INDUCTIVE REASONING IN THE JUSTIFICATION OF THE RESULT OF ADDING TWO EVEN NUMBERS

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In this paper we present an analysis of the inductive reasoning of twelve secondary students in a mathematical problem-solving context. Students were proposed to justify what is the result of adding two even numbers. Starting from the theoretical framework, which is based on Pólya's stages of inductive reasoning, and our empirical work, we created a category system that allowed us to make a qualitative data analysis. We show in this paper some of the results obtained in a previous study.

Proof appears as a real problem in different educational levels. On one hand, although pre-service teachers are accustomed to do many formal proofs, they have difficulties in proof teaching (Cañadas, Nieto & Pizarro, 2001). On the other hand, secondary students do not make as much progress as they are supposed to do. One possible reason lies on the fact that they cannot suddenly acquire the necessary reasoning skills for developing formal proof. They need a period of time to transform their daily reasoning into formal one (Jones, 1996). Some studies show that primary and secondary students are able to formulate conjectures, examine and justify them if they start working from particular cases (Lampert, 1990; Healy & Hoyles, 1998). These actions are part of inductive reasoning process. We are interested in analyzing this reasoning as a process instead that a result.

This paper consists of four main parts. First, we present the theoretical framework of the study reported. Second, we show the methodology used to analyze inductive reasoning process. Third, we present some sample data and finally, we discuss some results of the study.

THEORETICAL FRAMEWORK

Proof and reasoning

Many studies on mathematical proof use rigor as a criterion for establishing a rank between different types of proofs. The sort of reasoning and the type of language used determine rigor. Proofs where inductive reasoning is predominant must be in a lower level that proofs which involve deductive reasoning. In this sense, if we find a proof in which drawings or concrete numbers are involved and where inductive reasoning is predominant, it will be considered an informal proof. On the other hand, as much deductive reasoning and algebraic language involve in a proof, more formal it will be considered (Cañadas, Castro & Gómez, 2002).

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Due to the aforementioned relation between a proof and the reasoning involved, we will consider proof as a formal part of the reasoning process. Reasoning is to give reasons for explaining a fact. A chain of reasoning with some rigor characteristics (which includes sort of reasoning and language) leads to a formal proof.

Mathematical induction is a formal proof based more on deductive than on inductive reasoning, so, it is not considered part of inductive reasoning. We have to mention that some processes of inductive reasoning finish with mathematical induction but it does not always occur. The task proposed to the students in this study cannot be justified by mathematical induction.

Inductive reasoning

Inductive reasoning and deductive reasoning are the two traditional types of reasoning considered. Inductive reasoning is the natural reasoning that allows us to get the scientific knowledge (Pólya, 1967). Neubert & Binko (1992) connect inductive reasoning in Mathematics with patterns recognizing and its application to Numbers Theory. Pólya indicates inductive reasoning in mathematics teaching as a method to discover properties from phenomena and to find the regularities in a logical way.

Inductive reasoning in Mathematics Education is the reasoning process that begins with particular cases and gets the generalization from these cases. Pólya (1967) indicates four steps for a correct process of inductive reasoning: experiences with particular cases, conjecture formulation, conjecture proof and verification with new particular cases. Based on these steps and taking into account our previous study (Cañadas, 2002), we will consider the following actions related to the justification of a statement where inductive reasoning appears:

– Observation of particular cases. The starting point is the experiences with particular cases of the problem set out. We observe if the students used particular cases spontaneously, which types of particular cases they used and how many.

– Organization of particular cases. It can be used some strategies to systematize and facilitate work with particular cases. The most common strategy used is the organization of particular cases by data lists or tables (Allen, 2001; Grupo PI, 2002).

– Search and prediction of Patterns. Pattern is an important notion in Mathematics based on the idea of repeated and regular situation. Find patterns and use them is an important strategy in mathematical problem solving (Steen, 1988; Stacey, K. & Groves, 1989). Mathematics is considered by Keith (1994) as the patterns science.

– Conjectures formulation. A conjecture is a statement based on empirical facts, which has not been validated. In this report we seek that students formulate their own conjectures.

– Conjectures validation. The conjecture can be true for some particular cases. We are sure about the truth of such conjecture for those specific cases but not for other ones.

– Conjectures generalization. Based on the knowing that a conjecture is true for some particular cases, we hypothesize that the conjecture is also true for more particular cases apart from the first ones. Generalization is one of the actions considered by Pólya as one of the basis of inductive reasoning.

