“MATHEMATICS IS NOT A STALACTITE HANGING OVER A STALAGMITE” (W. KUYK) – THE PRODUCTIVE ROLE OF TEACHING

“Las matemáticas no son una estalactita colgando sobre una estalagmita” (W. Kuyk) – El rol productivo de la enseñanza

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Abstract

Willem Kuyk had denounced with this image the widely shared view that mathematics education grows only by receiving some drops from above, figuring there the supreme instance for teaching. Bruno Belhoste, in his paper of 1998, had argued against this view and pleaded for re-assessing the role of teaching mathematics in the development of mathematics. Yet, he conceptualised its role in a polarised manner: while speaking of contributions of mathematical research as “invention”, as “production”, he juxtaposed teaching as “reproduction”, as “socialisation”. I am arguing, instead, that the teaching of mathematics can transgress this type of reproduction in a much more decisive manner – and effectively did so throughout the historical development of mathematics.

I should like to discuss this productive role of teaching and in particular the methodological challenges for realising such analyses. Actually, historians of mathematics do not use to be very attached to methodological reflections, and, if ever dealing with them, use to reproduce the old dichotomy between internalism and externalism – quite contrary to historians of science who since quite a time overcame that dichotomy and use to study interactions between internal developments of ideas and broader cultural, social and political contexts. A key pattern for studying such interactions are institutions – i.e., institutions in which mathematicians are working. And, in general, the official major task there is teaching, forming new generations. Not only are such institutions at the crossroads of conceptual developments within the discipline and of contextual influences on the functioning of the institution, there also occurs the direct and concrete interaction between teaching and research. Not too rarely, it is the function of teaching which induces to innovations in mathematical concepts.

The most paradigmatic patterns for this functioning were higher education institutions from the French Revolution: due to the establishment of systems of public education, the higher degrees of teaching intensity entailed incentives for systematic revisions of mathematical concepts and their foundations. But there are also revealing cases of innovations in the discipline induced by institutions in earlier periods. Particularly telling studies have been made by Christine Proust on the functioning of the scribal schools (edubba) in Old Babylonian times where the masters, forming the scribal apprentices, transgressed the routine tasks of teaching by validating procedures, by solving new classes of problems.

Besides presenting and discussing pertinent cases for such interactions, the lecture will discuss another pertinent issue for interfaces between mathematical development and teaching: it is the notion of ‘element’ and of elementarisation of science. In fact, the notion of element connects the development of mathematics and the modes of teaching mathematics in a fundamental way. Since Euclid’s geometry textbook, the term ‘elements’ expresses the intention to give a systematic presentation of a mathematical theory, constructed from its basic components. While thus fixing the

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state of knowledge of mathematics or of one of its branches for a certain time and period, Felix Klein's notion of elementarisation dynamises the notion, emphasising the various stages where the meanwhile accumulated new results and branches led to a restructuration of the bases, defining a renewed structure of elements – the new architecture achieved by Bourbaki constituting a well-known case for textbooks. It is incited by teaching and serving for teaching that textbooks are contributing to the progress of mathematics.

Keywords: *Teaching of mathematics, mathematics history, Teacher Training.*

Resumen

Willem Huyk había denunciado con esta imagen la visión ampliamente compartida de que la educación matemática crece únicamente recibiendo algunas gotas de arriba, figurando allí la instancia suprema para la enseñanza. Bruno Belhoste, en su artículo de 1998, había argumentado en contra de este punto de vista e instado a reexaminar el rol que tiene la enseñanza de las matemáticas en el desarrollo mismo de las matemáticas. Sin embargo, conceptualizó ese rol de una manera polarizada: mientras hablaba de las contribuciones de la investigación en matemáticas como "invenciones" o "producciones", yuxtaponía la enseñanza como "reproducción", como "socialización". Por el contrario, yo argumento que la enseñanza de las matemáticas puede trasgredir esa reproducción de un modo mucho más decisivo, y que efectivamente así lo ha hecho a lo largo del desarrollo histórico de las matemáticas.

