ASSIGNING MATHEMATICS TASKS VERSUS PROVIDING PRE-FABRICATED MATHEMATICS IN ORDER TO SUPPORT LEARNING TO PROVE

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We present types of mathematics tasks that we propose to our students —future high school mathematics teachers— in a geometry course whose objective is learning to prove and whose enterprise is collectively building an axiomatic system for a portion of plane geometry. We pursue the achievement of the course objective by involving students in different types of tasks instead of providing them with pre-fabricated mathematics.

INTRODUCTION

As a result of a curriculum innovation process (Perry, Samper, Camargo, Echeverry, & Molina, 2006), that we have been implementing and adjusting since 2004, a plane geometry course for pre-service mathematics teachers was transformed from centering on the direct teaching of geometric content to focusing on learning to prove. At present, the general course objective is that students learn to prove, widen their vision about proof and its fundamental role in mathematics activity, and recognize proof as an explicative and argumentative resource for mathematics discourse. The course's enterprise is the collective construction of an axiomatic system for a portion of plane geometry theory that includes as themes: relations between points, lines, planes, angles, properties of triangles, congruency of triangles and quadrilaterals. There are drastic changes in the content management: content is not presented as something pre-fabricated and, therefore, neither the teacher nor a textbook are the source for the content that is studied; neither is the usual definition-theorems-application exercises sequence privileged. A great amount of the propositions that are proven are formulated, by the class community, as conjectures that arise from student productions when they solve the tasks the teacher proposes. Practically all the proofs carried out are done by the students with teacher support, in a greater or lesser degree. Some propositions are enounced and proven the instant they are recognized as indispensable to complete another proof that is being constructed. Definitions are introduced to satisfy a manifested necessity to determine exactly which geometric object is being considered; to define it, the starting point is the student's concept image, and then the careful analysis of the role each condition mentioned in the definition has. How have we been able to bring such a change in the class functioning? With respect to curriculum at the class level, there are various factors, that articulated, have made this change viable: the use of a dynamic geometry program always available in the class; the group or individual student work and the collective work as a community in different moments and with different purposes; the norms that regulate the use of the dynamic geometry program, the interaction in class and what is accepted as a correct proof; the teacher's role in managing the content; and the mathematics tasks in which we involve the students.

In this paper, we present the types of tasks through which the course is developed. This way, we give a partial answer to the question: "how do we involve students in the deductive systematization of some parts of mathematics, both in defining specific concepts and in axiomatizing a piece of mathematics?". Since the experience on which our contribution is based occurs at university level, this article is well placed in the seventh theme of ICMI Study 19.

BRIEF PRECISIONS

For us, *proving activity* includes two processes, not necessarily independent or separate. The first process consists of actions that sustain the production of a conjecture; these actions generally begin with the exploration of a situation to seek regularities, followed by the formulation of conjectures and the respective verification that the geometric fact enounced is true. Hereafter, the actions of the second process are concentrated on the search and organization of ideas that will become a *proof*. This last term refers to an argument of deductive nature based on a reference axiomatic system of which the proven statement can be part of.

Learning to prove is a process through which students acquire more capability to participate in proving activity in a genuine (i.e., voluntarily assuming their role in achieving the enterprise set in the course), autonomous (i.e., activating their resources to justify their own interventions and to understand those given by other members of the class community), and relevant form (i.e., make related contributions that are useful even if erroneous). Learning to prove in our course implies a great quantity of aspects that we group into three classes: (i) those related to the proving procedure itself (e.g., the use of conditionals in valid reasoning schemes, construction of a deductive chain that leads from the hypothesis to the thesis); (ii) those related to the proof within the framework of a reference axiomatic system (e.g., distinguishing the different types of propositions that conform an axiomatic system); (iii) those proper to proofs in geometry (e.g., visualization of figures on which proofs rest, the use of figures to obtain information, auxiliary constructions).

TYPES OF TASKS USED TO SUPPORT LEARNING TO PROVE

Related to the procedure of proving

Type 1. *Determine whether a specific set of postulates, definitions or theorems permit validating a given proposition*. With this type of task, precision is begun in the course about what is proving and how a proof is done. For example, students are asked whether the postulate that establishes the correspondence between points on the line and real numbers, conformed by two reciprocal conditionals, permits assuring the truth of the proposition: *Every line has at least two points*. Carrying out this task is an opportunity for students to start realizing how the reasoning scheme *modus ponendo ponens* is used, and how a deductive sequence of propositions is conformed that permits going necessarily from the hypothesis to the desired conclusion.

