GRAPHICAL REPRESENTATION AND GENERALIZATION IN SEQUENCES PROBLEMS

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In this paper we present different ways used by Secondary students to generalize when they try to solve problems involving sequences. 359 Spanish students solved generalization problems in a written test. These problems were posed through particular terms expressed in different representations. We present examples that illustrate different ways of achieving various types of generalization and how students express generalization. We identify graphical representation of generalization as a useful tool of getting other ways of expressing generalization, and we analyze its connection with other ways of expressing it.

INTRODUCTION

The number of works focused on the relation between the algebra and the expression of generalization has increased since the work of Mason, Graham, Pimm, and Gowar (1985). Some of this work deals with the idea that this connection does not seem to be direct for secondary students (e.g. Lee, 1996; Lee & Wheeler, 1987). Algebraic language is not the only way of expressing a generalization. For example, Mason and Pimm (1984) consider that natural language has a fundamental role in the generalization process, and Radford (2002) shows how some students used verbal and gestural means to express generalization. Currently we are engaged in a project in which some of the aims are related to the relation among generalization, and ways of achieving and expressing it.

In this paper, we focus attention on the generalization developed by students in a problem solving context and on the different representations used by students in generalization problems with different characteristics. In particular, this paper extends previous work by reporting on ways of achieving generalization through an inductive process, its relation with representations used, and how students in our investigation express generalization when they work on a written questionnaire constituted of problems involving sequences.

We first present the main ideas concerning our approach to generalization and different representations, focusing on different types of generalization. Secondly, we present our research questions. Then we present a general description of the methodology used. After this, we outline our findings and interpretations. We finally present the conclusions.

Cañadas, M. C., Castro, E., & Castro, E. (2011, february). *Graphical representation and generalization in sequences problems*. Paper presented at the CERME 7, Rzeszów, Poland.

ALGEBRAIC THINKING AND GENERALIZATION

Taking a semiotic approach, we consider that students are thinking algebraically when they “act in order to carry out the actions required by the generalizing task” (Radford, 2002, p. 258). From this perspective, generalization is achieved when students are able to identify a common pattern that arises from some particular cases and to apply this commonality to other particular cases.

We refer to *empirical generalization* in terms of Dörfler (1991). This type of generalization starts from work on particular cases and is very close to pattern identification. Cañadas and Castro (2007) developed a model to describe Secondary students’ inductive reasoning. This model is comprised of seven states, of which generalization is one. From our viewpoint, inductive reasoning is equivalent to what Pólya (1967) called *induction*. Different authors, including Pólya, assert that generalization is a key state in the process of acquiring mathematical knowledge (Neubert & Binko, 1992; Mill, 1858).

Thus generalization can be seen as “pattern generalization”, which is considered one of the prominent routes for introducing students to algebra (Mason, Graham, Pimm & Gowar, 1985; Radford, 2010, p. 37). However, it is assumed that algebra is not the only way of expressing a pattern nor is algebraic thinking the only way of forming a generalization.

REPRESENTATIONS

There is a general agreement amongst researchers of the need to distinguish between external and internal representations of students’ knowledge. In our research, we focus on external representations. These representations allow the students to express concepts and ideas, since ideas must be represented externally in order to communicate them (Duval, 1999; Hiebert & Carpenter, 1992). In this paper, we pay attention to the external representations produced by students that have a trace or tangible support even when this support has a high level of abstraction.

We also consider *multiple representations* (e.g. van Someren, 1998). Multiple representations have benefits on schema construction processes; but it is not always beneficial for learning (Kolloffel, Eysink, Jong, & Wilhelm, 2009). Figueiras and Cañadas (2010) distinguish two different kinds of multiple representations: (a) combined representation, which concerns the use of different representations (as mentioned by previous authors); and (b) synthetic representations, which are multiple representations but under the additional condition we must consider them as a whole to give sense to the student’s response (p. 3).

WAYS OF EXPRESSING THE GENERALIZATION

Algebra is one way of expressing a pattern, but it is not the only one. We concur with Radford’s idea of algebraic generalization. “It rests on the noticing of a local commonality that is then generalized to all the terms of the sequence and that serves as a warrant to build expressions of elements of the sequence that remain beyond the perceptual field" (Radford, 2010, p. 42). This author distinguishes this kind of generalization from *arithmetic generalization,* which is characterized by staying in the realm of arithmetic (p. 47). Students who generalize arithmetically have identified the pattern and are usually conscious that this pattern is unpractical for other terms of the sequence.