Me gustaría discutir este rol productivo de la enseñanza y, en particular, los desafíos metodológicos para dar cuenta de tal análisis. En realidad, los historiadores de las matemáticas no suelen estar muy ligados con reflexiones metodológicas y, si tienen que tratar con alguna, suelen reproducir la vieja dicotomía entre internalismo y externalismo – muy diferente del proceder de los historiadores de la ciencia quienes hace ya bastante tiempo que superaron esta dicotomía y suelen estudiar las interacciones entre el desarrollo interno de las ideas y un contexto cultural, social y político más amplio. Un modelo clave para estudiar esas interacciones son las instituciones, por ejemplo, las instituciones en las que los matemáticos están trabajando. Y, en general, la gran tarea oficial en ellas es la enseñanza, formar nuevas generaciones. No solo están estas instituciones en la intersección entre el desarrollo conceptual dentro de la disciplina y las influencias contextuales sobre el funcionamiento de la institución, también ocurren interacciones directas y concretas entre la docencia y la investigación. No es muy extraño que sea la función propia de la enseñanza la que induzca a innovaciones en los conceptos matemáticos.

Los modelos más paradigmáticos para este funcionamiento fueron las instituciones de educación superior provenientes de la Revolución Francesa: debido al establecimiento de sistemas de educación pública, la enseñanza en estos niveles implicó de forma intensa incentivos para realizar revisiones sistemáticas de los conceptos matemáticos y su fundamentación. Pero también hay otros casos reveladores de innovaciones en la disciplina inducidos por otras instituciones en periodos anteriores. En particular, estudios reveladores han sido desarrollados por Christine Proust en relación con el funcionamiento de las escuelas de escribas (edubba) en tiempos de la Antigua Babilonia donde los maestros, formando a los escribas aprendices, trasgredían las tareas rutinarias de la enseñanza por medio de la validación de procedimientos o la resolución de nuevas clases de problemas.

Además de la presentación y discusión de casos pertinentes sobre esas interacciones, discutiré otro asunto pertinente para la interrelación entre el desarrollo matemático y la enseñanza: la noción de "elemento" o de elementarización de la ciencia. De hecho, la noción de elemento conecta el desarrollo de la matemática y los modos de enseñarla de un modo fundamental. Desde el libro de Geometría de Euclides, el término "elementos" expresa la intención de proporcionar una presentación sistemática de la teoría matemática, construida desde sus componentes básicos. Mientras se fija el estado del conocimiento de las matemáticas o de una de sus ramas en un cierto *tiempo o periodo, la noción de elementarización de Felix Klein dinamizó la propia noción, haciendo hincapié en las diversas etapas o fases en las que la acumulación de nuevos resultados y ramas realizada hasta el momento condujeron a una reestructuración de las bases, definiendo una estructura renovada de elementos. La nueva arquitectura desarrollada por Bourbaki constituye un caso bien conocido para manuales y libros de texto. Incitados por la docencia y por el propósito de servir para ella, los manuales están contribuyendo al progreso de las matemáticas.*

Palabras clave: *Enseñanza de las matemáticas, historia de las matemáticas, formación de profesores.*

INTRODUCTION

Willem Kuyk had denounced with this image – stalactite *versus* stalagmite - the widely shared view that mathematics education grows only by receiving some drops from above, figuring there the supreme instance for teaching. Bruno Belhoste had published a paper in 1998, arguing intensely against this view and pleading for re-assessing the role of teaching mathematics in the development of mathematics:

> Contre ce préjugé, je voudrais défendre le point de vue selon lequel la mise en commun du savoir mathématique, c’est-à-dire sa socialisation au sein de communautés de spécialistes et de communautés d’utilisateurs, qu’elles soient savantes ou de métier, voire même dans l’ensemble du corps social, constitue un aspect essentiel de l’activité mathématique, partie intégrante de l’activité d’invention (Belhoste, 1998, p. 289).

