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Functional thinking and generalisation in third year of primary school

Eder Pinto and María C. Cañadas

University of Granada, Spain

epinto@correo.ugr.es / mconsu@ugr.es

This paper focuses on functional thinking as an approximation to algebraic thinking in third-year primary-school students. It describes a study with a class group of 24 Spanish pupils displaying functional thinking to solve a contextualised problem, identifying the type of functional relationships distinguished by these students and the ability to generalise observed in some of them. It contains an analysis of the information collected in one questionnaire, which is part of a teaching experiment. The students distinguished two types of functional relationships, correspondence and covariation, predominantly the former. Three students generalised as well.

Keywords: Algebraic thinking, generalisation, functional relationship, functional thinking.

Introduction

The idea of introducing algebraic notions in the elementary and even in the pre-school curriculum began to gain acceptance in the early nineteen nineties, when the emphasis was on what students were able to learn (Kaput, 2008). That led to the early algebra proposal, which seeks to further and enhance algebraic thinking among the younger pupils through approximations by working on classroom algebra-associated elements intended to help secondary school students perform the tasks expected of them. Generalisation lies at the heart of algebraic thinking: arithmetic operations can be viewed as functions and algebraic symbolism supports such thinking (Blanton, Levi, Crites, & Dougherty, 2011).

Functional thinking is a type of algebraic thinking, which is focussed on functions, regarded as the relationship between two co-varying quantities. The growing research interest in this type of thinking is attributable to its many advantages as an introduction to algebra (Blanton & Kaput, 2011).

Studies on functional thinking address different aspects. Some of the foremost include: (a) functional relationships drawn by students (Cañadas & Morales, 2016), (b) patterns and generalisation (Brizuela & Lara-Roth, 2002), and (c) representational strategies and systems (Carraher & Schliemann, 2007).

This paper addresses a topic not covered by previous studies concerning the different types of functional relationships identified by third year primary school students (hereafter, P3). The objectives pursued are: (a) to identify P3 pupils exhibiting functional thinking, (b) to describe the generalisation observed, and (c) to describe the functional relationships identified by students who generalise.

Functional thinking and functional relationships

Consensus has yet to be reached around the definition of algebraic thinking (Cañadas, Dooley, Hodgen, & Oldenburg, 2012). Algebraic thinking is regarded as an educational objective that affords, for instance, opportunities: (a) to generalise; and (b) to enable students to use symbols to represent ideas, which helps them solve problems, communicate and justify their ideas (Kaput, 2008).

Functional thinking is regarded as a cognitive activity "that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationship (individual incidences) to generalisations of that relationship across instances" (Smith, 2008, p. 143). Such thinking involves the construction, description and reasoning with and about functions and includes generalizing about inter-related variables (Blanton, 2008).

Based on the functional relationships established from a mathematical perspective, Smith (2008) proposed three types of approximation for working with functions: (a) recurrence, which entail finding the variation or pattern of variation in a series of values for a variable in a way such that a specific value can be obtained based on the preceding value or values; (b) correspondence, and (c) covariation. We particularly focus on the correspondence and covariation relationships because the recurrence does not involve values of more than one variable. Correspondence, stresses the relationship between the pairs $(a, f(a))$ for the variable; and covariation focuses on how a change in the values of one variable entails a change in the values of another. We show an example of these two functional relationships in Figure 1.

Correspondence. "The focus would be on the relation between x and y, which might be described as twice x plus six, or algebraically as: $2x+6$".

Variation between two quantities	
x	y
1	8
2	10
3	12
…	…

Covariation. "The focus is on corresponding changes in the individual variable". Example, in the table 1, when x increases by 1, y increases by 2.

Figure 1. Example of functional relationship (Smith, 2008, p. 146-147)

These relationships not only concern the generalization as the representations of the general relationship, they can also refer to the pattern observed in particular cases of the two variables. In recent studies, Blanton, Brizuela, Murphy, Sawrey, and Newman-Owens (2015), and Stephens, Fonger, Knuth, Strachota, and Isler (2016) described the types of functional relationships and the levels of sophistication with which subjects think about such relationships. One indicator for establishing such levels is the kind of functional relationship. Some findings showed that students in the early years evolve from the ability to establish recurrent relationships, the most basic area worked with, primarily in pre-school and early primary education, to the understanding of correspondence and covariation. Cañadas & Morales (2016) observed correspondence and covariation relationships in P1 pupils. These pupils' replies showed no evidence of the recurrence relationship. Moreover, as pupils generalise, correspondence and covariation relationship were observed more frequently, particularly the former.