In the context of analyzing the generalization process in problems involving sequences in a written problem solving test, Cañadas and Castro (2007) distinguish between algebraic and verbal representations as two ways of expressing the general term of a sequence. The first way concerns the use of symbols and numbers, in which each term of the sequence can be obtained by substituting the symbols with concrete numbers; and the second one refers to the use of natural language to express the generalization. These authors left an open question related to the role of graphical representation in the generalization procedure and the expression of such generalization. We tackle this question in this paper.

RESEARCH QUESTIONS

We break down our research interests into three research questions for this paper, which concerns two central aspects of the generalization: (a) generalization process, and (b) generalization expression. These questions are:

- What is the role of graphical representation in the generalization process?
- How do the students express the generalizations achieved?
- What are the features of graphical expression of the generalization?

METHODOLOGY

Students

We took 359 students in years 9 and 10 of four State Spanish Schools whose teachers were close to us.

We obtained information about students' educational experiences related to generalization, problem solving, sequences, and algebra from four sources: (a) Spanish curriculum, (b) informal interviews with students' teachers, (c) mathematics textbooks used by students, and (d) students' notebooks.

Spanish Secondary curriculum does not include the generalization process explicitly. It includes reasoning as one of its main objectives. However, it contains just some actions related to inductive reasoning, such as: (a) to recognize numerical regularities, (b) to find strategies to support students' own argumentations, and (c) to formulate and to prove conjectures (Boletín Oficial del Estado, 2003).

Students had previously studied sequences. They had worked on problems using inductive reasoning, usually involving sequences, on occasion. These kinds of problems are usually presented with particular cases expressed numerically and are most of them de-contextualized. Students had begun the study of algebra between one or two years before the research commenced (depending on the year they studied by the time of this research). These lessons included work related to interpretation of formula and algebraic expressions, and first grade equations. We consider that these students had the experience required to focus on the research questions posed. Specifically, the students had sufficient content knowledge of sequences. On the other hand, our analysis of their previous educational experiences demonstrates that they were not used to solving the kind of problems posed.

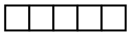
Problems Posed

We prepared a written questionnaire with six problems involving linear and quadratic sequences. We asked students to work individually on this questionnaire for an hour. The problems of the questionnaire were selected according to our research objective and using the characteristics that arose through *subject matter analysis* (Gómez, 2007) of natural number sequences: (a) the order of the sequence, (b) the representation used in the statements, and (c) the task proposed.

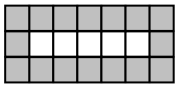
In this paper we will focus on three problems which involve linear and quadratic sequences [1], and with different representations used in the statements. Each problem was focused on a "far generalization" task (Stacey, 1989). So, particular cases were presented in the problems statements to lead the students to generalize at some point. Each problem had a complementary task consisting of justifying their responses [2]. Since sequences are a particular kind of function, we took into account the four representation systems traditionally considered for functions: (a) graphical, (b) numerical, (c) verbal and (d) algebraic (Janvier, 1987). In accordance with our research objectives, problems lead the students to work on information given through particular cases expressed in a graphical, numerical or verbal context.

In what follows, we focus on three of the six problems: problems 3, 4, and 5.

Imagine some white squares tiles and some grey square tiles. They are all the same size. We make a row of white tiles:



We surround the white tiles by a single layer of grey tiles.



- How many grey tiles do you need to surround a row of 1320 white tiles?
- Justify your answer.

Figure 1: Problem 3

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We are organizing the first round of a competition. Each team has to play two matches against the rest of the participating teams (first and second leg). Depending on whether the competition is local or national; we will have 22 or 230 teams.

- Calculate the number of matches depending if there are 22 teams and if there are 230 teams.
- Justify your answer.

Figure 2: Problem 4

We have the following numerical sequence:

1, 4, 7, 10, ...

- Write down the number that should be in position 234 of this sequence.
- Justify your answer.

