# Detecting psychological obstacles to teaching and learning the topics of ratio and proportion in sixth grade primary pupils

## Elena Fabiola Ruiz[1] and Jose Luis Lupiáñez[2]

[1]School of Higher Studies in Computation, National Polytechnical Institute, Mexico, D.F.
[2] Department of Mathematics Didactics, University of Granada

[1] **Mexico**

[2] **Spain**

*Elena Fabiola Ruiz Ledesma*. Av. Renacimiento No. 120 Edif 3 depto. 507. Col Ampliación Petrolera. Delegación azcapotzalco. 02710. Mexico. E-mail:  elen_fruiz@yahoo.com.mx

# Abstract

**Introduction**: The research presented here consists of reviewing the strategies used by sixth-graders when solving simple, direct ratio and proportion problems, so as to recognize the cognitive process of their thinking, and determine how they structure their response when faced with a problem situation.

**Method**:  The study took place at a public primary school in Mexico city, where we worked with 29 subjects enrolled in the sixth grade. Methodological instruments were: direct classroom observation; indirect observation through informal chats with the teacher and the students, review of their text books and notebooks; design and validation of an exploratory questionnaire through a pilot run, and its final application.

**Results**: This section presents results from the questionnaire and the kind of strategies used in working through each task. A comparison is then made with some of the investigations that were mentioned in the introduction.

**Discussion** or **Conclusion**: Heavy reliance on algorithms was observed, but they were not used meaningfully. Scholastic education has not fully exploited students' qualitative thinking about proportionality, as was observed when students focused on only one dimension of a figure that they were asked to reduce or enlarge.

**Keywords:** Ratio, Proportion, strategies, proportional qualitative thinking, proportional quantitative thinking.

# Resumen

**Introducción**: El problema de investigación que se plantea en este artículo, consiste en revisar las estrategias que emplean los estudiantes de sexto grado de primaria, al resolver actividades de razón y proporción simple y directa, para reconocer los procesos cognitivos del pensamiento de los alumnos y poder determinar cómo estructuran sus respuestas ante situaciones problemáticas.

**Método**: En esta parte se describen los siguientes aspectos: El escenario donde tuvo lugar el estudio que fue una escuela pública en la ciudad de México, los sujetos con los que se trabajó que fueron 29 estudiantes que cursaban sexto grado de educación primaria, y los instrumentos metodológicos empleados: la observación directa en el aula, la observación indirecta a través de pláticas informales con el profesor del grupo y los estudiantes, así como la revisión de sus cuadernos y sus libros de texto; el diseño, validación, a través del piloteo y la aplicación de un cuestionario de naturaleza exploratoria

**Resultados**: En esta sección se presentan los resultados que se obtuvieron del cuestionario así como el tipo de estrategias que emplearon en cada tarea y se hace una comparación con algunas de las investigaciones que se mencionan en la parte de introducción.

**Discusión** o **Conclusión**: Se observó una fuerte carga sobre los algoritmos, pero carentes de significado. La enseñanza escolar no ha explotado al máximo el pensamiento cualitativo de los estudiantes en torno a la proporcionalidad, lo cual se observó cuando manifestaron que se centraban en una de las dimensiones de las figuras que se les pedía reducir o amplificar.
Se reconoció la familiaridad que muchos alumnos tenían con el dibujo a escala, cuando una parte del dibujo ya estaba hecha, por lo que se ha señalado el predominio de la "ley del cierre" que prevalece en ellos.

**Palabras Clave:** Razón, proporción, estrategias, pensamiento proporcional cualitativo, pensamiento proporcional cuantitativo

## Introduction

Studies such as that by Guerrero, Gil and Blanco (2006) indicate that we need to understand and analyze how the student internalizes certain beliefs when learning mathematics and when interacting with his or her environment, these beliefs leading to success or failure as a learner. But success or failure in learning also has to do with how the teacher carries out instruction. In this regard, Thurston, Grant and Topping (2006) indicate that the inadequate knowledge and lack of confidence of some who teach science in primary school influence how the student engages in learning. Teachers' beliefs and their lack of adequate understanding both influence a student's learning. Other factors include cognitive processes and solution strategies that may constitute obstacles to a student's learning; this general class of obstacles pertains to the psychology of pedagogy.