Type 2. *Starting from a plan or ordered sequence of key statements to prove a given proposition, write a complete proof of the proposition*. This type of task, proposed principally at the beginning of the course, requires that students include the missing sufficient conditions of every conditional that is involved in the given sequence or plan, the theoretic justification for each statement, indicating which numbered statements intervene when obtaining the partial conclusions that make up the proof. For example, the following plan is presented: "If 0 is the coordinate of F and b the coordinate of G, $b > 0$, I look for the point H on the line for which the coordinate is $5b$. This way, G is between F and H", that must be used to write a complete proof of the proposition *For each pair of points F and G on a line, there exists a point H on the line such that G is between F and H.* With the plan, students are given a guide that should conduce them through a suitable path for the proof and the delegated work intends to concentrate their attention on details such as: which are the given premises in the proposition that must be proved, which postulate, theorem or definition guides the proof and which statements must be made to be able to use it.

Type 3. *Critically examine a proof written on the blackboard by one or two students.* Although students know their interventions in class are always possible and desired, on occasions, the responsibility of accepting or not a proof is delegated explicitly to them. This type of task compels recognizing key issues, generally problematic, which have been highlighted throughout the course. For example, the use of an element of the reference axiomatic system as warrant for a conclusion when not all the sufficient conditions of the respective conditional are on hand; the existence of an object is justified through the corresponding definition; the inclusion of statements that are not used in a proof or a sequence of statements that could be replaced by a proposition that has already been incorporated in the axiomatic system, which makes a proof longer than it should be.

Type 4. *Generate an ordered sequence of key statements that outline a route for the proof of a proposition*. This occurs either when a conjecture is generated as a solution to a construction problem or when, especially towards the end of the course, a theorem is proved because its proof follows easily from another. For example, the following problem is presented: *Given three non collinear points A, B and C, determine, if possible, a point D such that* $\overline{AB}$ *and* $\overline{CD}$ *bisect each other*. As part of the solution, the student must describe in detail his construction process and validate each step within the reference axiomatic system, which becomes a resource to outline the proof. In this type of task, students are asked to enounce the proposition in the if-then format, and occasionally, to give a synthetic formulation as a mean to give sense to the geometric fact treated. In the example, the statement is *Three non collinear points determine two segments which bisect each other.*

Related to the proof within the framework of a reference axiomatic system

Type 1. *Produce a diagram of dependency relations between the different propositions that make up a portion of the axiomatic system related to a specific topic.* In certain moments of the development of the theory, students are involved in the revision of what has been done related to a specific topic with the purpose of reconstructing the network of the propositions incorporated into the system, signaling, for each proposition, those it depends on and those that depend on it. This type of task fosters, on one hand, discriminating between postulates, theorems and definitions, and on the other, recognizing not the relation between hypothesis and thesis of a conditional but relations between the proposition and the rest of the theory.

Type 2. *Decide if a proposition, product of an exploration or search for statements that are required to complete a proof, is going to become a postulate, definition or theorem of the axiomatic system.* This type of task contributes to the discrimination of postulates, definitions and theorems that make up the system, and to establish the possibility of their use in proofs. For example, searching for a way to prove that vertical angles are congruent, a student suggested the possibility of affirming that the sum of the measurements of two angles that form a linear pair is 180°. Since this was not yet an element of the system, it was discussed whether it should be assumed as a postulate, definition or theorem. Trying to decide if it could be a definition, the class community noticed that its reciprocal was not true, and therefore discarded that possibility. To decide if the statement could be a theorem, they looked for propositions in the axiomatic system so far developed that could lead to concluding that the sum of the measurements of the angles was 180°, parting from having angles that form a linear pair; since none were found, they discarded this option. Finally, and given that the proposition was fundamental for the proof in question, and for future proofs, they decided to incorporate it as a postulate.

Type 3. *Produce a set of propositions and prove them, establishing dependency relations between them, thus forming a portion of the axiomatic system relative to a particular topic.* This type of task is initiated by proposing one or more open-ended problems that demand students' involvement in an exploratory activity, with a dynamic geometry program, that must lead to the formulation of a conjecture. Once all conjectures have been communicated, these are revised to determine conviction with respect to its truthfulness, examine whether its enunciation is clear and complete, and if necessary, carry out the pertinent modifications; moreover, during this revision process, the required definitions are elaborated. Then, with the indispensable teacher support to establish the sequence in which the conjectures are to be proven, students either produce a plan to construct the proof —that each one must finish as homework— or they collectively construct the proof. For example, in the first version of the course, problems like "Determine the quadrilaterals for which one diagonal bisects the other diagonal", "What happens in a triangle, with the segment that joins the midpoints of two sides?", and "In a quadrilateral, we choose midpoints of opposite sides or of adjacent sides. What can be said about the segment that joins them?", gave rise to the definition of parallelogram, kite, isosceles trapezoid, among others; and propositions like *In a kite, the diagonals are perpendicular and only one bisects the other, If a quadrilateral has one pair of opposite sides that are both congruent and parallel, then it is a parallelogram, The length of the segment that joins midpoints of the non-parallel sides of a trapezoid is equal to the half sum of the lengths of the bases.* Not all the theorems proved arose as conjectures from the initial exploration; some were generated during the proof of another theorem as a needed proposition to complete the proof that was being done, and others appeared when asked whether the reciprocal of the theorem was or not true.

