Fostering Relational Thinking While Negotiating the Meaning of the Equal Sign

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The Standards of the National Council of Teachers of Mathematics (2000) recommend that algebraic thinking should be developed beginning in the elementary grades. Teachers can do this while teaching other important components of the elementary curriculum by helping children attend to patterns, relations and properties of operations in all mathematical activities. True/False and open number sentences provide a good context to achieve this goal while also providing information about students' knowledge of operations and properties. In this article, we relate an experience in which we have used number sentences to begin to develop algebraic thinking.

Integral to children's work in algebra is an understanding of the equals sign, but, unfortunately, children often have serious misconceptions about its meaning. (Behr, Erlwanger & Nichols, 1980; Falkner, Levi, & Carpenter, 1999; and Saenz-Ludlow & Walgamuth, 1998). Children tend to perceive the equal sign as a stimulus for an answer and react negatively to number sentences that challenge their conceptions of the equal sign. For example, children often change 3 + 2 = 2 + 3 to 3 + 2 + 2 + 3 = 10 and 8 + 4 = __ + 5 to 8 + 4 = 12 + 5 =17 for expressing a string of operations. Older students continue to have difficulty using the equal sign correctly (Mevarech & Yitschak, 1983; Byers & Herscovich, 1977).

Understanding the equal sign is associated with what Carpenter, Franke and Levi (2003) call relational thinking. When students employ relational thinking, they can solve number sentences by focusing on the relations between the numbers in the equation instead of performing all the computations. For example, in the sentence 27 + 48 – 48 = __, students might recognize that adding and then subtracting 48 will leave 27 unaffected, therefore avoiding computation. This particular problem does not require a broad understanding of the equal sign because all the computation takes place on the left. Other equations, such as 8 + 4 = __ + 5, can only be solved if students have a broad understanding of the equal sign. Students can solve this sentence using relational thinking by noticing that 5 is one more than 4, so the unknown number has to be one less than 8.

This more sophisticated approach to solving number sentences has been observed in some elementary grade students (Carpenter, Franke and Levi, 2003; Koehler, 2004), but little evidence is available about the sequence of instructional activities that led to this thinking nor is there evidence of how readily students developed it.

Working with third-graders, we set out to explore the following questions: How do students' conceptions of the equal sign evolve when considering and discussing varied True/False number sentences? Do students develop relational thinking while we negotiate the meaning of the equal sign? Do students retain the new interpretation of the equal sign over time?

We worked with eighteen students over five sessions which took place during the students' regular school time (see table 1 for schedule of activities). The class was ethnically and linguistically diverse. Five students spoke a second language and two of them had significant difficulty understanding English. During the previous months we, as guest teachers, had worked with the class on a weekly basis doing a variety of mathematics activities. The classroom teacher was always present and sometimes collaborated with us in helping the students.

DAY 1 What do students understand regarding the use of the equal sign?

We began by giving the students an individual assessment with some open number sentences to determine their understanding of the equal sign. All but the second sentence could easily be solved using relational thinking by looking across the equal sign to compare the numbers. None involved computation that would be challenging for most third grade children. The students had difficulty interpreting these open number sentences. No student gave more than one correct answer, revealing they all held the misconception that the equal sign is a stimulus for an answer (see table 2).

After the students handed in their answers, we discussed two of the problems. All of the students thought the answer to 8 + 4 = ___ + 5 was 12. We told them that "mathematicians" would disagree. When we spotlighted the presence of 5 on the right side, they suggested the answer 17 (adding all the numbers). Then, when we said that "mathematicians" would still disagree, a student proposed to modify the sequence to 5 + 8 + 4 = 12. Finally, we explained that "mathematicians" use the equal sign to show that the whole expression on one side is equal to the whole expression on the other side. A student then gave the answer 7. Our discussion of the second problem, 14 + __ = 13 + 4, followed a similar pattern.

The use of the equal sign in these sentences seemed unnatural to the students. We discussed that they probably had never seen sentences like this before. One student specifically asked why the equal sign was in the middle. Through this discussion, we stressed mathematicians' interpretation of the equal sign to indicate it as a convention.

