DESIGN-BASED RESEARCH AND LOCAL INSTRUCTION THEORIES IN MATHEMATICS EDUCATION

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Abstract

This paper sketches the approach of design-based research in mathematics education with results of two innovation-oriented projects. The projects investigated how students can be involved in the development of mathematical concepts and skills by using design tools related to guided reinvention and emergent modelling. Both studies combine design-based research with a prominent role for the hypothetical learning trajectory as a research instrument in the three phases of design research (design, teaching experiment, retrospective analysis). Each of the phases of the research cycles is addressed: the preliminary phase in which a hypothetical learning trajectory and instructional activities are designed, the teaching experiment phase and the phase of the retrospective analysis. I conclude with a reflection on design-based research as an approach to study innovative teaching approaches that offers researchers to take into account contextual factors and that create opportunities for others to adapt the results to their research or teaching practice.

INTRODUCTION

One of the salient characteristics of learning mathematics is being introduced in a world full of symbols. This is not merely an external characteristic, as mathematical symbols are an integral part of mathematics. It is hard to think about measurement without the use of unit measures, or to understand calculus without pointing to rates of change in graphs. This intertwinement of meaning and visual representations poses a problem for mathematics education. Experts - like teachers and instructional designers - tend to see these symbols as carriers of meaning. For them, symbols and graphs are transparent; they can “see the mathematics through it”, so to speak. Students, however, often do not have the necessary mathematical background to interpret those symbolic representations in that manner. As a consequence, teachers will have to explain to the students what there is to see, and how to reason with those symbolic representations. This, Cobb et al. (1992) point out, leads to proceduralising and algorithmetising and the loss of meaning - or to, as van Oers (2000) calls it, “pseudo mathematics”.

To find a way out of this dilemma, one may consider the history of mathematics to investigate how meaning and symbols emerged. It turns out that mathematical symbols did not arrive ready-made, with full-fletched meaning. Instead, one can discern a reflexive process in which symbolising and the development of meaning co-evolve (Meira, 1995). Symbolising, here, refers to inventing and using a series of symbols. In relation to this, Latour (1990) and others (e.g. Roth y McGinn, 1998) speak of a “cascade of inscriptions”. This notion of a cascade of inscriptions has its counterpart in semiotic concepts as “chain of signification” (Walkerdine, 1988; Whitson, 1997).

A challenge for mathematics educators is to develop mathematics education that is in line with this dynamic conception of symbolising and development of meaning. The task of researchers is to shed light on the key elements of this type of mathematics education. In order to investigate the possibilities of such a new and innovative approach to mathematics education is that the instructional materials are not available yet. Moreover, research into the topic requires a process in which the design of instructional activities and teaching experiments are intertwined with the development of instructional theories for specific topics in mathematics. In this paper I will discuss a project on the introduction of the basic principles of calculus with the aim to illustrate the

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characteristics of such a design-based research approach. This approach tries to ensure a systematic process of design and analysis that offers opportunities to generalize findings over specific contexts. This project has a dual goal:

- on the one hand, answering the question on how to develop and investigate an innovative teaching and learning process for calculus, and
- on the other hand, investigating the reflexive relations between symbol use and the development of meaning.

Given these goals, design-based research or developmental research (Gravemeijer, 1994, 1998) seems to be an appropriate research method. Following Brown (1992), Cobb, Confrey, diSessa, Lehrer and Schauble (2003) refer to this type of research as design experiments, which they elucidate in the following manner:

> Prototypically, design experiments entail both "engineering" particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them. This designed context is subject to test and revision, and successive iterations that result play a role similar to that of systematic variation in experiment (Cobb, Confrey, diSessa, Lehrer y Schauble, 2003, p. 9).

In this description, the two central aspects of this paper come to the fore in (a) the design of means of support for particular forms of learning, and (b) the study of those forms of learning. In the research project under discussion, the backbone of the design is formed by the design, development and revision of a hypothetical learning trajectory.

DESIGN-BASED RESEARCH

The design-based research approach has a cyclic character in which thought experiments and teaching experiments alternate. A cycle consists of three phases: the preliminary design phase, the teaching experiment phase, and the phase of retrospective analysis. A second characteristic of design-based research is the importance of the development of a learning trajectory that is made tangible in instructional activities. The design of instructional activities is more than a necessity for carrying out teaching experiments. The design process forces the researcher to make explicit choices, hypotheses and expectations that otherwise might have remained implicit. The development of the design also indicates how the emphasis within the theoretical development may shift and how the researcher's insights and hypotheses develop. As Edelson argues, design is a meaningful part of the research methodology:

> (...) design research explicitly exploits the design process as an opportunity to advance the researchers understanding of teaching, learning, and educational systems. Design research may still incorporate the same types of outcome-based evaluation that characterise traditional theory testing, however, it recognizes design as an important approach to research in its own right. (Edelson, 2002, p.107)

This is particularly the case when the theoretical framework involved is under construction.

