The generalized tanh method to obtain exact solutions of nonlinear partial differential equation

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Abstract

In this paper, we present the generalized tanh method to obtain exact solutions of nonlinear partial differential equations, and we obtain solitons and exact solutions of some important equations of the mathematical physics.

Key words: Nonlinear differential equation; Travelling Wave Solution; Generalized Extended tanh Method; Symbolic Computation; Mathematica.

Introduction

The search of exact solutions of partial differential equations is of great importance, because these equations appear in complex physics phenomena, mechanics, chemistry, biology and engineering. A variety of powerful and direct methods have been developed in this direction. In this paper, we consider the *generalized tanh method* [1]. In particular cases we apply the mentioned method to obtain exact solution of some important equations such that: a reaction–diffusion equation, double sine–Gordon equation, and the $(2+1)$ dimensional sine–Gordon equation.

1. The generalized tanh method

The simplest classes of exact solutions described by ordinary differential equations involve *travelling–wave* solutions. *The tavelling–wave* solutions have by definition the form

$$u(x, y, \ldots t) = v(\xi), \quad \xi = \lambda_1 x + \lambda_2 y + \ldots + \lambda_n t. \tag{1.1}$$

where λ_i $(i = 1, 2, \ldots, n)$ are constants. The travelling–wave solutions occur for equations that do not explicitly involve independent variables,

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0 \tag{1.2}$$

Substituting (1.1) into (1.2), we obtain an ordinary differential equation for the function $v(\xi)$:

$$P(v, v', v'', \ldots) = 0 \tag{1.3}$$

The next crucial step is to introduce a new variable $\phi = \phi(\xi)$ which is a solution of the Riccati equation

$$\phi' = \phi^2 + k \tag{1.4}$$

whose solutions are given by

$$\phi(\xi) = \begin{cases} -\dfrac{1}{\xi}, & k = 0 \\ \sqrt{k}\tan(\sqrt{k}\xi - c) & k > 0 \\ -\sqrt{k}\cot(\sqrt{k}\xi - c) & k > 0 \\ -\sqrt{-k}\tanh(\sqrt{-k}\xi - c) & k < 0 \\ -\sqrt{-k}\coth(\sqrt{-k}\xi - c) & k < 0 \end{cases} \tag{1.5}$$

The generalized tanh method can be described as follows:

- Step 1. Introduce the transformation $u = v(\xi)$ with $\xi = x + \lambda t$, or $\xi = x + \gamma y + \lambda t$, which transforms (1.2) into (1.3).
- Step 2. We seek solutions of (1,3) in the form

  $$u(x,t) = v(\xi) = \sum_{i=0}^{m} a_i \phi^i, \tag{1.6}$$

  where $\phi = \phi(\xi)$ is a solution of (1.4). The exponent m must be determined before the a_i can be computed. Substituting $v(\xi)$ in (1.3) the coefficients of every power of ϕ must vanish. In particular, the highest degree term must vanish. We obtain m.
- Step 3. To generate the system for the unknown coefficients a_i and parameters λ, γ and k, substitute $v(\xi)$ in (1.3) and use the relation (1.4).
- Step 4. The most difficult step of the method is to analyze and solve the algebraic system.
- Step 5. Substituting the solution from step 4 in $v(\xi)$ and reversing step 1. We obtain the explicit solutions in the original variables.

1.1. Exact solutions for the double sine-Gordon equation

This is the equation

$$u_{xt} = \sin u + \sin 2u. \tag{1.7}$$

First introduce the transformations

$$\sin u = \frac{V - V^{-1}}{2i},\ \sin 2u = \frac{V^2 - V^{-2}}{2i},\ V = e^{iu}, \tag{1.8}$$

after which we obtain the equation

$$2V\ \ V_{xt} - 2V_xV_t - V^4 - V^3 + V + 1 = 0. \tag{1.9}$$

The substitution $V = v(\xi) = v(x + \lambda t)$ in (1,9) gives us the equation

$$2\lambda v v'' - 2\lambda(v')^2 - v^4 - v^3 + v + 1 = 0. \tag{1.10}$$

We seek solution of (1.10) in the form

$$v(\xi) = \sum_{i=0}^{m} a_i \phi^i, \tag{1.11}$$

where $\phi = \phi(\xi)$ is solution of (1.4). Substituting (1.11) and (1.4) into (1.10) and balancing $v\,v''$ with v^4 we obtain $m = 1$. Therefore

$$v = a_0 + a_1\varphi \tag{1.12}$$

Substituting (1.12) and (1.4) into (1.10) and equaling the coefficients of $\phi^i (i = 0, 1, \ldots)$ to zero we get the system

$$\begin{cases} 1 + a_0 - a_3^0 - a_0^4 - 2k^2\lambda a_1^2 = 0. \\ a_1 + 4a_0a_1\lambda k - 3a_0^2a_1 - 4a_0^3 = 0. \\ -3a_0a_1^2 - 6a_0^2a_1^2 = 0. \\ 4a_0a_1\lambda - a_1^3 - 4a_0a_1^3 = 0. \\ 2a_1^2\lambda - a_1^4 = 0. \end{cases} \tag{1.13}$$