Yet, he conceptualised the role of teaching in a polarised manner: while speaking of contributions of mathematical research as “invention”, as “production”, he juxtaposed teaching as “reproduction”, as “socialisation”. In fact, he understood teaching as dissemination, as socialisation:

> L’enseignement constitue lui-même une modalité particulière de la socialisation du savoir dans laquelle le récepteur est en situation d’apprentissage, ce qui implique une mise en forme didactique et l’invention d’activités spécifiques (Belhoste, 1998, p. 290).

The problem seems to be that he restricted the meaning of “enseignement” to the side of the learner, not conceiving of the other pole, the teacher, realising the teaching.

And while he denounced the lack of attention for teaching, he associated it with “reproduction”: “Sous cette indifférence se cache en fait l’idée fausse que la production mathématique peut être séparée *a priori* par l’historien des conditions de sa reproduction” (Belhoste, 1998, p. 289).

I am arguing, instead, that the teaching of mathematics can transgress this type of reproduction in a much more decisive manner – and effectively did so throughout the historical development of mathematics.

I should like to discuss this productive role of teaching and in particular the methodological challenges for realising such analyses. Actually, historians of mathematics do not use to be very attached to methodological reflections, and, if ever dealing with them, use to reproduce the old dichotomy between internalism and externalism – quite contrary to historians of science who since quite a time overcame that dichotomy and use to study interactions between internal developments of ideas and broader cultural, social and political contexts.

Characteristic is the volume *Writing the History of Mathematics* (Dauben & Scriba, 2002). Although composed by eminent historians of mathematics, the volume documents that mathematics historiography is still strongly marked by the opposition between “internal” and “external” approaches, while a new German *Handbuch Wissenschaftsgeschichte* of 2017 declares this dispute as overcome and is open to much broader conceptual approaches, understanding science as just one form of knowledge—history of science being hence a part of *Wissensgeschichte*, the history of knowledge (Sommer, Müller-Wille, & Reinhardt, 2017, p. 3). In fact, this handbook realised an ambitious endeavour to reflect the methodology of history of science research. It presents, in particular, systematic chapters on recent research approaches. Pertinent for my issue here of interfaces is the chapter on cultural sciences and science history (Brandt, 2017).

IMPACT BY INSTITUTIONAL ANALYSES

A key pattern for studying such interactions are institutions – i.e., institutions in which mathematicians are working. And, in general, the official major task there is teaching, forming new generations. Not only are such institutions at the crossroads of conceptual developments within the discipline and of contextual influences on the functioning of the institution, there also occurs the direct and concrete interaction between teaching and research. Not too rarely, it is the function of teaching which induces innovations in mathematical concepts.

The most paradigmatic patterns for this functioning were higher education institutions from the French Revolution: due to the establishment of systems of public education, the higher degrees of teaching intensity entailed incentives for systematic revisions of mathematical concepts and their foundations.

An Old Babylonian institution: the *edubba*

But there are also revealing cases of innovations in the discipline induced by institutions in earlier periods. Particularly telling studies have been made by Christine Proust on the functioning of the scribal schools (*edubba*) in Old Babylonian times where the masters, forming the scribal apprentices, transgressed the routine tasks of teaching by validating procedures, by solving new classes of problems. While teaching the apprentices, the masters perfected and developed the practices of arithmetic and geometry already established.