Hypothetical learning trajectory

Within each macro level research cycle, three phases are distinguished: the preliminary design phase, the teaching experiment phase, and the phase of retrospective analysis. The first phase of preliminary design includes two related parts, the development of a Hypothetical Learning Trajectory (HLT) and the design of instructional activities. The notion of a "Hypothetical learning trajectory" is taken from Simon (1995). Originally, Simon used the HLT for designing and planning short teaching cycles of one or two lessons. In our study, however, a HLT is developed for teaching experiments that lasted for a longer sequences of lessons. As a consequence, the HLT comes close to the concept of a local instruction theory (Gravemeijer, 1994).

The development of an HLT involves the choice or design of instructional activities in relation to the assessment of the starting level of understanding, the formulation of the end goal and the conjectured mental activities of the students. Essential in Simon's notion of a HLT is that it is hypothetical; when the instructional activities are acted out, the teacher – or researcher in our case – will be looking for evidence of whether these conjectures can be verified, or should be rejected.

For the design of the student activities, their motivation and the estimation of their mental effects, the designer makes full use of his domain specific knowledge, his repertoire of activities and representations, his teaching experience, and his view on the teaching and learning of the topic. After a field test by means of a teaching experiment, the HLT will usually be adapted and changed. These changes, based on the experiences in the classroom, start a new round through the mathematical teaching cycle, and, in terms of the design research approach, the next research cycle.

The concept of the HLT may seem to suggest that all students follow the same learning trajectory at the same speed. This is not how the HLT should be understood. Rather than a rigid structure, the HLT represents a learning route that is broader than one single track and has a particular bandwidth.

With an emphasis on the mental activities of the students and on the motivation of the expected results by the designer, the HLT concept is an adequate research instrument for monitoring the development of the designed instructional activities and the accompanying hypotheses. It provides a means of capturing the researcher's thinking and helps in getting from problem analysis to design solutions.

Design of instructional activities

The preliminary design phase of the design research cycles includes the development of the HLT and the instructional activities. The expectations of the students' mental activities established in the HLT are elaborated into specific key activities in the instructional materials.

The design of instructional activities in these studies included the development of student text booklets and teacher guides. While designing these materials, choices and intentions were captured and motivated, to inform the teacher and to keep track of the development of the designer's insights. When the materials were about to be finalised, these aims and expectations were described at the task level. Key items, that embodied the main phases in the HLT, were identified. These items reflected the relevant aspects of the intended learning process and were based on the conceptual analysis of the topic. The identification of key items guided observations and prepared for the retrospective data analysis. Finally, teacher guides as well as observation instructions were written, to make intentions and expectations clear to teachers and observers. During the design phase, products were presented to colleagues, teachers and observers. This led to feedback that forced the researcher to become explicit about goals and aims, and that provided opportunities for improving all the materials.

While designing instructional activities, the key question is what meaningful problems may foster students' cognitive development according to the goals of the HLT. Three design principles guided the design process: guided reinvention, didactical phenomenology and emergent models.

The design principle of *guided reinvention* involves reconstructing the natural way of developing a mathematical concept from a given problem situation. A method for this can be to try to think how you would approach a problem situation if it were new to you. In practice, this is not always easy to do, because as a domain expert it is hard to think as if you were a freshman. The history of the domain can be informative on specific difficulties concerning concept development (e.g. Gravemeijer y Doorman, 1999).

The second design principle, *didactical phenomenology*, was developed by Freudenthal (Freudenthal, 1983). Didactical phenomenology aims at confronting the students with phenomena that “beg to be organised” by means of mathematical structures. In that way, students are invited to build up mathematical concepts. Meaningful contexts, from real life or “experientially real” in another way, are sources for generating such phenomena (de Lange, Burrill, Romberg, y van Reeuwijk, 1993; Treffers, 1987). The question, therefore, is to find meaningful problem contexts that may foster the development of the targeted mathematical objects. The context should be perceived as natural and meaningful, and offer an orientation basis for mathematisation.

The last remark leads to the third design principle, the use of *emergent models* (Gravemeijer e.a., 2000; Van den Heuvel-Panhuizen, 2003). In the design phase we try to find problem situations that lead to models that initially represent the concrete problem situation, but in the meantime have the potential to develop into general models for an abstract world of mathematical objects and relations.