Generalisation

Generalisation is one of the core processes in algebra (Kaput, 2008). All pupils can generalise and abstract from specifics, for this activity is "entirely natural, pleasurable, and part of human sense-making" (Mason, Graham, & Johnston-Wilder, 2005, p. 2). Generalisation is said to have been attained when a statement is made that applies to all the instances in a given class.

Although algebraic symbolism is the characteristic representation for algebra, there are other ways of representing the generalization, specially when concerns elementary students. Carraher, Martinez, & Schliemann (2008) focused on third year primary school students' generalization and how they express it. These students generalise functional relationships (correspondence and covariation). The authors highlight that students should learn to generalise solving mathematical problems that allow them to look for and observe patterns, relationships, and structures. In this way, students have the possibility to get new informations and reflect about the generalisations produces by themselves and their partners.

Method

This study forms part of a broader teaching experiment focusing on Spanish P3 students' functional thinking. The contextualised problem posed in each session involved a linear function. The fourth and last session is discussed hereunder.

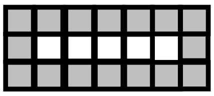
Subjects and data collection

The subjects were 24 P3 pupils (8-9 years old), intentionally selected on the grounds of school and teacher availability. These students had not worked with problems involving functional relationships prior to the study, except in the first three sessions of the same project in which they were introduced to problems involving two linear functions: $f(x)=x+5$ and $f(x)=x+3$. All the sessions were guided by a teacher-researcher.

In the first part of the session, we introduced the tiles problem[1] to the students, asking them questions concerning particular cases, in order to assure that they understand the situation and the questions. In this problem, the function involved is $f(x)=2x+6$. This paper focuses on the results from a written questionnaire that had to be answered individually in connection with the problem posed. The way in which the problem was posed and questions used are presented in Figure 2. In questions Q1, Q2, Q3, Q4.A and Q4.B pupils were asked about specific non-consecutive cases, whilst the fifth (Q5) asked the pupils to generalise the relationship between the dependent and independent variables (white and grey tiles, respectively). The students were furnished with manipulative material: white and grey paper tiles.

[1] The well-known tiles problem used here has been applied by a number of researchers in the context of classroom algebra (e.g., Küchemann, 1981).

A school wants to renovate the ground of all its corridors because it is already very damaged. The management team decides to pave the corridor with white tiles and grey tiles. All tiles are squares and have the same size. The tiles are being placed in each corridor so that you can see in the picture below.



The school ask a company for renovating the different corridors of the school. We ask you to help the workers to answer some questions that they need to answer for their work.

Q1. How many grey tiles are needed for the floor of a corridor in which 5 white tiles are placed? How do you know that?

Q2. Some corridors are longer than others. Therefore, the workers need different number of tiles for each corridor. How many grey tiles are needed for a floor corridor in which 8 white tiles are placed? How do you know that?

Q3. How many grey tiles are needed for a floor corridor in which 10 white tiles are placed? How do you know that?

Q4A. How many grey tiles are needed for a corridor floor in which 100 white tiles are placed? How do you know that?

Q4B. Now, do it in a different way and explain it below.

Q5. The workers always place the white tiles and then the grey tiles first. How do you know how many grey tiles you need if you have already placed the white tiles?

Figure 2. Tiles problem

Analytical categories and data analysis

Following our research objectives, we used information from the theoretical framework and previous studies to design the categories used in data analysis. Moreover, we were aware of possible modifications needed as long as we performed a preliminary data analysis in order to adapt them to our specific data. Two categories were established: (a) functional relationships, and (b) generalisation. Finally both categories were related because the generalisation involved at least one of the functional relationships.

The category of functional relationship covered the type of functional relationships identified by the pupils: (a) correspondence, and (b) covariation. Functional thinking was deemed to be present in pupils' replies when at least one functional relationship was drawn in at least two of the questions posed. This criterion pursued to avoid those students who used a computation strategy but not neccesarily a relationship between variables.

The second category dealt with the presence or absence of generalisation and how it was reached and expressed in any of the five questions posed. More specifically, it focused on the students' replies to Q5 (regarding generalisation), because is the only question in which students generalised.