Christine Proust reflected about the relation between teaching and research in a book chapter with the revealing title: “Does a master always write for his students? Some evidence from Old Babylonian scribal schools”. In this chapter, she analyses carefully the different types of cuneiform mathematical texts hitherto detected and accessible. The focus of the analysis is the period called Old Babylonian, i.e. from about 2000 to 1700 BC. The particular importance of her analysis is the re-assessment of traditional classifications of these collections. The more than 2000 clay tablets containing mathematical texts used to be classified as “table-texts” and as “problem-texts” – she characterises the two types as school texts, being “exercises written by young scribes during the elementary education” – and as “erudite texts”, “as opposed to the texts written by young pupils”; since all these texts were produced as activities of scribal schools, they all can doubtless be attributed to a teaching context (Proust, 2014, p. 70). Proust confirms that the mathematical texts were written by masters in the scribal schools, referring to a result of Jens Høyrup:

> The mathematical texts are school texts. […] Their authors […] were teachers of computation, at times teachers of pure, inapplicable computation, and plausibly specialists of this branch of scribal education; but they remained teachers, teachers of scribal school students who were later to end up applying mathematics to engineering, managerial, accounting, or notarial tasks. (Høyrup, 2002, p. 8)

Proust questions the traditional conviction that all these texts were produced for teaching purposes – and in particular that the erudite texts were all directed by masters to their students in the schools. For this re-analysis, she classifies the clay tablets from this period into three groups and analyses their function.

- The first type are school tablets containing stereotypical texts, which are similar to all scribal schools in Mesopotamia and which enable to reconstruct the school curriculum. The curriculum could thus be identified to be structured in an elementary and in an intermediate level. For a characteristic clay tablet, she resumes: “[…] is a school text in the sense that it was written by an apprentice scribe as he learned mathematics. In addition, its content has a clearly didactic function: the text is designed to teach precise mathematical knowledge" (Proust, 2014, p. 77).

- Proust emphasises that the school texts are not necessarily "puerile" texts. Among the "elementary" tablets there is the group of metrological tables and of multiplication tables. These tables were not necessarily created for teaching purposes. They might not have been produced originally for teaching, but for use in practical activities, and were later introduced in teaching, too – like in Modern Times the logarithmic tables and the trigonometric tables.
- The traditional group of "erudite" texts is divided by her in two different types. One (thus the second type) is called procedure texts, hence "lists of problem statements followed by their resolution and catalogues: lists of problem statements with no indication of their resolution" (Proust, 2014, p. 72). They look, at first, like texts composed by masters for their students. But they might have served also other purposes. Proust qualifies them as of "advanced teaching" since they work in a more sophisticated manner with the knowledge established in the elementary texts. A second group within this second type are the so-called catalogues: collections of problems, giving only their solution without the procedure for the resolution. Proust interprets them as used by the masters for classifying, ordering and arranging the knowledge: "These compilations could have been developed by masters to streamline the organization of the mathematical training curriculum, and maybe to classify and archive educational texts, thus constituting the first libraries" (Proust, 2014, p. 87).
- The third group, presented as "texts written by masters for their peers", show that the scribal school masters were also able "to develop projects not directly related to their teaching activities" (Proust, 2014). These are the "mathematical series" texts, i.e. written on several numbered tables, contain very long lists of problem statements – more than 1500 in total. Their writing techniques evidence a virtuosity in their writing techniques. Different from the problem statements in the catalogues which are widely encountered in the various locations of cuneiform findings, the problem statements put in the series texts are rather unique in the known mathematical collections: "One can even wonder if some of them were ever intended to be solved" (Proust, 2014, p. 91).
- Proust's intriguing conclusion refers to the existence of a community of masters, thus masters producing texts for their peers – and this due to the institutionalization of the scribal training:

  > The activities of masters included teaching and other objectives, such as communication between peers. These components are strongly interconnected, and yet they do not completely overlap. Developments in mathematics are the result of both the activity of teaching and interaction within a community of scholars.
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  > In the Old Babylonian period, education went hand in hand with creative activity, supported by a very active milieu. […] A network of long- distance links between the scribes seems to have existed, as shown by the similarities of the content of school tablets found through the Ancient Near East.
  >
  > […] Old Babylonian scribal schools were the places where the learning of cuneiform writing and arithmetic took place, but some of them were also intellectual centers. Some texts, written by the students themselves clearly reflect elementary teaching activities, others, written by the masters, bear witness to the activity of teaching, while others still show communication between scholars (Proust, 2014, p. 92)

IMPACT OF ESTABLISHING PUBLIC EDUCATION

As already emphasised, the impact of teaching on the production of knowledge can be studied and revealed more profoundly in particular after the French Revolution, with its enormous effect of establishing public education systems – and thus stimulating systematisation of knowledge and reflection upon its foundations.