– General conjectures justification. The first step in the way to confirm or reject a general conjecture is validating it with particular cases. These cases never draw us to confirm the veracity of the general conjecture. It is necessary to give reasons that explain the conjecture with the intention of convincing another person to justify the generalization. We look for a fair examination of the conjecture and, if it is necessary, we will do a formal proof as the latest justification that guarantee the veracity of such conjecture.

METHODOLOGY

To observe the students working, we chose a methodology based on individual interviews. The interviewer (represented by “I” in collection and data analysis) was one of the researchers. She had an interview plan which allowed her to guide students by questions so that we can observe their reasoning in getting the problem solution (Cohen & Manion, 1990).

Due to curriculum indications and our research objective, this study involved secondary students. We interviewed twelve Spanish students (six girls and six boys) from the four years that comprise Secondary Education in Spain. We interviewed three students from each year. Academic results were the main criterion to choose the three students from each year in order to have a wide variety of answer. For presenting and analyzing the data, we symbolize the students as 1, 2, 3 and 4 depending on the year s/he belongs to. We assigned A, B or C for high, medium or low academic results. For example, 3A is a third year student whose academic results are higher than her/his classmates.

In Mathematics Education, problem solving is a highly formative task due to the knowledge, skills and reasoning that it arises (Segovia & Rico, 2001). Spanish curriculum recognizes problem solving as one of the main objectives in secondary mathematics because it is supposed to develop reasoning abilities and provide typical attitudes and habits of mathematical work (Boletín Oficial del Estado, 2003). In this report, problem solving is used to arise the reasoning of secondary students.

The task we proposed to these students was to justify the result obtained when adding two even numbers. The main aspect was the justifications of their reasoning, paying more attention to the process instead of the result. This task was adequate because it came up actions related to inductive reasoning (generalization, patterns…) and it refers to Numbers Theory. Moreover, we took into account that the students belonged to an educational level where they are supposed to be able to solve the task.

DATA COLLECTION AND ANALYSIS

We collected data in three ways: the interview was recorded in an audio tape, we gave worksheets to the students so that they could write their work if they wanted and the interviewer took notes during and after each interview about relevant research aspects impossible to be registered in the tape.

We completed the interviewer's notes with the students' worksheets. We transcribed the tapes and introduce them, together with the interviewer notes, in a qualitative data analysis program, Nud*ist revision (N4). This program allowed us to see the data in a structured way and to discover details, patterns and relations that would be more complicated by hand.

In the following table we show the category system used in this study. The categories emerged from the actions mentioned in the theoretical framework. For almost all the categories, we considered subcategories which emerged from other studies (Goetting, 1995; Edwards, 1999; Miyakawa, 2002).

CATEGORIES	SUBCATEGORIES
Observation of particular cases	Spontaneity Proposed by interviewer Number of particular cases Sort of particular cases
Organization of particular cases	Systematic way Tables
Search and prediction of patterns	
Conjectures formulation	Use of school knowledge
Conjectures validation	
Conjectures generalization	Characterization of even numbers
General conjectures justification	Justification necessity Based on particular cases General case Kind of language

RESULTS

All the students gave an answer equivalent to "the result of adding two even numbers is another even number". We based on the aforementioned category system to present the results. All the subcategories did not appear in the analysis of this task. We summarize some of the main results in two tables and then we will make some comments in relation with the determinated categories. The first table refers to particular cases and the second one shows students' advance forward the general case.

Table 1. Particular cases

	Observation		Organization	Patterns
	Spont	2 is one addend		
1C		X		X
1B	X	X		
1A	X			X
2C	X			
2B			X	
2A				
3C			X	X
3B	X			X
3A				X
4C	X	X		
4B	X	X		
4A				X

- Observation of particular cases. Just two students (1B and 2B) mentioned particular cases before giving the correct answer. All the students stated that the result of adding two even numbers is another even number. As we can see in the table, six students turned to particular cases in a spontaneous way, without any interviewer's indication. Finally, when they were required to justify their conjectures, all the students used specific numbers, although in different ways. 1A, 2A, 3A and 4A used particular cases just as support of their reasoning because they look for the justification in the general case. The rest of them used particular cases when they try to justify their conjectures.

All the students used 2 as a highlighted number in their justification. Four of them (1C, 1B, 4C and 4B) considered 2 as one of the addends in all their sums. All the students mentioned in their reasoning small and close numbers. No students mentioned even numbers higher than 18 as addends. They tried to work with higher numbers when the interviewer guided them.