We obtain the following solutions of (1.13):

$$a_0 = -\frac{1}{2}\quad, \lambda = \frac{1}{2}a_1^2, \quad k = \frac{3}{4a_1^2},$$

where $a_1 \neq 0$ is an arbitrary constant. Since $k > 0$, according to (1.5) we get

$$u_1(x,t) = \arccos\left(\frac{5 + 3\cot^2\theta + 2\sqrt{3}\cot\theta}{4(1 + \sqrt{3}\cot\theta)}\right) \quad (k > 0)$$

$$u_2(x,t) = \arccos\left(\frac{5 - 2\sqrt{3}\tan\theta + 3\tan^2\theta}{4(-1 + \sqrt{3}\tan\theta)}\right) \quad (k > 0)$$

$$\theta = \sqrt{\frac{3}{4a_1^2}}\xi = \sqrt{\frac{3}{4a_1^2}}\left(x + \frac{a_1^2}{2}t\right).$$

The solution $u_1(x,t)$ is not considered in [1].

1.2. Solutions for the (2+1)-dimensional sine-Gordon equation

This is the equation

$$u_{tt} - u_{xx} - u_{yy} + m^2 \operatorname{sen} u = 0. \tag{1.14}$$

First we introduce the transformations

$$V = e^{iu}, V(x,\, y,\, t) = v(\xi), \xi = x + \gamma y + \lambda t. \tag{1.15}$$

We get the required form

$$2(\lambda^2 - \gamma^2 - 1)(vv'' + v'^2) + (v^3 - v) = 0. \tag{1.16}$$

Balancing vv'' and v^3 in (1,16), we obtain

$$v = a_0 + a_1\phi + a_2\phi^2$$

The algebraic system is

$$\begin{cases} -(m^2a_0) + m^2{a_0}^3 + 2k^2{a_1}^2 + 2\gamma^2k^2{a_1}^2 - 2k^2\lambda^2{a_1}^2 - 4k^2a_0a_2 - 4\lambda^2k^2a_0a_1 + 4k^2\lambda^2a_0a_2 = 0 \\ -(m^2a_1) - 4ka_0a_1 - 4\lambda^2ka_0a_1 + 4k\lambda^2a_0a_1 + 3m^2a_0a_1 + 4k^2a_1a_2 + 4\gamma^2k^2a_1a_2 - 4k^2\lambda^2a_1a_2 = 0 \\ -4a_0a_1 - 4\gamma^2a_0a_1 + 4\lambda^2a_0a_1 + m^2a_1^3 - 4ka_1a_2 - 4\gamma^2ka_1a_2 + 4k\lambda^2a_1a_2 + 6m^2a_0a_1a_2 = 0 \\ 3m^2a_0{a_1}^2 - m^2a_2 - 16ka_0a_2 - 16\gamma^2ka_0a_2 + 16k\lambda^2a_0a_2 + 3m^2{a_0}^2a_2 + 4k^2{a_2}^2 + \\ \qquad 4\gamma^2k^2{a_2}^2 - 4k^2\lambda^2{a_2}^2 = 0 \\ -2{a_1}^2 - 2\gamma^2{a_1}^2 + 2\lambda^2{a_1}^2 - 12a_0a_2 - 12\gamma^2a_0a_2 + 12\lambda^2a_0a_2 + 3m^2{a_1}^2a_2 + 3m^2a_0{a_2}^2 = 0 \\ -8a_1a_2 - 8\gamma^2a_1a_2 + 8\lambda^2a_1a_2 + 3m^2a_1{a_2}^2 = 0 \\ -4{a_2}^2 - 4\gamma^2{a_2}^2 + 4\lambda^2{a_2}^2 + m^2{a_2}^3 = 0. \end{cases} \tag{1.17}$$

We obtain the following solutions:

$$u_1 = \arccos\left(\pm\frac{\tan^4(\sqrt{k}\theta) + 1}{2\tan^2(\sqrt{k}\theta)}\right) \quad (k > 0)$$

$$u_2 = \arccos\left(\pm\frac{\tanh^4(\sqrt{-k}\theta) + 1}{2\tanh^2(\sqrt{-k}\theta)}\right) \quad (k < 0)$$

$$u_3 = \arccos\left(\frac{1}{2}(a_0 - ka_2\tanh^2(\sqrt{-k}\theta)) + \frac{1}{a_0 - ka_2\tanh^2(\sqrt{-k}\theta)}\right) \quad (k < 0)$$

where $\theta = x + \lambda t \mp \frac{1}{2}y\sqrt{-4 \pm \frac{m^2}{k} + 4\lambda^2}$ and a_0, a_2 are constans.