Teaching experiments

The second phase of the design research cycle is the phase of the teaching experiment, in which the prior expectations embedded in the HLT and the instructional activities are confronted with classroom reality. The term “teaching experiment” is borrowed from Steffe (Steffe y Thompson, 2000). The word “experiment” is not referring to an experimental group – control group design. In this section we explain how the teaching experiments were carried out; in particular, we pay attention to the data sampling techniques used during the teaching experiments.

The research questions share a process character: they concern the development of understanding of mathematical concepts. Therefore, we focussed on data that reflected the learning process and provided insight into the thinking of the students. The main sources of data were observations of student behaviour and interviews with students. The observations took place on three levels: classroom level, group level and individual level. Observations at classroom level concerned classroom discussions, explanations and demonstrations that were audio and video taped. These plenary observations were completed by written data from students, such as handed in tasks and notebooks.

Observations at group level took place while the students were working on the instructional activities in pairs or small groups. Short interviews were held with pairs of students. In addition to this, the observers made field notes.

The lessons were evaluated with the teachers. In particular, the organisation of the next lesson and the content of the plenary parts were discussed. Also, decisions were taken about skipping (parts of) tasks because of time pressure. Such decisions were written down in the teaching experiment logbook.

Retrospective analysis

The third phase of a design research cycle is the phase of retrospective analysis. It includes data analysis, reflection on the findings and the formulation of the feed-forward for the next research cycle.

The first step of the retrospective analysis concerned *elaborating on the data*. A selection from video and audiotapes was made by event sampling. Criteria for the selection were the relevance of the fragment for the research questions and for the HLT of the teaching experiment in particular. Data concerning key items was always selected and these selections were transcribed verbatim. The written work from the students was surveyed and analysed, especially the work on key items, tests and hand-in tasks. Results were summarised in partial analyses. This phase of the analysis consisted of *working through the protocols* with an open approach that was inspired by the constant comparative method (Glaser y Strauss, 1967; Strauss y Corbin, 1988). Remarkable events or trends were noted as conjectures and were confronted with the expectations based on the HLT and the instructional activities.

The second phase of analysis concerned *looking for trends* by means of sorting events and analysing patterns. The findings were summarised illustrated by prototypical observations. These conjectures were tested by surveying the data to find counterexamples or other interpretations, and by data triangulation: we analysed the other data sources, and in particular the written student material, to find instances that confirmed or rejected the conjectures. Analysis of the written materials often evoked a reconsideration of the protocols. Analysis was continued in this way until saturation, which meant that no new elements were added to the analysis and no conclusions were subject to change.

The third phase in analysing the data was the *interpretation of the findings* and the comparison with the preliminary expectations of the HLT. Also, explanations for the differences between expectations and findings were developed. These conclusions and interpretations functioned as feed-forward for the formulation of new hypotheses for the next cycle in the research.

AN EXEMPLARY CASE: THE BASIC PRINCIPLES OF THE MATHEMATICS OF CHANGE[1]

Background and design of HLT

The aim of this project is to find out how students can learn the basic principles of calculus and kinematics by modelling motion. Nowadays, graphs are used in calculus and kinematics education as representations for describing change of velocity or distance travelled during a time interval. Students are expected to give meaning to the relation between distance travelled and velocity through characteristics of these graphs such as area and slope. The use of such instructional materials is based on a representational view (Cobb et al., 1992), which assumes that instructional materials can represent scientific knowledge, and that scientific concepts can be made accessible without fully taking into account the limitations of the knowledgebase of the students into which they have to be integrated.

Cobb et al. oppose this view. In line with their reasoning, we claim that symbolisations and knowledge of motion can co-evolve in a learning process. Theories on symbolising give rise to heuristics for designing a learning route within which the mathematical and scientific knowledge emerges from the activity of the students (Gravemeijer et al. 2000). In this route, the creation, use and adaptation of various graphical representations are interwoven with learners' activities in a series of science-practices, from modelling discrete measurements to reasoning with continuous models of motion. Our focus is on students' contributions during these practices, and how we can build upon their contributions towards the intended attainment targets. Consequently, for understanding their reasoning we use the design-based research approach of planning and testing the envisioned trajectory in classroom situations for investigating *how* a trajectory works and can be improved, instead of trying to decide *whether* it works.

The learning route – inspired by the domain history – is tried out and revised during teaching experiments in three tenth-grade classes. We collected data by video and audio taping whole class discussions and group work. The videotapes were used to analyse students' discourses and students' written materials with respect to the conjectured teaching and learning process.