Figure 3: Problem 5

The first problem, presented in a graphical context through a generic example, is a familiar generalization problem that has been presented in many different versions since Küchemann's study (1981). Problem 4 is presented in a verbal context, and problem 5 in a numerical one.

FINDINGS AND INTERPRETATION

We first used a quantitative data analysis to identify the stages of inductive reasoning model performed by each student in his/her response to each problem. Table 1 shows the number of students who expressed generalization using different representations.

Generalization			
Arithmetic	Algebraic	Verbal	Graphic
Problem 3			
125	3	57	11
Problem 4			
174	1	69	0
Problem 5			
222	57	26	0

Table 1. Representations used by students when generalizing

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The generalization most frequently used by the students was the arithmetical one. They generalized the pattern, got the commonality of the particular cases, and used the generalization to calculate the number of grey tiles for the required case. However, they were not able to provide an expression of *any* term of the sequence. In what follows, we mainly focus on graphical representation of generalization.

The students did not use graphical representation in the solving process except in problem 3, where the particular case was graphically presented in the problem itself. The use of graphical representation is is strategy is illustrated in one the student's production in Figure 4.

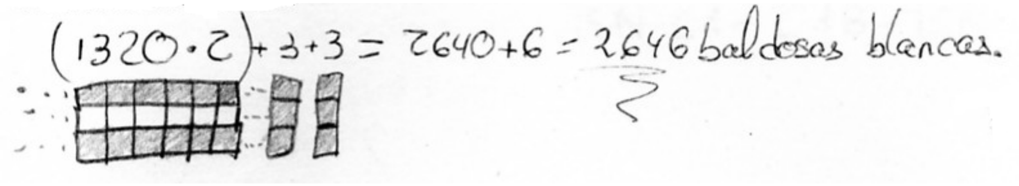
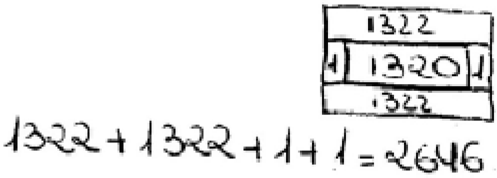


Figure 4. Generalization arithmetically and graphically expressed [Note: *Baldosas blancas* means white tiles in English]

We can differentiate two parts in the student's drawing. First, the student considered the two vertical sides made by grey tiles and identified that these sides remain with the same number of grey tiles independent of the number of white tiles. The second part of the drawing is the part of the figure that has white and grey tiles, and the student interpreted that he needed to double the number of grey tiles to surround the white tiles. The student expressed graphically the pattern of the sequence, showing it in a different particular case from the one shown in the statement (for six white tiles instead of five). The suspension points show this student's awareness that the commonality applies to other terms of the sequence. In this context, the suspension points could be interpreted as "I should repeat *this* as many times as necessary". The student applied the commonality to the 1320 tiles in the arithmetic expression. Therefore, on one hand, the graphical representation helped the student to get the generalization; and on the other hand, the student identified the commonality in the graphical expression beyond the generic example shown in the statement. This is why we consider that the generalization is expressed graphically.

Others students, went beyond graphical generalization to verbal or algebraic generalization. We illustrate this with the example shown in Figure 5.



"I have to add two to the number of white tiles to the top, and two more to the bottom; plus two more to each side" (Authors' student's response translation)

Figure 5. Generalization graphical and verbally expressed

As we observe in Figure 5, this student drew a graphical representation which represents the common features from the generic example shown in the statement, without tracing the lines that separate different tiles. S/he used this generalization to calculate the number of tiles that the problem required. Therefore, this is another example of graphical generalization as well as arithmetical. S/he also provided a general verbal expression for any given number of white tiles, using natural language.

Moreover, some of the students used graphical and algebraic representation together to give sense to the generalization. We present an example typical of these students in Figure 6.

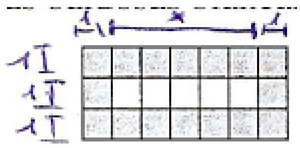


Figure 6. Generalization graphical and algebraically expressed

This student's diagram shows how s/he identified the data that remain constant and the data that change depending on the number of white tiles considered.