DAY 2 How do students' conceptions of the equal sign evolve when considering and discussing varied True/False number sentences?

Two months later, we assessed the students to determine how many students had adopted the mathematicians' interpretation of the equal sign. We chose true/false sentences to challenge students' computational mindset. When students saw an operation they wanted to complete that operation even before looking at the right side of the equal sign. Considering true/false sentences was a good way to force them to look to the whole sentences while challenging their conceptions of the equal sign. Our resource for these sentences was Carpenter, Levi and Franke (2003).

In table 3 we show the number of correct responses for each sentence. Three students responded appropriately to all of the sentences with the exception of the first two, and when correcting false sentences they wrote sentences of the form a + b = c + d. We inferred that these three students remembered and understood our previous discussion. Six students began accepting "backwards" sentences (10 = 4 + 6) but not sentences in the form a + b = c + d. The assessment showed that two-thirds of the class continued to have a misconception about the equal sign despite our previous discussion.

We had assumed that pointing out their misconceptions might be enough to change students' minds about the equal sign. Clearly it was not. Students needed opportunities to come to their own understanding of mathematicians' uses of the equal sign by sharing their thinking with one another. We used the assessment questions to spark discussion and began by asking students the meaning of the equal sign. Students said: "It is like if you have a scale and you have to put the same amount of both, on each side, for it to be equal," and "It means equal to, the same amount". We continued to negotiate the meaning of the equal sign with the students by discussing opposite opinions about each sentence and different ways of fixing the false sentences. For example, when discussing the sentence 2 + 2 + 2 = 3 + 3, some students affirmed it was true and one explained “it is true because 2 + 2 + 2 does equal 6 and so does 3 + 3”. Other students said that they thought it was false and explained “I thought it should be 2 + 3 = 5,” and “I thought it was false because it has the equal sign in the middle”.

During the discussion a student said “34= 34 + 12 is false because 34 + 12 would be more than 34”. The student did not report adding the numbers to decide if the sentence was true or false but compared both quantities 34 and 34 + 12, showing the beginning of relational thinking. We could see the students struggling to make sense of some of the sentences, hearing comments like “they want to trick you!” indicating we had successfully created dissonance, from which learning might result.

What sentences do students generate?

To further assess and stimulate students’ conception of the equal sign, we asked them to write true sentences of the form _ + _ = _ + _, _ - _ = _ - _ or _ + _ = _ - _. These templates might confuse students who interpret the underline as a variable and assume the same number has to go in each blank. This did not cause a problem for our students. Writing number sentences was a good activity for helping them to clarify and consolidate their understanding. All but two of the students were able to do it, although we helped four of them to get started because they were writing sentences of the form a + b = c.

In this activity, students could generate more or less difficult sentences depending on their own choice. The students used multiplication and division as well as addition and subtraction in their sentences (see fig. 1 and 2). In many cases like in figure 2, students multiplied or divided by 1 or in other cases they added 0. This was an easy way for them to generate sentences which gave us insight about their knowledge of the identity property. While these sentences showed understanding of the equal sign they did not show relational thinking. Some students wrote sentences that did show relational thinking. A few sentences took the form a + b = (a – 1) + (b + 1) as in 51 + 51 = 50 + 52. One student used relational thinking to decompose addends in changing 90 to a string of addends (see fig. 3).

One of the students showed a tendency to write the answer of the operation in the middle of the sentence (separating both sides) (see fig. 4). We also found this pattern in another student’s work on Day 2. This seems to show a necessity to have the “answer” in the equation and could be the bridge between the more familiar a ± b = c form to the less familiar a ± b = c ± d form.

<u>DAY 3</u>: DO students develop relational thinking while we negotiate the meaning of the equal sign?

Because we wanted to promote relational thinking and we recognized that students' understanding of the equal sign was still fragile, two weeks later we planned a discussion of some true/false sentences (see fig. 5), some of which had been previously written by the students. We asked the students if they could solve the sentences without doing the arithmetic in an attempt to promote the use and verbalization of relational thinking. Students got engaged in sharing their thinking.