Teaching experiment phase

We illustrate the change in how students think and talk about a model with the following episode. The trajectory starts with questions about a weather forecast. The teacher discusses the change of position of a hurricane with students: when will it reach land? This problem is posed as a leading question throughout the unit as a context for the need of grasping change. After the emergence of time series as useful tools for describing change of position, students work with situations that are described by stroboscopic photographs. The idea is that students come up with measurements of displacements, and that it makes sense to display them graphically for finding and extrapolating patterns. Two types of discrete graphs are discussed: graphs of displacements (distances between successive positions) and graphs of the total distance travelled. Note that discrete graphs are not introduced as an arbitrary symbol system, but emerge as models of discrete approximations of a motion, that link up with prior activities and students' experiences. By using the computer program Flash students are able to investigate many situations. During these activities their attention shifts from describing specific situations to properties of these discrete graphs and the relation with kinematical concepts.

Our findings confirm such a change in reasoning. In the beginning students refer to distances between successive positions. After a while they reason using the global shapes and properties of graphs and motion. An example of such reasoning concerns an exercise about a zebra that is running at constant speed and a cheetah that starts hunting the zebra. The question is whether the cheetah will catch up with the zebra. In the graphs the successive measurements of the zebra and the cheetah are displayed. The following discussion takes place between an observer and two students (Rob and Anna).

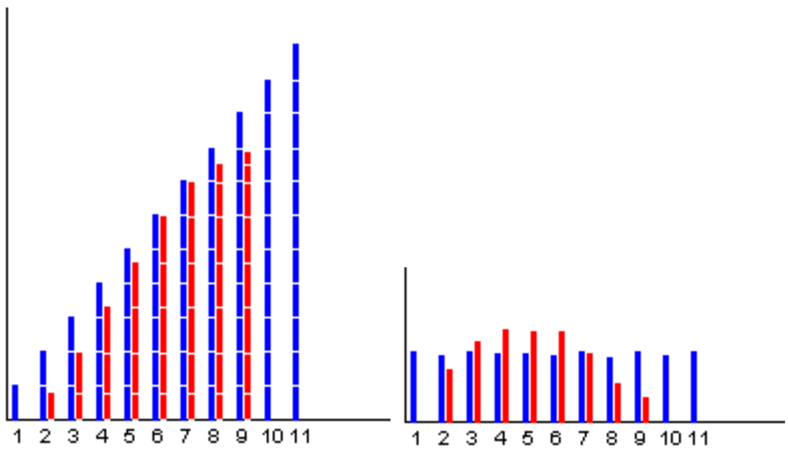


Figure 1. Distance-travelled and displacement graphs in Flash

Observer: Oh yes. So why did you choose the one for the total distance [left graph in Fig. 1]?

Rob: Because it's the total distance that they cover and then you can-

Anna: Then you can see if they catch up with each other.

Observer: And can't you see that in the other [right graph]? There you can also see that the red [grey] catches up with blue [dark grey]?

Rob: Yes, but -

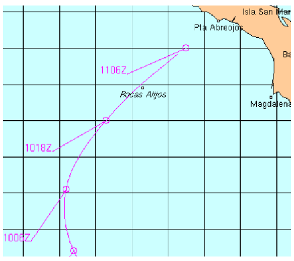
Anna: Yes, but that's at one moment. That only means that it's going faster at that moment but not that it'll catch up with the zebra.

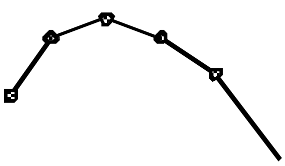
Retrospective analysis

A difference between the displacements graph and the distance-travelled graph is the difference between the interpretations of the horizontal (time) axis. A value in the distance-travelled graph represents a distance from the start until the corresponding time, while a value in the displacements graph represents a distance in the corresponding time interval. Anna's last observation is an important step in the process of building the model of a velocity-time graph (and everything that comes with it).

The qualitative analyses show that during the practices students re-invent and develop graphical symbolisations, as well as the language and the scientific concepts that come with them. However, these inventions only became explicit after interventions by an observer or by the teacher. Additionally, we found that the teacher had a crucial role during the classroom discussions. It was not always easy to organise the discussions in line with the intended process. Sometimes the teacher reacted to students' contributions in terms of the inscriptions or concepts aimed at. In those cases students awaited further explanation. The discussions appeared to be especially productive when the teacher organised classroom discussions about students' contributions in such a way that the students themselves posed the problems that had to be solved, and reflected on their answers. In a second teaching experiment we arranged a setting where the teacher had more information about the possible contributions of the students and the way in which they could be organised. Additionally, we designed activities for classroom discussions. The HLT for the second teaching experiment is summarized in Figure 2. This summary shows the development of the tools that students used in their reasoning about change and how the transparency of the tools, the image that students have, is created by previous activities with preliminary representations. Furthermore, the figure illustrates the interaction between the development of these tools and the development of meaning related to the mathematical concepts of change.