The three students' representations shown in Figures 4, 5, and 6 are typical examples of three groups that allow us to classify the eleven students that generalize graphically in problem 3. In the first group are the students who used suspension points to notice that the drawing would continue in the way indicated. These students used combined representation (graphical and arithmetical). In the second group are the students who used specific numbers in tiles of different sizes (the size of the tile is bigger when the number is bigger). In the third group each tile is represented by the number *1* and the number of tiles that depends on the number of white tiles is represented by *x*. Students in groups 2 and 3 used synthetic representations because at least two representations are considered as a whole to give sense to the generalization.

Most of the students, who expressed the generalization in problem 3, as well as in problem 4, did it verbally. For example, one student's response to problem 4 was, "The result is the number of matches that play each team against the rest of them, multiplied by two".

Cañadas, M. C., Castro, E., & Castro, E. (2011, february). *Graphical representation and generalization in sequences problems*. Paper presented at the CERME 7, Rzeszów, Poland.

Most of the students generalized algebraically in problem 5. Some students even tried to use a formula which was familiar to them. The students tended to present a correct formula and used it to calculate the number in the positions requested ($a_n = a_1 + (n-1)d$). This indicated that these students expressed the generalization algebraically. Some of these students worked on particular cases in the generalization process.

DISCUSSION

We have identified four ways of expressing the generalization: (a) arithmetical, (b) algebraic, (c) graphical, and (d) verbal. Most of the student who got the generalization, expressed it arithmetically. This result is consistent with findings from the earlier study of Becker and Rivera (2005) and with what we could expect due to students' previous knowledge.

This paper contributes to understanding the use of graphical representation in generalization (Mason & Pimm, 1984; Radford, 2002; Cañadas, 2007). This kind of representation illustrates what is common to all terms using particular cases but does not provide a general expression of any term of the sequence. Students seem particularly disposed to using it when the problem was posed using graphical representation, but not in other cases. Graphical representation of the generalization appeared to help students to generalize algebraically or verbally and thus, using these expressions, obtain particular case of the sequences. Generally students who utilized this kind of representation used it in combination with other sorts of representation. In this sense, we can consider graphical representation as a way of developing algebraic thinking and of expressing the generalization verbally or algebraically. This idea complements previous work which is mainly focused on other ways of generalization.

Graphical representation is sometimes enough for students to answer the question posed because they see the general pattern in the drawing. However, some of them revert to verbal generalization when they try to justify the answer, as Cañadas (2007) noticed. This is the main reason why verbal generalization is frequent in these problems (see Table 1).

Students generalized algebraically more frequently in the problem where particular cases were expressed numerically. This seems to be a consequence of what students were accustomed to in class. Only a low number of students generalized algebraically in problems where the statements were presented in unfamiliar representation. In particular, the lowest frequency of generalization was found in problem 4, which was presented in the least familiar way. This suggests that it is more difficult for the students to establish a relationship between algebra and generalization problems in non-numerical contexts and that, at some point, the idea of generalization has not moved from one context to others.

Cañadas, M. C., Castro, E., & Castro, E. (2011, february). *Graphical representation and generalization in sequences problems*. Paper presented at the CERME 7, Rzeszów, Poland.

Unless we and other researchers have made an effort to identify and describe different kind of generalizations, one conclusion of this paper is that sometimes it is quite difficult to distinguish among them. In most cases where students used graphical generalization, they used a combined-multiple representation or synthetic representation. Multiple representations in generalization seem to be useful for students to express the generalization. The distinction between the two kinds of multiple representations of Figueiras and Cañadas (2010) is a powerful way to describe how students reach generalization. This paper shows that synthetic representation appeared in tasks with graphical representation in the statement. Graphical representation can be considered the primary in the sense that it is the one that promotes the appearance of the other(s).

As a practical consequence, it would be desirable to use tasks in different contexts to guide students to algebra as a way of generalization. Work on generalization tasks starting from particular cases expressed in different representations would be enriching to students and would promote algebraic thinking capabilities because they would relate algebra with representations different from the numerical one.

Acknowledgement

This work has been supported by the Spanish National Plan I+D+i grants EDU2009-11337, funded by the Ministry of Education and Science and co-financed by FEDER.

NOTES

1. The questionnaire is reproduced in Cañadas (2007, Appendix B).
2. This second task allowed us to develop other part of our objectives, which is beyond the scope of this paper.

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