Solution u_3 is not considered in [1] and u_1 and u_2 have some different form from those that are given in [1].

1.3. The Dodd-Bullough-Mikhailov equation

This is the equation

$$u_{xt} + pe^{u} + qe^{-2u} = 0. \tag{1.18}$$

From the transformation $u = \ln V$, $V = v(\xi)$ and $\xi = x + \lambda t$, we get the equation

$$\lambda vv'' - \lambda(v)^2 + pv^3 + q = 0. \tag{1.19}$$

Balancing vv'' and v^3, we obtain

$$v = a_0 + a_1\phi + a_2\phi^2$$

The algebraic system is

$$\begin{cases} q + pa_0{}^3 - k^2\lambda a_1{}^2 + 2k^2\lambda a_0 a_2 = 0 \\ 2k\lambda a_0 a_1 + 3pa_0{}^2 a_1 - 2k^2\lambda a_1 a_2 = 0 \\ 2\lambda a_0 a_1 + pa_1^3 + 2k\lambda a_1 a_2 + 6pa_0 a_1 a_2 = 0 \\ 3pa_0 a_1{}^2 + 8k\lambda a_0 a_2 + 3pa_0{}^2 a_2 - 2k^2\lambda a_2{}^2 = 0 \\ \lambda a_1{}^2 + 6\lambda a_0 a_2 + 3pa_1{}^2 a_2 + 3pa_0 a_2{}^2 = 0 \\ 4\lambda a_1 a_2 + 3pa_1 a_2{}^2 = 0 \\ 2\lambda a_2 + pa_2{}^3 = 0. \end{cases} \tag{1.20}$$

The solutions are given by:

$$u_1 = \ln\left(\frac{q^{\frac{1}{3}}(1 + 3\cot^2\sqrt{k}\theta)}{2p^{\frac{1}{3}}}\right) \quad (k > 0)$$

$$u_2 = \ln\left(\frac{q^{\frac{1}{3}}(1 + 3\tan^2\sqrt{k}\theta)}{2p^{\frac{1}{3}}}\right) \quad (k > 0)$$

$$u_3 = \ln\left(\frac{q^{\frac{1}{3}}(1 - 3\coth^2\sqrt{-k}\theta)}{2p^{\frac{1}{3}}}\right) \quad (k < 0)$$

$$u_4 = \ln\left(\frac{q^{\frac{1}{3}}(1 - 3\tanh^2\sqrt{-k}\theta)}{2p^{\frac{1}{3}}}\right) \quad (k < 0)$$

where $\theta = x - \dfrac{3p^{\frac{2}{3}}q^{\frac{1}{3}}}{4k}t$. These solutions are different from the ones that are considered in [1].

1.4. The reaction-diffusion equation

Consider the reaction-diffusion equation (see [2])

$$u_{tt} + \alpha u_{xx} + \beta u + \gamma u^3 = 0. \tag{1.21}$$

The transformation

$$u(x;t) = v(\xi).\ \xi = x + \lambda t$$

reduces(1.21) to

$$v'' + k_1 v + k_2 v^3 0. \qquad (1.22)$$

where

$$k_1 = \frac{\beta}{\alpha + \lambda^2}, \qquad k_2 = \frac{\gamma}{\alpha + \lambda^2}.$$

The algebraic system is

$$\begin{cases} \beta a_0 + \gamma {a_0}^3 = 0 \\ 3\gamma a_0 {a_1}^2 = 0 \\ \beta a_1 + 2\alpha k a_1 + 2k\lambda^2 a_1 + 3\gamma {a_0}^2 a_1 = 0 \\ 2\alpha a_1 + 2\lambda^2 a_1 + \gamma {a_1}^3 = 0 \end{cases} \qquad (1.23)$$

The solutions are:

$$u_1 = \mp \frac{\sqrt{\beta}\cot\sqrt{k}\theta}{\sqrt{\gamma}} \quad (k > 0;\ \beta > 0)$$

$$u_2 = \mp \frac{\sqrt{\beta}\tan\sqrt{k}\theta}{\sqrt{\gamma}} \quad (k > 0;\ \beta > 0)$$

$$u_3 = \mp \frac{i\sqrt{\beta}\coth\sqrt{k}i\theta}{\sqrt{\gamma}} \quad (k < 0;\ \beta < 0)$$

$$u_4 = \mp \frac{i\sqrt{\beta}\tanh\sqrt{k}i\theta}{\sqrt{\gamma}} \quad (k < 0;\ \beta < 0)$$

where $\theta = x \pm \sqrt{\frac{-2k\alpha - \beta}{2k}}t$.

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