A particular revealing aspect is the re-assessment of the role of textbook authors. Usually, textbook writing is understood as work for divulgation and disseminating of science, thus not contributing to the progress of science. It was in particular Thomas Kuhn's otherwise famous book on Scientific Revolutions, of 1962, which negated emphatically a productive role of textbooks (Kuhn, 1962). Yet, the period after the French Revolution is rich in providing contrary cases.

Then, the role of the textbook author also became investigated and even credited. While the share of textbook composition in establishing the elements of science was valued, the textbook author was also assessed in his productive contribution to science. A first such crediting was published in 1796, in a review of the second edition of J. A. J. Cousin's calculus textbook: *Leçons de Calcul Différentiel et de Calcul Intégral* (1796). The review was published in *La Décade*, the journal of the *idéologues*. Its anonymous author assumed the novel stance of attributing to a textbook author the rank of "inventor"—a notion in the discourse on science that designated an innovative scientist since Clairaut and d'Alembert:

> The author of an elementary book attains the rank of an inventor if he can present the elements, first, in the best order, in the most simple and the most clear manner: if he removes from the science all its technical wrapping and if he illustrates after each step the space traversed in such a manner that the student always knows well where he is (quoted from Schubring, 1987, p. 43).

And Sylvestre-François Lacroix (1765-1843), the prolific and successful textbook author since the first periods of the French Revolution, was distinguished even by the *Institut*—the new form of the Academy of Sciences since the Revolution—in being attributed a rank equal to an inventor. The distinction had been given in the *Institut's* report on the project presented by Lacroix to publish a treatise on the differential and integral calculus. In fact, he published this treatise as a three volumes textbook from 1797 to 1799. The report explained:

> To present difficult theories with clarity, to connect them with other known theories, to dismantle some of the systematic or erroneous parts which might have obscured them at the time of their emergence, to spread an equal degree of enlightenment and precision over the whole; or, put shortly: to produce a book which is at the same time elementary and up to the mark in science. This is the objective which Citizen Lacroix has taken to himself and which he could not have attained without engaging himself in profound research and by progressing often at the same level as the inventors (quoted from: ibid.).

TEACHER TRAINING AT HIGHER EDUCATION INSTITUTIONS

Since the times of Old Babylonia and its *edubba*, mathematics has been taught in all cultures known. Yet, the forms of teaching were not necessarily institutionalised ones: over large periods and epochs, teaching might have occurred in private forms, as apprenticeship instructed by a master – or even in autodidact forms. And institutionalised forms did not need to be continuous ones – due to the invasion by a warrior people, the *edubba* did not continue after the abrupt end of the Old Babylonian period. And the School of Computation, founded in 656 in China for training the state functionaries suffered many closings and ended functioning definitely by 1120.

For Greek and Roman times, one has no evidence of an institutionalised teaching of mathematics, it occurred rather in private forms. For the classical period of Islamic civilisation, there existed one institution, the *madrasa* where – while forming muftis and khadis, for religious and law practices – mathematics could be taught as auxiliary science. In the (West-) European Middle Ages, this form was transmitted and developed: teaching the *quadrivium* as a minor subject and by generalists, without a proper qualification in mathematics. In Pre-Modern times occurred the split in Western Europe between Protestant and Catholic educational systems. In Catholic states, dominated by the Jesuits, higher education mathematics teaching was abolished – except France, where courses in mathematics were given at the *Collège Royal*, but without any degrees and diplomas. Later in the eighteenth century, mathematics was introduced for engineering studies, as a “service subject”. In Protestant states, the mathematics professorships introduced as major innovation during the Renaissance, were maintained and somewhat expanded, but serving either as a service subject, too, or as a propaedeutic for studies at the higher, i.e. professional faculties.