This article focuses on detecting such psychological obstacles to learning as seen in sixth-grade pupils (11- and 12-year-olds), particularly in the topic of ratio and proportion. The research problem consists of reviewing strategies used by sixth-graders in solving ratio problems and simple, direct proportion problems, so as to recognize the cognitive processes involved in how they think and be able to determine how they structure their response when faced with a problem situationº.

In order to recognize cognitive processes in the students' thinking, we must review the strategies they use; these will allow us to recover certain trains of thought used when addressing ratio and proportion tasks. Strategies exhibit a wide variety of resources and different modes of representation, such as tables, drawings and numeric representations (Gueudet, 2007).

*Justification*

At the curriculum level, topics are introduced in primary or elementary school; if these are successfully learned, they enable the student to move on to comprehension of concepts they will work with in higher levels of education. Such is the case with the topics of ratio and proportion, their teaching and learning begins in primary education, and they are foundational to the acquisition of key concepts (Clark, Berenson & Cavey, 2003; Ruiz, 2002; Ruiz, Lupiáñez & Valdemoros, 2002).

On the other hand, poor understanding of ratio and proportion contribute to poor use of arithmetic knowledge, such as in managing multiplication problems, not to mention limiting and distorting concepts to be addressed in secondary education, such as the study of functions.

In the following section we will focus on several studies that address cognitive aspects related to learning the concepts of ratio and proportion and to proportional thinking; these will be useful in interpreting the results of our investigation. First we look at contributions from Jean Piaget, foundational to this topic and continuing to be the object of analysis today; second, we look at research which traces schoolchildren's cognitive processes in problem-solving strategies when they work on tasks using ratio and proportion.

*Piaget and the qualitative roots of proportional thinking*

Piaget (1978) focused his work on the subject's thinking by analyzing his or her expressions, and examining the spheres wherein intellectual operations are explained. In order to determine when and how these intellectual operations are formed, Piaget (1978) exploited a combination of two methods: *problem-solving* and *concept formation.* He monitored the stages of intellectual development as far as the stage of *formal operations*, which led him to understand the fundamentals which he found in dealing with ratio and proportion topics specifically.

Piaget and Inhelder (1978a) indicate that in studying the development of a child's thinking, they often found the problem of how proportions come to be understood. In the case of velocity, this idea is observed whenever two successive movements have to be compared over two different times and distances, for example, 5 cm in 1 second and 10 cm in 2 seconds. Similarly, with children's judgment in the case of probability, the idea of proportion is directly involved, for example, when 2 cases are positive out of a total of 4 cases, and 3 cases are positive out of a total of 6 cases. Neither situation is fully developed until the stage of formal operations, but one can observe that proportion exists earlier in simpler cases, like laboratory situations that make use of the balance, communicating vessels, flexed rods, etc.

Piaget and Inhelder (1978b) indicate that the subject is able to think in terms of qualitative proportionality when he or she understands that an increase in an independent variable

gives the same result as a decrease in the dependent variable. In other words, when they understand that an element of compensation is required. Piaget (1978), based on experiments he carried out, indicates that *the child acquires qualitative identity before quantitative conservation* and makes a distinction between qualitative comparisons and true quantification. For Piaget, undoubtedly, the notion of proportion always begins in a qualitative, logical way, before becoming quantitatively structured.

Piaget and Inhelder (1978a) maintain that in 11- to 12-year-olds, one can observe the notion of proportion in different spheres, such as: spatial proportions (similar figures), relationships between weight and the length of the arms of a balance scale, and in figuring probabilities. In the case of the balance scale, the subject can understand through manipulating the device that it is possible to maintain a balance by having two equal weights at the same distance from the center, but that balance is also maintained by decreasing one weight and moving it further from the center, or by increasing the other weight and moving it toward the center. Understanding this proportionality (both direct and inverse) takes place initially in the qualitative sense: "increasing the weight or the distance is the same thing", then in simple metric forms: "decreasing the weight by extending the length is equivalent to increasing the weight by decreasing the length".