When discussing the first sentence 20 + 20 = 20 + 20, students claimed "it is true because they are the same numbers", and "you don't need to write the answer". One student claimed "it is false because the equal sign is in the middle". This last comment reminded us that not all students had adopted the mathematicians' interpretation of the equal sign. In the sentence 7 + 15 = 100 + 100, all the students were sure it was false. One explained: "it is false because 7 + 15 is small and 100 + 100 is 200", and "7 + 15 is not even 100". Students also provided steps towards relational thinking in their comments about other sentences, showing the kind of thinking we were hoping to foster. They also explained "[51 + 51 = 50 + 52] is true because if you take the 1 from the 51 to the other 51 you get 50 + 52", "[15 + 2 = 15 + 3] is false because 3 is bigger than 2".

We observed a significant growth in students' understanding because they used the equal sign with a broader interpretation when correcting the false number sentences, proposing sentences like 7 + 193 = 100 + 100, 10 – 7 = 7 – 4 and 15 + 3 = 15 + 3.

The discussion of the last sentence, 3 + 3 + 3 = 9 + 2 = 11, was especially interesting. We tried to challenge the students' understanding of the equal sign by playing the Devils' Advocate. One student claimed, "I think it is false because 3 +3 + 3 = 9 and 9 + 2 = 11," to which we responded, "Isn't that what it said there?" Regardless of our attempts to cause confusion about the incorrectness of the sentence, some students' knowledge of the equal sign was firm enough to maintain their opinion about its incorrectness, and they defended their opinion throughout the discussion. One said, "3 + 3 + 3 does not equal 11." Other students became confused saying, "I am not sure… It is in part true, and it also seems false," and, "It is true because 3 + 3 + 3 = 9 and 9 + 2 = 11." Because there was a difference of opinion, we finally explained that mathematicians would say it was false because 3 + 3 + 3 does not equal 9 + 2. We explained that mathematicians use arrows when showing a string of operations like in 3 + 3 +3 → 9 + 2 →11.

Our final activity on this day was to assess students' understanding of the equal sign with some written questions (see fig. 6). Twelve of the eighteen students solved at least 5 of the 6 items correctly. Three students continued to have the misconception of the equal sign as a stimulus to give an answer but showed acceptance of "backwards" sentences ($c = a \pm b$). Another three students did not successfully perform the assessment activity. As a result of our discussion and activity on Day 2 and this discussion on Day 3, nine more students seemed to have constructed a broader understanding of the equal sign.

<u>DAY 4:</u> Once students correctly interpret the equal sign, do they employ relational thinking to evaluate number sentences?

Two weeks later we had a discussion to elicit relational thinking and consolidate students' understanding of the equal sign. We discussed true/false sentences (see fig. 7) and more students verbalized relational thinking. In all but one of the sentences (34 + 28 = 30 + 20 + 4 + 8) students gave explanations based on relational thinking such as "[27 + 48 – 48 = 27] is true because there is a plus 48 and a minus 48. That's going to be zero", "[103 + 205 = 105 + 203] is true because 5 + 3 = 8 and there are two eights matching, so they are both the same", and "[12 – 7 = 13 – 8] is true because they added one to the seven and they added one to the twelve". In these cases students did not compute to determine their response. On the other hand, in all the sentences some students also gave justifications based on the computation of the operations on each side, showing that students were not exclusively using relational thinking.

During this discussion seven of the students' contributions to the discussion displayed relational thinking. Only one comment evidenced some remaining misconception about the use of the equal sign. From the previous days we knew this student tended to consider the equal sign as a stimulus for an answer and during the discussions he did not seem to become aware of a broader conception.

<u>DAY 5</u>: Do students retain the new interpretation of the equal sign over time?

To determine the durability of students' understanding of the equal sign, two months later we gave students an assessment which was similar to the one administered on the first day. Twelve of the fifteen students correctly answered 5 of the 7 sentences which we felt showed they understood the meaning of the equal sign. Another two presented the misconception of the equal sign as a stimulus to give an answer but showed acceptance of "backwards" sentences ($c = a \pm b$). One student did not solve the assessment correctly and did not show a clear conception of the equal sign.