In Modern Times, it was at first in only one state, in Prussia, that mathematics became established at its universities as a proper study course, leading to a professional qualification: to be a teacher of mathematics at its secondary schools: the *Gymnasien*. Thanks to prepare thus, the first time, for a professional career in mathematics, the mathematics professors at the Prussian universities – being, before these reforms from 1810 on, rather generalists teaching encyclopaedic courses – themselves specialised in mathematics, turning from being teachers to the combined role of teacher and researcher. Hence, mathematics achieved autonomy at these Prussian universities, and this thanks to their task of teacher education.

Yet, this profound change was due to contexts specific for Prussia, and thus not directly generalizable:

- After the defeat in the 1806 war with Napoleonic France, the reforms occurred in a specific cultural and social context, characterised in particular by neo-humanism, a cultural conception, valorising knowledge as an organic unity, comprising mathematics and the sciences as essential components;
- Consequently, mathematics became one of the three main disciplines in the Gymnasium curriculum, with a comfortable number of weekly hours, and as a further consequence, the stipulation to have at least two mathematics teachers at each Gymnasium;
- The mathematics teachers being formed at the level of scientists, they enjoyed high social status, thanks to the neo-humanist spirit present in the Prussian culture;
- Since the Prussian territory was large, there were at the beginning ca. 90 *Gymnasien* – a number steadily increasing -, entailing a permanent demand for teachers educated at the universities (see Schubring, 1991).

Nevertheless, social changes in the other German states, mainly after the revolutionary period of 1848/49, induced structural reforms in their secondary school systems, too, and thus reforming their universities and leading to an autonomy of mathematics and emergence of pure mathematics research. Teacher education remained to constitute the essentially unique professional career provided by mathematical studies at German universities until the 1940s at least. A second career pattern, for mathematicians in industry and for research careers, the mathematics diploma, emerged only in 1942.

Therefore, teacher education proved to be not due only to the Prussian case, as structural basis for enabling institutionalised research contexts. This can be confirmed even by a rather extremely different case: the emergence of research structures for mathematics in Brazil.

In fact, Brazil provides an extreme case. In this country, universities were founded only from the 1930s on. How had higher education functioned earlier on? It was basically institutionalised according to the French model of *écoles spéciales*: separate faculties for law and medicine, and a military academy, later complemented by polytechnic schools, for training engineers. Mathematics was taught, yes, at higher education level, but only in these institutions for engineers – thus, providing service courses, without proper study courses. And how could one become a mathematics teacher at a secondary school (a *colégio*)? For a vacant position, interested people would have to pass a *concours*, being prepared either autodidactically, or as a “collateral” effect of engineering studies.

Universities were founded in Brazil only from the 1930s on, due to changed social and cultural conditions. And then, in the first two universities—the *Universidade de São Paulo* (USP) and *Universidade do Distrito Federal* (UDF), resp. the *Universidade do Brasil* (in Rio de Janeiro) —it was the study course for the *magistério*, the teaching profession, within the equivalent of a Philosophy Faculty, which enabled a "take-off" of practicing mathematical research.

The first university to be founded was the USP, in 1934. Its distinctive new feature was a Faculty, which basically resembled the German Faculties of Philosophy: the FFCL—*Faculdade de Filosofia, Ciências e Letras*—which constituted in fact the kernel of disciplinary development. The founding decree of the USP, of 25 January 1934, in art. 5, § 1, stipulated the introduction of the teaching licence for those trained to become teachers at secondary schools as the "*licença para o magistério secundário"*. The degree afforded studies of a scientific discipline at the FFCL and accompanying pedagogical studies at the Institute of Education, attached to the Faculty. It is even more revealing, that the statutes projected doctoral studies; for such studies, only students having the *licenciado* diploma were mentioned to be admitted for an additional two years of studies (§ 12 of the decree).[xiv] Hence, a direct continuation was established: studying for a teaching license, and possible continuation for a doctorate.