Piaget (1978) described the child's advance in reasoning that appears as he or she nears adolescence as *formal reasoning,* having different characteristics from those specified as *concrete reasoning*. Piaget always accepted a logic of the concrete, which is placed in the concrete operations stage. Proportional reasoning, together with the ability to formulate hypotheses and work with a certain number of variables indicate that the student is in the stage of formal reasoning; this is when the subject must reflect and make abstractions in order to understand ratios as relationships between quantities and to link them to other ratios. The full solution of a problem involving proportions demonstrates formal reasoning, while an incomplete solution shows the level of concrete reasoning.

*Students' cognitive development and problem-solving strategies*

Karplus, Pulos and Stage (1983) were among the first researchers to categorize children's responses, not as indicators of a general stage of reasoning, but rather as demonstrating a level of comprehension of proportion. These authors hold that additive reasoning is not

something that appears in an unvarying developmental sequence; rather, it is highly influenced by instruction and represents an effort on the students' part to address a task effectively, not just by doing so mechanically.

Elsewhere, Markovits, Hershkowits and Bruckheimer (1986) analyze Missing Value problems and Comparison problems[1], and conclude that one of the reasons that school children make errors in solving these problems is their perfunctory, unconscious use of the rule of three. Nesher and Sukenik (1989), for their part, indicate that one of the dominant errors in strategies used by children at different ages is the additive strategy, where the relationship in ratios is seen as the difference between terms, instead of understanding its multiplicative nature.

Hart (1988), after analyzing methods used by secondary students in solving ratio and proportion problems, found that most students find these problems difficult to solve, and a large number of students use additive methods without implementing multiplication, and they avoid use of fractions. Hart comments that proportion problems often require recognizing a fractional scalar factor, followed by multiplication by that factor, so she considers that the comprehension of fractions and of proportions are linked.

Hart highlights the development of proportional thinking as having great value for the evaluation of teaching and learning. She considers that certain levels of generalization such as handling ratios, ways to generate equivalencies, etc., come about when multiplicative strategies are used. With regard to this, she emphasizes that if teaching begins with the additive, it can create difficulties that affect how far the subject matures in being able to use multiplication as a pre-requisite to equivalency.

Freudenthal (1983) labels ratios as numeric entities linked to proportions, and he makes reference to the logical basis of ratios being a function of ordered pairs of numbers or magnitudes, a framework within which they have an equivalency relationship. For Freudenthal, internal and external ratios must be taken into account when teaching, where the former

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[1] This classification was taken from earlier studies where Missing Value problems are defined as those where 3 numbers appear (a,b and c) and the task consists of finding the fourth value "x", such that a/b=c/x. Comparison problems are those that present 4 numbers, a,b,c and d, and the task is to determine whether there is any proportional relationship between them.

are relationships established between different values of the same magnitude, and the latter are relationships between values of different magnitudes.

Finally, we point to studies by Streefland (1984a, 1984b), whose research emphasizes that early teaching of ratio and proportion should begin with qualitative levels of recognition, and make use of didactic resources that encourage development of perception patterns, as a support to the corresponding quantification processes.

The original aspect of Streefland's work is that he uses a story called "With the Giant's regards" to establish comparisons between the giant's world and the human world, in order to set up relationships that lead to the notion of ratio. As qualitative reasoning develops, there is an advance in thinking and the child can come to incorporate more elements for an analysis that makes possible joint consideration of different factors.

Like Piaget (1978), Streefland holds that the qualitative appears before the quantitative, but he takes it further, into the sphere of teaching. This contribution is what we will use in this study, since first of all it is of value to understand how students who are reaching the end of primary education organize qualitative components (Ruiz & Valdemoros, 2004).

**Method**

*Participants*

The study took place in a natural setting, with a group of sixth-grade pupils in public school, morning shift. The school forms part of a community where most urban services are available, and it was chosen since its characteristics are typical of a large number of urban schools. It is located in a district of Mexico City which is dynamic and under constant growth, but has a low socio-economic status.

The group involved in this study was made up of twenty-nine students who were reaching the end of primary education. Their schoolwork was based on the official Program of the Secretariat of Public Education (2001a), equivalent to the Ministry or Department of

Education in other countries. The students were eleven-year-olds who were typically non-participative.

In order to make our inquiry into students' cognitive processes (thinking, solution strategies, perceptual information, language, semiotic registers, etc.), we made use of direct and indirect observations as well as both qualitative and quantitative analysis of a questionnaire.