We included one additional item 238 + 49 = __ + 40 + 9 to assess the use of relational thinking, asking for an explanation. Four students did not have time to complete the problem. 7 of the 15 students successfully solved the problem with four of these providing clear explanations reflective of relational thinking. One wrote, "cause 40 + 9 = 49 then you add 238 then it makes the same answer." Another wrote, "You split 49 in 40 and 9 and it's the same." Other successful students provided explanations that were more difficult to interpret. We believed these students employed relational thinking because they did not subtract, and we could not imagine another way they could have solved this problem. In addition to these seven students who were successful on the 238 + 49 item, there were four other students who had expressed relational thinking on Day 4. These 11 students had begun to use relational thinking in their analysis of the equations.

By looking at the evolution of the students from Day 3 to 5, we observed that two of the students regressed in their understanding of the equal sign, answering incorrectly at least five of the seven sentences on Day 5 by considering the equal sign as a stimulus for an answer. Three other students improved in their understanding of the equal sign from Day 3 to Day 5. They solved correctly most of the sentences in the assessment on Day 5. The rest of the students showed a stable performance from Day 3 to 5.

CONCLUSIONS

We successfully helped the students to broaden their conceptions through the different tasks. Students' conceptions evolved from a "stimulus for an answer" to the acceptance of "backwards" sentences, to understanding the equal sign as indicating equality of expressions. The variety of sentences we considered challenged students' understanding and forced them to think in new ways about the symbol. Asking the students to write their own sentences was particularly helpful for assimilating and consolidating their broadened conceptions. We observed that telling the students what the equal sign means was not sufficient for helping them to adopt the conventional understanding of the equal sign. Rather, engaging them in discussion in which they had to defend their opinions was critical to their development.

We were only partially successful at initiating relational thinking. Getting students to step back and look at the whole equation is a challenge because the students tended to focus on computation, proceeding from left to right as they read number sentences. Understanding equations requires looking to the whole sentence, beginning at the equal sign (which is in the middle) and then looking to both sides. We observed that some sentences like $11 + 3 = 4 + \_\_$ forced the students to step back and try to make sense of the whole sentence because they know that $11 + 3 \neq 4$. In the future we plan to use more sentences of this form to stimulate relational thinking. Discussion is critical to foster relational thinking because it takes the emphasis off of the computing answers and places the emphasis on noticing patterns.

We were convinced by our foray into true/false and open number sentences that they are a fruitful place to initiate algebraic thinking. The beauty of these tasks is that they have multiple entry points. In solving them, students can use computation or they can begin to consider relations between the numbers and operations. These multiple approaches engender the kind of mathematical discussion that supports reflection. We concluded that negotiating new interpretations of the equal sign by exposing students to these unfamiliar number sentences was an important first step on the road to algebraic thinking.

References

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Session	Day 1	Day 2	Day 3	Day 4	Day 5
Date	11-20-03	2-5-04	2-19-04	3-4-04	5-13-04
Number of students in class	13	15	18	18	15
Session activities	- written assessment - brief discussion	- written assessment - discussion - written activity - brief discussion	- discussion - written assessment	- discussion	- written assessment
Number sentences used	Open number sentences	True/false number sentences	True/false number sentences	True/false number sentences	Open number sentences

Table 1: Schedule and organization of the lessons.

Sentences on Day 1	**Correct answer**	**Most common incorrect answer**	**Other answers**
8 + 4 = __ + 5	7 (0)	12 (13)	None
__ = 25 – 12	13 (3)	7 (2)	5, 17, 18, 20, 25, 35
14 + __ = 13 + 4	3 (2)	1 (5)	-1, 0, 3, 17, 31
12 + 7 = 7 + __	12 (3)	26 (3)	0, 3 ,5, 6, 7
13 - 7 = __ - 6	12 (0)	6 (11)	3, 7
__ + 4 = 5 + 7	8 (0)	1 (9)	2, 3, 12

Table 2. In parentheses we indicate the number of students who gave each response. N = 13

Sentences on Day 2	**Correct answer**	**Number of students responding**		
		True	**False**	**No answer**
3 = 3	T	5	9	1
7 = 12	F	2	10	3
10 = 4 + 6	T	9	5	1
2 + 2 + 2 = 3 + 3	T	5	9	1
34 = 34 + 12	F	2	11	2
99 + 4 = 4 + 9	F	3	8	4
37 + 14 = 38 + 13	T	5	6	4

Table 3. The number of students who responded correctly is highlighted. N= 15.