Results and discussion

The 24 pupils' written responses to the questionnaires were analysed. All the students answered the first three questions, 23 the fourth one, and 16 the fifth one (generalisation). Those findings were interpreted to mean that more students answered the first four questions because they involved specific, non-consecutive instances and small numbers. Similarly, the high rate of blank answers to Q5 was conjectured to be due to the complexity involved in generalizing the functional relationship.

The findings set out below are organised in keeping with the objectives pursued. The students are referred to with the letter S followed by a number, from 1 to 24.

Functional relationships

Eleven students exhibited functional thinking, identifying a functional relationship in at least two of the questions asked. The other 13 students, in contrast, did not.

Among the students exhibiting functional thinking, seven students distinguished only correspondence relationships in their replies, and four identified both functional relationships (correspondence and covariation).

A representative example of students who used only the correspondence relationship is S22. We show this student's responses to the first four questions in Figure 3.

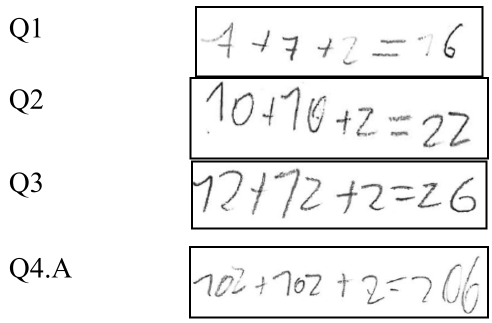


Figure 3. S22's responses

For instance, in Q4.A, he took the number of white tiles (100) and added 2 (100+2). Then he found the number of grey tiles needed for the bottom and top rows by adding 102 twice (102 + 102). Lastly, he added 2, the ones on the right and left of the white tiles (102 + 102 + 2). He used that same functional relationship for 5, 8 and 10 white tiles in Q1, Q2, and Q3, respectively. In all three cases, this student related pairs of values (a, f(a)) to the a values in each specific case and established a relationship with the number of grey tiles: 16, 22 and 26, respectively.

Four pupils identified two functional relationships in their answers to the questions on the questionnaire: correspondence and covariation. None of the students recognised more than one functional relationship in their answers to a given question.

S3 is a student who identified correspondence and covariation relationships. In Q2, she answered that 22 grey tiles are necessary for 8 white tiles, using a counting strategy. In Q3, S3 answered, "if 8 [white tiles] need 20 [grey tiles], there are 20 + 2 = 22". Although this answer is wrong, she used the previous response to work on (adding two to the previous response). We observe that the student focused on the variation between the number of white tiles (between 8 and 10, there is an increase of 2 white tiles) in order to calculate the number of grey tiles, considering that such increase is also 2. Therefore, she focused on how variation in values of the number of white tiles influence in a variation in values of grey tiles, which is the notion of covariation relationship.

Generalisation

We find generalization evidence in Q5. In previous questions, students referred to the relationship between variables through particular cases involved.

Three of the 11 pupils who exhibited functional thinking showed the generalisation in their replies to Q5. One of them, S9, generalised appropriately to the problem posed. In contrast, the other two students —S11 and S22— generalised incorrectly. In what follows, we present examples of the students who generalized, describing when they got it and what kind of relationship generalised.

S9 used a numerical representation to calculate the number of grey tiles in the first four questions. In Q5, he stated "you double the number white tiles and then you add 6". He used different representation to the verbal one in other questions. This fact evidences the importance of the verbal representation in the development of functional thinking in the same way as Kieran, Pang, Schifter, and Fong Ng (2016) noted. Student S9 generalised the correspondence relationship that he also identified in questions concerning particular cases.

S11 found a correspondence relationship in the questions concerning particular cases. In his reply to Q5 he noted: "if there are 50 tiles, then I add 50+50 and then the ones on the sides, 3+3, 106 in all". The student used a particular case to answer the question but he evidenced that he recognised the fixed number of grey tiles (3+3). This fact shows generalization at an initial stage: although his answer is not complete, he is approaching to the generalization of the relationship because he identified the function constant (6). S11 used different relationships to determine the number of grey tiles in Q2, Q3, and Q4, focussing on a correspondence relationship between the variables involved. He "generalise" the correspondence relationship in Q5.