*Instruments*

Our methodological instruments were: direct classroom observation, indirect observation through informal conversations with the students and their teacher, through reviewing their notebooks and text books, and application of an exploratory questionnaire that had been used in a pilot study in an earlier version.

We learned the types of problems that students were solving from observation – whether their presentation in a text book (Secretariat of Public Education, 2001b) or by the group's teacher – as well as observing how students worked on them. From this observation we conclude that their approach to diverse situations can be understood through a diversification of problems; many solution strategies were poor because situations were being imposed. Similarly, a heavy emphasis on algorithms was observed, but these were lacking in meaning.

*Procedure*

The ratio and proportion tasks that were designed for the questionnaire were submitted to different criteria (Misallidov, 2003). Some tasks refer to very elementary, proportional variation problems, while others attempt to discover how well students were able to recognize qualitative and quantitative proportional relations between quantities, particularly where, under observation, they appeared to establish these relations mechanically, mimicking whatever the teacher did. Generally speaking, the objectives of the questionnaire were as follows:

1. Determine the subject's level in terms of how qualitative components and quantification processes of proportional relations are organized.
2. Make an in-depth inquiry to explore both what the student exhibits as well as what is only suggested.
3. Recover the sequence of the student's thinking.

4. Demonstrate how the student approaches problem-solving in these cases, both in his or her solution strategies and in the use of representations.

The questionnaire was divided into two blocks. The first section was made up of five tasks, and attempted to uncover justifications with more of a qualitative value. We were interested in seeing how thinking was constructed qualitatively, apart from explicit quantities involved in the proportional relations that were given. Three of the tasks made use of a grid (graph paper), in order to encourage transferral to quantification. The second block included quantified ratio and proportion tasks, where certain values were supplied and other values were to be determined. Some of these tasks involved using a table as a means of representation in order to recognize external or internal ratios.

Table 1 summarizes the specific objective of each block and of each task included.

*Regarding the pilot study with the initial questionnaire*

The pilot study of the questionnaire allowed us to make several types of adjustments:

1. To ensure that the tasks were readable and comprehensible.
2. To confirm the degree of difficulty, since neither extreme was desirable (very easy or very difficult).
3. To recognize what aspects might be considered more universal.

After a first, pilot application of the questionnaire, we observed that the tasks included had a certain connection with students' experience with their teacher and their program of study. The tasks were not totally solved by all students, but there were no students who were completely lost or who did not know what to do.

The pilot study also revealed that, overall, the information provided through drawings facilitated students' comprehension, so we assume there was a history of their use in earlier school grades (use of drawings to scale).

Based on results of the pilot study, adjustments were made in how the tasks were presented. We took this restructuring further than what the school institution had established to that point, attempting to evoke experiences that were familiar and common to the participating students.

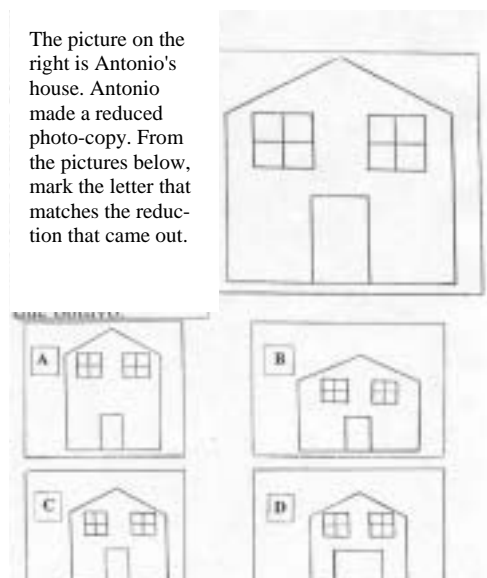**Table 1.** *Objectives for each block and each task in the initial questionnaire*

| Block I | Block II |
|---|---|

*Specific objective of each block*

To investigate the student's mathematical knowledge and know-how with regard to qualitative components and elementary quantitative aspects of proportional relationships.

To investigate how the student processes his thinking around the quantification of proportional relationships.

*Specific objective of each task*

1. Find out whether the student can recognize the reduction of a drawing in all its components, and so be able to express whether it still has its original shape, based on visual discrimination.