I: ok… and what happens for example, with 1784 and 2320?

- Organization of particular cases. There were no students who organized particular cases. Just two of the students (2A and 3B) used written language for particular cases, listing them.

- Search and prediction of patterns. The interviewer guided the students' work with particular cases so that they could find regular situation for even numbers. Half of the students (1C, 1A, 3C, 3B, 3A and 4A) noticed that even numbers ended by an even digit, which is the most frequent characterization used by the students. We found among them the students who finally justify correctly their conjecture.

- Conjectures formulation. All the students formulated the right conjecture. They were convinced that the sum of two even numbers is another even number.

In this study, the inductive process before the conjecture formulation is short. One reason for this is that all the students seemed to previously know the result of the proposed task.

Table 2. General case

	Characterization		General Justification			
			Particular cases	Language		
	Termination	Divisib		Natural	Written	Algeb
1C	X			X		
1B						
1A	X			X		
2C			X			
2B		X	X			X
2A		X			X	X
3C	X		X			
3B	X			X	X	
3A	X	X	X			
4C		X				
4B		X				X
4A				X		

- Conjectures validation. 2B, 2C, 3C, 3A based their justification on particular cases and they did not make progress to general justification.

I: imagine that you have to convince me that the result is always an even number.
2C: Then... we had to... maybe... in a practical way, hadn't we? For example, tell me an even number.
I: ok, one thousand and seven hundred.
2C: One thousand and seven hundred… and another even number, for example four. You see? Then, I add both numbers and the result is an even number.

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- Conjectures generalization. All the students except 2C based their reasoning on even numbers characterizations. 3A noticed the common termination of even numbers but did not advance in this sense. Four students (2B, 2A, 4C and 4B) detected this pattern of even numbers. Then, they used the characterization of even numbers as divisors or multiples of number 2. 2B, 2A and 4B turned to algebraic language to express this characterization and write 2x to represent an even number.

The difficulties in the task appeared when they were required to justify this conjecture. After getting the general conjecture, two students (4C and 4A) proved it with new particular cases but they did not justify it.

- General conjectures justification. Three students (1A, 2C and 4B) recognized the necessity of justifying their result by their own. The rest of them saw the result as an evident statement obtained from particular cases, without any additional justification to be convinced of its truth.

Some of the students used the termination characterization (1C, 1A, 3B, 3C and 4A) of even numbers. They based their justifications on even numbers smaller than 10 and they argued that all the sums made with numbers whose termination was 0, 2, 4, 6 or 8, gave as result another number that ended in 0, 2, 4, 5, or 8. This means that the result was another even number.

2B, 2A, 3A, 4C and 4B mentioned divisibility criterion. 2A, 4C and 4B used this criterion and the common factor concept. Just one of them (2A) justified her conjecture with algebraic language:

> 2x (represents) an even number
> 2y (represents) another even number
> 2x + 2y = 2 (x + y) and this is another even number because it appears multiplied by 2.

Another of these students (4B) made the right justification spoken:

> Because all the numbers can be decomposed in a number multiplied by two. Then, for adding two even numbers, you can get the number two as a common factor and you have the number two multiplied by a number and then, this is an even number too.

Finally, seven students justified the general conjecture correctly using natural, written or algebraic language (1C, 1A, 2B, 2A, 3B, 4B and 4A).

DISCUSIONS

We made a categories and subcategories system basing on the theoretical framework and previous studies related to inductive reasoning but due to specific characteristics of this report, some of the considered subcategories did not come into view in the data analysis.

Inductive reasoning appeared in an implicit or explicit way in all the students. Students turned to particular cases when they try to justify their conjecture, so, we can conclude that inductive reasoning appears naturally on these educational levels.

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Although all of the students stated directly the result of adding two even numbers, the justification for the general case was considered difficult by them. Many students considered evident the conjecture from particular cases and did not think of the necessity of a general conjecture justification to validate their statements. This was an obstacle because it did not allow them to make progress in their reasoning forward the generalization.

The seven students who justified the general conjecture found a mathematical pattern from particular cases obtained from the characterization used in their justifications. This confirms that searching pattern is a relevant step in inductive reasoning process and an important strategy in mathematical problem solving.

We did not notice significant differences among students' reasoning of different educational levels. It happened in the same way with students with different academic results belonging to the same year. We just detected some differences in the way they expressed their argumentations. First year students used natural language and not used neither written nor algebraic language.

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