S22's reply illustrates another way to generalise in Q5: "add 6". This generalisation was incomplete, for she recognised the number of grey tiles that remains constant (left and right sides), but not the number on the top and bottom rows, even though in the preceding questions she distinguished the pattern for determining the number of grey tiles given a certain number of white tiles (see Figure 1). Moreover, S22 used the correspondence relationships to answer the first four questions (see Figure 3). On the contrary, this student used a co-variation relationship in Q5 because he identified the neccessity of adding 6 to calculate the number of grey tiles given any number of white tiles.

Conclusion

The students exhibiting functional thinking (those recognizing at least one functional relationship in at least two questions) could be detected on the grounds of the relationships they identified.

The correspondence was the functional relationship predominantly observed in the students' answers, followed by covariation. This holds particular significance, specifically by: the pupils' age, the specific demands of the tiles problem and the functional relationships distinguished. The prevalence of the correspondence relationship in the first four questions, which involved familiar specific cases, seemed to be connected with the pupils' broader experience with areas such as numerical patterns. Additionally, we conjecture that this functional relationship could be induced by the problem context because each particular case involved in one question is not connected with other particular cases.

Covariation was observed in Q5, which sought to induce the pupils to express the general relationship between the variables involved; the preceding questions could be answered with no need to generalise.

According to Blanton, Brizuela, Gardiner, Sawrey, and Newman-Owens (2015), functional thinking involves (among others) drawing general patterns from relationships between quantities that co-vary and representing and justifying such relationships in different ways with a number of representational systems. The results of their study are supplemented by the present findings, further to which P3 pupils naturally (for they had not worked on this area in the classroom) identified more than mere recurrence, establishing relationships (correspondence and covariance) involving the values of both variables. Whilst influenced by the type of problem, these P3 students were found to be able to distinguish correspondence and covariation relationships, even though they were not always able recognise a general pattern.

In a future line of research the way generalisation is expressed will be studied in greater depth, along with pupils' arguments and explanations. Student interviews are regarded as a suitable tool for obtaining a fuller description of how inter-variable relationships are expressed.

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References

Blanton, M. L. (2008). *Algebra and the elementary classroom: Transforming thinking, transforming practice*. Portsmouth, NH: Heinemann.

Blanton, M. L., Levi, L., Crites, T., & Dougherty, B. J. (2011). *Developing essential understanding of algebraic thinking for teaching mathematics in grades 3-5.* Reston, VA: NCTM.

Blanton, M. L., & Kaput, J. (2011). Functional thinking as a route into algebra in the elementary grades. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 5-23). Heidelberg, Germany: Springer.

Blanton, M., Brizuela, B. M., Murphy, A., Sawrey, K., & Newman-Owens, A. (2015). A learning trajectory in 6-year-olds' thinking about generalizing functional relationship. *Journal for Research in Mathematics Education*, *46*, 511−559.

Brizuela, B., & Lara-Roth, S. (2002). Additive relations and function tables. *Journal of Mathematical Behavior*, *20*, 309−319.

Cañadas, M. C., & Morales, R. (2016). Functional relationships identified by first graders. In C. Csíkos, A. Rausch, & J. Szitányi. (Eds.), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 131-138). Szeged, Hungary: PME.

Cañadas, M. C., Dooley, T., Hodgen, J., & Oldenburg, R. (2012). CERME7 Working Group 3: Algebraic thinking. *Research in Mathematics Education*, *14*, 189−190.

Carraher, D., & Schliemann, A. (2007). Early algebra and algebraic reasoning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 669−705). Reston, VA: NCTM.

Carraher, D., Martinez, M., & Schliemann, A. (2008). Early algebra and mathematical generalization. *ZDM — The International Journal on Mathematics Education, 40*, 3−22.

Kaput, J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5−17). New York, NY: Lawrence Erlbaum.

Kieran, C., Pang, J., Schifter, D., & Fong Ng, S. (2016). *Early algebra. Research into its nature, its learning, its teaching.* New York, NY: Springer Verlag.

Küchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics: 11-16* (pp. 102−119). London, UK: John Murray.

Mason, J., Graham, A., & Johnston-Wilder, S. (2005). *Developing thinking in algebra*. London, UK: Paul Chapman Publishing.

Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 133−163). New York, NY: Lawrence Erlbaum.

Stephens, A., Fonger, N., Knuth, E., Strachota, S., & Isler, I. (2016, July). *Elementary students' generalization and representation of functional relationships: A learning progressions approach.* Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.