2. Find out whether the student visually recognizes the ratio existing between dimensions of rectangles and whether the child can draw what is missing based on this visual recognition.

3. Find out whether the student can reproduce a figure at a given scale (s/he was asked to reproduce a figure linearly twice the size of the original).

4. Find out whether the student can complete a figure which is reduced from an original, maintaining its proportionality (keeping its original shape), find out what strategy s/he made use of and what type of representation was used to carry out the task.

5. Find out whether the student can complete the enlargement of a figure, maintaining its proportionality, and have him/her explain in writing how it was done.

6a. Find out whether the student can complete a geometric figure where s/he knows the value of the segment that represents the figure's width and the values for height and width of a figure that ought to be proportional, and the strategy that is used. The task was embedded in a similarity relationship. Review whether s/he relates magnitudes on the same scale (internal ratios).

6b. Find out whether the student can complete a geometric figure, where s/he knows the value of the segment that represents the figure's width and the values for height and width of a figure that ought to be proportional, and the strategy that is used. The task was embedded in a similarity relationship. Review whether s/he relates magnitudes on different scales (external ratios).

7 (a and b). Determine what quantities correspond to data shown in a table, resulting from a hypothetical situation. Review the strategy used to fill in the table and whether a relationship was established between the table and the questions asked.

8. Given a task that is illustrated with drawings (buckets of paint), complete the unknown data based on three given values, and review the strategy used.

9. Determine the missing value in a problem, based on the model of buying objects familiar to the student (bags of candy), given 3 data points. Review what strategy is used.

10 (a and b). Find out whether the student can invent a problem related to three given quantities in a table. Review whether s/he can fill in the table and solve the problem presented.

Some of the tasks were also balanced out in terms of how the information was presented; on one hand, we sought to reduce irrelevant information that would distract students, and on the other hand, we modified problems that were heavily laden with data in the form of text by substituting part of the text with drawings.

Some of the terminology used in the instructions was also changed in certain problems, since it seemed to be a source of difficulty. One task was added in order to determine what was more familiar to students when required to establish relationships between magnitudes presented – whether internal or external ratios, based on the distinction Freudenthal describes (1983).

Below we present the tasks from the initial questionnaire, including modifications made after the pilot study. The first two items explore qualitative aspects in the area of proportionality.



The picture on the right is Antonio's house. Antonio made a reduced photo-copy. From the pictures below, mark the letter that matches the reduction that came out.
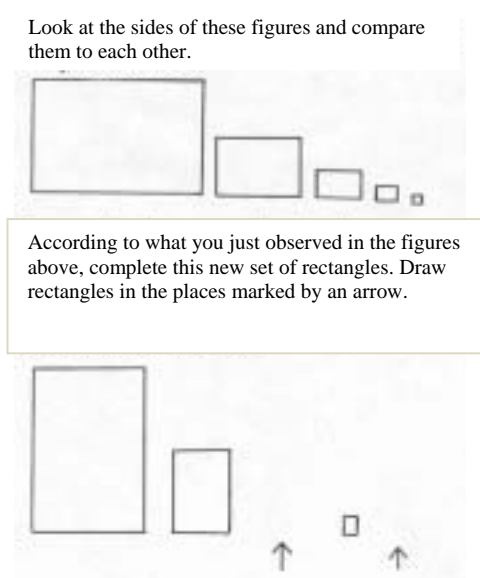
Write down what you did, step the step, in order to figure this out.

*Figure 1*. Task 1

Look at the sides of these figures and compare them to each other.

According to what you just observed in the figures above, complete this new set of rectangles. Draw rectangles in the places marked by an arrow.
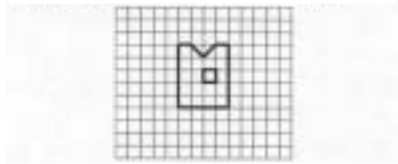
Write down step the step how you compared the sides and what you were able to observe.
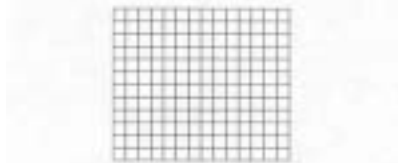
*Figure 2*. Task 2

The next three focus on making the transition from qualitative to quantitative proportional thinking:

Mrs. Saucedo makes knitted vests, and she has been asked to make one that is an enlargement of the following:
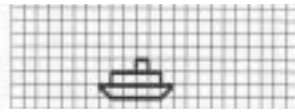
In the space below, draw the new vest, making each side of the vest twice as big as in the sample.
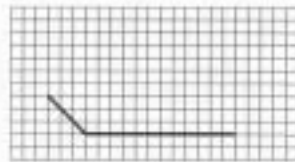
Now write down the steps that you took in order to draw it.