Related to proper issues of proving in geometry

Type 1. *Obtain or use information that a graphic representation on paper or product of a dynamic geometry construction provides.* With this type of task we expect students to use the graphic representations of the objects, involved in a statement, to find useful geometric relations, but discriminating between information that can be considered true about the figure and that which is not. In a paper representation, for example, complying with norms established (system of symbols, of conventions), the task of carefully examining the figure that represented vertical angles, to find geometric relations that would permit proving they were congruent, gave the clue needed for the proof. Since the only acceptable information that could be deduced from the figure was betweenness of points, the students had to justify the existence of two pairs of opposite rays and thereof of linear pair angles. On the other hand, in dynamic geometry, the identification of invariance or the variance of certain properties by dragging became a fundamental element for discovering new properties; discard others or establish which properties depend on others. For example, students investigated the position of $\overrightarrow{BK}$ for which the bisectors of angles *KBA* and *KBC*, that compose a linear pair, form the angle with greatest measure, and realized that such bisectors always form a right angle, because in any position of $\overrightarrow{BK}$ two pairs of congruent angles are formed whose sum is 180°. So they concluded that the measure of the angle determined by the bisectors is 90°.

Type 2. *Find an appropriate auxiliary construction that directs a proof process.* A type of task particularly frequent in the fifth version of the course consists in finding the auxiliary construction that can be useful to enlarge the set of propositions to be used in a proof. To carry out the task, the teacher organizes the proposed constructions and the class analyses the benefits of one or another, leading to the appropriate one. For example, to prove that two right triangles *ABC* and *DEF* are congruent, given that their hypotenuse and a leg are congruent, a student's first idea was to construct a triangle that shared a leg with ΔABC that also had two congruent sides with it, to be able to use the known congruency criteria, but he never referred to ΔDEF. The teacher pointed out, as an important idea, the construction of a triangle "stuck to" another triangle. Then another student suggested constructing a ΔGHI congruent to ΔABC with triangles *GHI* and *DEF* sharing the congruent leg; this way, he expected to use the transitive property to prove that $\Delta ABC \cong \Delta DEF$. The teacher explained that this proposal was better than the first, but could not be used because it was impossible to justify the betweenness property of some points. Finally, another student proposed constructing ΔGHI as suggested by his classmate, but using the non-congruent leg. This way the inconvenience presented previously was overcome and they found the how to carry out the proof.

Type 3. *Recognize and use certain figures of the axiomatic system as resource to find a way to prove a proposition.* In a portion of the axiomatic system associated to angles, triangles and quadrilaterals, there are some geometric figures that become fundamental pieces of the proof process because their properties are a source for the use of elements of the system. Identifying or constructing, in a given figure, an isosceles triangle, two congruent triangles, the external angle of a triangle or a parallelogram is part of the required expertise to guarantee properties that lead to the desired conclusion. For example, in the proof of the congruency of two right triangles with hypotenuse and a leg congruent, students take advantage of their knowledge of isosceles triangles and congruency criteria to carry out the proof.

FINAL REMARKS

The types of tasks described exemplify the effort carried out in planning the class to genuinely involve students in the collective construction of the axiomatic system. Due to lack of space, we can not amplify the information about how the teacher manages the tasks, essential element to obtain from them the greatest benefit in the generation of a participative climate. The sensitivity to find in student's expressions and ideas the source to key propositions for the system and the path towards their proof, since these are not necessarily exposed in the appropriate language, joined to flexible thinking capable of sacrificing organization and rigor, proper of an advanced mathematical discourse, in pro of favoring student proving activity is a determining aspect of the success of this curriculum innovation.

REFERENCE

Perry, P., Samper, C., Camargo, L., Echeverry, A. y Molina, Ó. (2006). Innovación en la enseñanza de la demostración en un curso de geometría para formación inicial de profesores. Paper presented in SIEM XVII, 17-21 November, Universidad Autónoma del Estado de México, Toluca, Estado de México.