In this approach the construction and interpretation of graphs and the scientific concepts are rooted in the activities of the students through emerging models. This ensures that the mathematical and physical concepts aimed at are connected to students' understanding of everyday phenomena. On the basis of our findings we conclude that classroom discussions where students discuss their solutions and pose new problems to be solved, are essential for a learning process during which symbolisations and knowledge of motion co-evolve.

<table>
<tr><th>Tool</th><th>Imagery</th><th>Activity</th><th>Concepts</th></tr>
<tr><td rowspan="2">

time series (e.g. satellite photos y stroboscopic pictures)</td><td rowspan="2">real world representations signify real world situations</td><td>predicting motion (e.g. in the context of weather predictions)</td><td>displacements in equal time intervals as an aid for describing and predicting change</td></tr>
<tr><td colspan="2">should result in a feeling that the ability to predict motion with discrete data is an important issue</td></tr>
<tr><td rowspan="2">trace graphs of successive locations
</td><td rowspan="2">signifies a series of successive displacements in equal time intervals</td><td>compare, look for patterns in displacements and make predictions by extrapolating these patterns</td><td>displacements as a measure of speed, of changing positions, but difficult to extrapolate</td></tr>
<tr><td colspan="2">resulting in a willingness to find other ways to display displacements for viewing and extrapolating patterns in them</td></tr>
</table>

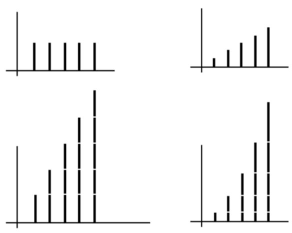
<table>
<tr>
<td rowspan="2">discrete 2-dim graphs
</td>
<td rowspan="2">signifies patterns in displacements of trace graphs
(and cumulative)</td>
<td>compare patterns and use graphs for reasoning and making predictions about motion (also at certain moments: interpolate graphs)
refine your measurements for a better prediction: displacements decrease</td>
<td>displacements depicting patterns in motion;
linear line of summit in graph of displacements or graph of distances traveled;
problems with predictions of instantaneous velocity</td>
</tr>
<tr>
<td colspan="2">should result in the need to know more about the relation between sums and differences, and in the need to know how to determine and depict velocity</td>
</tr>
</table>

Figure 2: The HLT for the second teaching experiment

DISCUSSION

With this case study I tried to illustrate how an innovative approach for a topic in mathematics can be investigated. The use of semiotic theories turned out useful for analysing the relationship between symbolising and development of meaning. Mathematics education asks for a careful design of teaching-learning trajectories involving an intertwined process of symbol introduction and meaning making. During that process, students get opportunities for creating their own constructions and reflecting on them. Realistic contexts proved important in that.

With respect to the methodology of design-based research, the Hypothetical Learning Trajectory appeared to be a useful instrument in all phases of design research. During the design phase it is the theoretically grounded vision of the learning process, which is specified for concrete instructional activities. During the teaching experiments, the HLT offers a framework for decisions during the teaching experiment and guides observations and data collections. In the retrospective analysis phase, the HLT serves as a guideline for data selection and offered conjectures that could be tested during the analysis. The final HLT is a reconstruction of a sequence of concepts, tools and instructional activities, which constitute the effective elements of a learning trajectory. In this manner, the result is a well-considered and empirically grounded local instruction theory for the basic principles of calculus. The HLT, together with a description of the cyclic process of design and research, enables others to retrace the learning process of the research team. Understanding the how and why of the specified steps makes it possible to let that learning process become your own and to adapt findings to your own context.

With this example I illustrated design-based research as a systematic approach for innovation-oriented studies. The close connections between design and theory development offer teachers and researchers opportunities to translate the results to their own teaching or researching practice.

The intertwinement of teaching experiments and theoretical reflections guided by a HLT emphasize the contribution to theory development within design-based research. This process, as presented in this paper, is intrinsically different from what tries to be achieved with lesson studies.

Notes

This paper is an adaption from: Bakker, A., Doorman, M. y Drijvers, P. (2003*). Design research on how IT may support the development of symbols and meaning in mathematics education*. Paper presented at the Onderwijs Research Dagen (ORD), Kerkrade, The Netherlands.

[1] This case is based upon: Doorman y Gravemeijer (2009).

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