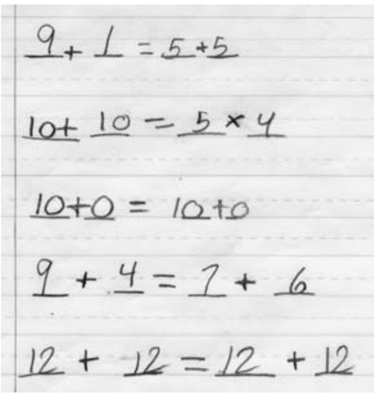

Figure 1

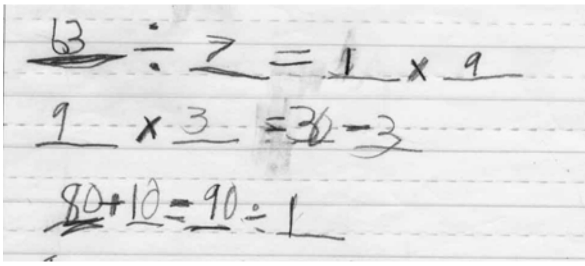

Figure 2

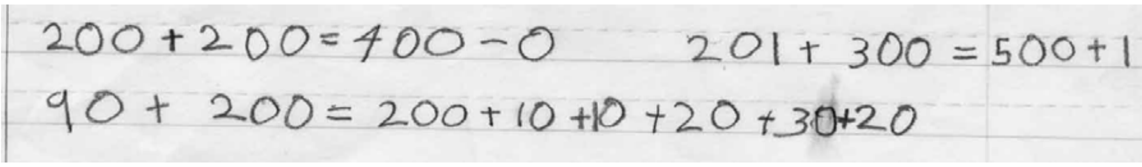

Figure 3

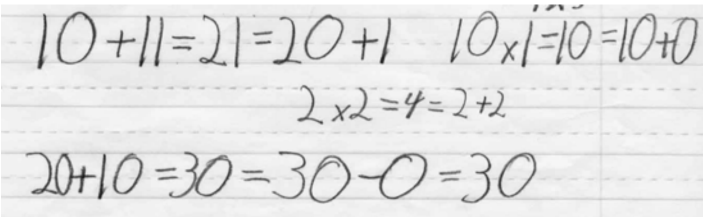

Figure 4

True/false sentences for the discussion on Day 3

20 + 20 = 20 + 20,	6 – 6 = 1 – 1,
10 x 10 =100 = 90 + 10,	10 – 7 = 10 – 4,
7 + 15 = 100 + 100,	51 + 51 = 50 + 52,
12 + 11 = 11 + 12,	5 + 1 = 7 – 1,
15 + 2 = 15 + 3,	3 + 3 + 3 = 9 + 2 = 11
3 x 5 = 15 ÷ 1,	

Figure 5

Assessment activity on Day 3

1. Fill the blank with a number which makes the number sentence true.

   5 + 1 = __ + 2

   4 + __ = 2 + 2 + 2

   __ + 0 = 30 – 10

2. Decide whether the number sentence is true or false.

9 = 5 + 4	T	F
3 + 7 = 10 + 6	T	F
8 = 8	T	F

3. Write a sentence that is true.

Figure 6

True/false sentences for the discussion on Day 4

37 + 23 = 142

27 + 48 – 48 = 27

34 + 28 = 30 + 20 + 4 + 8

76 = 50 - 14

4 x 5 = 5 +5 +5 + 4

20 + 15= 20 + 10 + 5

103 + 205 = 105 + 203

12 – 7 = 13 – 8

Figure 7