At the UDF, founded in 1935 - it became later the likewise important *Universidade Federal do Rio de Janeiro* -, there was also a new Faculty besides the integration of various former professional schools, like the polytechnic schools, which was at first called *Escola de Ciências*. It had as its principal function the formation of teachers for secondary schools. The § 25 of the founding statutes, of 5 April 1935, attributed the function of providing study courses for the "candidato ao professorado secundário das ciências" in four different courses: for teachers of mathematics, physics, chemistry, and natural sciences. Doctoral studies were not yet instituted.

A research question which I am studying in this context is: what was the role of mathematics teacher education in France, in the 19th century, for the development of research? Usually, one credits France to have realised high levels of mathematical research, since about the period of Descartes and the creation of the Academy of Sciences in Paris, in 1666. After the French Revolution, this situation changed structurally, and it was quite different from Prussia and Germany: there existed now four types of institutions with teaching mathematics at higher education level:

- The *École polytechnique*, founded in 1794. It provided high level mathematical teaching, but not for forming in mathematics, but as preparatory studies for various engineering professions.
- The *facultés des sciences*, founded in 1808/1810, serving basically at the beginning as propaedeutic for the medicine faculties, and with a minor function for educating teachers, but with courses not beyond the college level.
- The *École normale supérieure* (ENS), founded in 1810. This institution should prepare, yes, teachers for the secondary schools, but was in its first decades also not of a high level and not dedicated to research.
- And there was the *Collège Royal*, founded in 1525, the only continuously functioning institution. But it offered free lectures, without a professional compromise or deferring of degrees – and an institution exclusively for teaching.

There was, in a parallel manner and in an independent structure, not connected to teaching, the *Institut*, the transformed former Academy, charged with research. But it suffered increasingly during the 19th century the often-discussed decline of science in France, which one can attribute, in the case of mathematics, to the dominance of the *École polytechnique*, with its focus on applications and engineering, and to the weak structures of the science faculties. This tendency became reverted by the last third of the 19th century, due to the expansion of the education system: the *facultés des sciences* succeeded in achieving a more independent status, and its professors were increasingly good mathematicians. But it was in particular the ENS which transformed, from the 1880s, into an institution forming highly qualified mathematics teachers, contributing decisively to now strengthened research in mathematics, initiated by the doctoral theses of its graduates (see Gispert, 1989).

ELEMENTS AND ELEMENTARISATION

In this last part, I will discuss another pertinent issue for interfaces between mathematical development and teaching: it is the notion of 'element' and of elementarisation of science. 'Elementarisation' here not means, as in common-day-language, trivialising knowledge, but to reveal the basic essence of knowledge and to structure knowledge from its basic constituents. In fact, the notion of element connects the development of mathematics and the modes of teaching mathematics in a fundamental way. Since Euclid's geometry textbook, the term 'elements' expresses the intention to give a systematic presentation of a mathematical theory, constructed from its basic components. This notion of elements and of elementarisation has been introduced and reflected in a paradigmatic manner during the Enlightenment, by d'Alembert (d'Alembert, 1755). It became the basis of the great project of the French Revolution to elaborate *livres élémentaires,* to realise this demand of the Enlightenment, to make scientific knowledge accessible for all, as the basis for a rational mode in the society.

While thus fixing the state of knowledge of mathematics or of one of its branches for a certain time and period, Felix Klein's notion of elementarisation dynamises the notion, emphasising the various stages where the meanwhile accumulated new results and branches led to a restructuration of the bases, defining a renewed structure of elements – the new architecture achieved by Bourbaki constituting a well-known case for textbooks (Bourbaki, 1948). It is incited by teaching and serving for teaching that textbooks are contributing to the progress of mathematics.

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