for the number's development.

The students are missing a simple expression, like 3 or 5, to express the number, so that they do not have to resort to a polynomial expression, which they link to development. There is a very clear understanding difficulty: we are asking for the nth number, and accordingly, they would rather answer with a single answer. Many students feel akward with the expression *of a relational structure that determines the general term*. When such a structure comes up, the expression is that of the development of the general term, *but it is not the general term*

TA.- If we say "n", we are just stating the place it is in the sequence. That is not correct, because the place does not coincide with the number of dots. So, in the third place there are not 3 dots, but 5, and so on.

Two students are discussing the matter, and they are asked to repeat their discussion to the whole clas, so that the other students can give their opinions.

T.- Please, tell us what you are saying.

SA.- I am saying both expressions are equally valid, n + (n-1) and 2n -1, because both are the same and both express the number of dots in the nth number. It is not possible to write n, because by so doing, we would be giving the position of each number.

SA.- I do not agree with her. I want to give a simple expression and not a development.

TA.- It is not possible to give the dot number at the nth number by means of a simple expression. We have got to either choose n + (n-1), thus clearly saying there are two rows, one of which has got a dot less; and 2n-1, that seems to say the two lines are the same and that they share one dot.

SA.- Then, I could say n + (n+1), as in this expression we are saying there are two rows, one of which has got one dot more.

TA.- Just think about it, and say whether it is correct or not.

SA.- Oh, yes. It is. Because there always will be one more dot in the row than in the column.

SA.- I do not agree. That does not coincide with what we have in the first numbers.

It is then when the whole class give their opinions, taking sides for each girl. They are all talking at the same time, and they all think they are right.

SA.- I think both of them are right. We have just said that the nth place represents all numbers. So, it doesn't matter which expression we choose.

Students carry on arguing. Teacher tries to get a bit of silence.

T.- One of you is going to tell me what you are saying.

SA.- We are supposed to say what the nth place is. So, I think it is n +(n-1). If we have a look at the other places, we have this: the second number is 3 ,that is, 2 +(2-1); the third is 5, 3 +(3-1).

TQ.- And what is the first place like?

SA.- 1 +(1-1)

SA.- I think it is n +(n-1), as say, the 4th place is 4 +(4-1).

SA.- That is the same as 2n - 1.

SA.-The nth place represents all the different numbers. I think we just have to note down n.

TQ.- So, if we note down -n, do you think we express the number of dots there are?

Some students keep saying no, and the same reasoning repeats.

A.- Let's think again on the information we've got: n stands for the number position; the number of dots is given by one of these two expressions: 2n -1, or n + (n-1). In fact, both indicate the same number of dots. The only difference between both lies in the development we are considering for the numbers.

If we consider: 1 +(1-1); 2 +(2-1); 3 +(3-2);...the expression is n +(n-1); if we consider 2.1-1; 2.2-1. 2.3-1,... the expression is 2n-1.

TQ.- I repeat the question: which of the two expressions suits better the dot sequence?

SA.- I think both are the same.

SA.- Both.

SA.- As the two rows have the same dot number, and they share one dot...it is 2n -1.

SA.- We will always have a row with one more dot than the column.
So, I think it is n +(n-1).

Analysis.

We have been considering the general term as occupying the nth position in a sucesion. The display is apparently simple: two same dot number rows, with a shared dot. How do we represent that number? Two are the options: some students propose "n", while some others note down either n + (n-1) or 2n -1. What is relevant here is that the two last expressions do not equal a number but a number development that is expressing a relational structure common to all the sequence numbers.

A conclusion must be drawn that *there is no number expression of the general term, but a type of number development for the general term*. When students are requested to give the "general term in a sequence" quite a few of them understand "any number", but not the expression of a general number sequence development.

A deficient expression is greatly impeding the understanding of the question. The teacher is not well aware that there are two equivalent symbolic systems: numbers and arithmetical developments. The teacher considers the two as the same thing, which is not the case of the students because they do not have that information.

The "n" value for the general term in the sequence is an additional difficulty : are n +(n-1) and (n+1) + n the same?

Conclusion.

The mere statistical analysis of the learner incorrect answers in classroom activities, and the relations between them have been the only direction in error research for quite a long time. We believe that such studies are relevant and necessary in a first stage in error and difficulty research. Our research is trying to design learning situations of mathematical concepts that may promote the student skills to enable them to express and critically discuss in class the degree of their conceptual understanding. The elements that make possible the understanding of a given conceptual field can be determined by the analysis of the expression's inconsistencies and limitations. Action-research, class discussions, case studies and clinical interviews have been the methodological strategies aplied in our research.

There are some conceptual elements in Number Thinking that are not much used. Figurate number representation are, as it has been in the case studied, a symbolic representation system different from the decimal number system. This representation shows certain number properties that are not otherwise evidently represented. The use of new representation systems that may boost learner visual comprehension constitutes an important aspect in our research field.

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