*Figure 3.* **Task 3**

Now, Mr. Escalante has been asked to make an enlargement of the following original drawing:
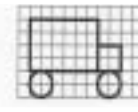
Below you can see that part of the drawing has been enlarged. Complete the rest of this enlargement, keeping it the same shape as the original.
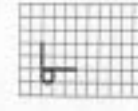
In the space below, explain how you did it.

*Figure 4.* **Task 4**

Mr. Escalante is a draftsman, and he has been asked to make a reduction of the following original drawing:

Below you can see that part of the drawing is already reduced. Complete the rest of this reduction, without changing its shape.

In the space below, explain how you did it.

*Figure 5.* **Task 5**

The eight remaining tasks from the questionnaire focus on the quantitative arena of the student's proportional thinking.

A carpenter cut a wooden board in a rectangular shape, 9 cm long and 6 cm wide. He needs to cut another board with the same shape as the first, but with different measurements. If the second board should measure 3 cm long, how wide should it be? _____ cm.

Complete the figure that represents the second wooden board.

Step by step, explain how you solved it.

*Figure 6.* **Task 6a**

A carpenter cut a wooden board in a rectangular shape, 12 cm long and 6 cm wide. He needs to cut another board with the same shape as the first, but with different measurements. If the second board should measure 5 cm long, how wide should it be? _____ cm.

Complete the figure that represents the second wooden board.

Step by step, explain how you solved it.

*Figure 7.* **Task 6b**

**Read and complete**

Mrs. Saucedo is having guests over for a snack and she plans to make hot chocolate. Help her figure out how many chocolate bars she needs for 2 liters of milk _____, and how many for 5 liters of milk _____.

Also, help her figure out how many liters of milk she should buy for 6 chocolate bars: _____ liters. Fill in the following table to help you.

| Chocolate bars | Liters of milk |
|---|---|
| 2 | 1 |
| | 2 |
| 6 | |
| | |
| | 5 |

Explain what you did in order to answer the questions

Luis is going to help his dad paint the house, and they need several buckets of paint. They know that 20 liters costs 300 pesos. Help them calculate the price of the other buckets they need. Fill in:
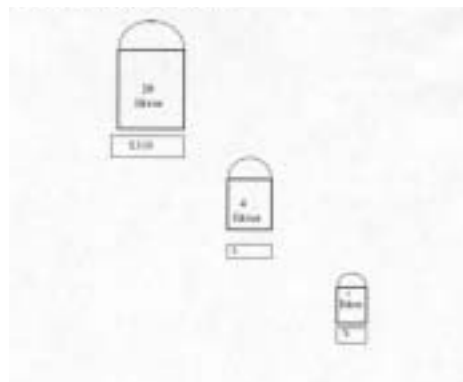


In the space below, write down what you did in order to figure it out.

*Figure 8*. **Tasks 7a and 7b**          *Figure 9*. **Task 8**

Andres bought 4 bags of candy and paid 120 pesos for them:



Julián bought 7 bags of the same candy:



How much did he pay for them? _____

How did you solve the problem?

Some amounts are provided in the following table:

| Number of packets | Number of stamps |
|---|---|
| | |
| | |
| 3 | 15 |
| | |
| 5 | |

Create a problem that has to do with the amounts that are provided, and solve it.

Fill in the table and explain how you did it.

*Figure 10*. **Task 9**          *Figure 11*. **Tasks 10a and 10b**

**Results**

Overall, of the twenty-nine students who completed the questionnaire, twenty-six solved all the tasks on the questionnaire, and the other three left from one to three activities unanswered. Twelve of the twenty-nine answered more than half the tasks correctly, but no single student answered correctly all thirteen tasks which made up the questionnaire. One student was able to solve twelve, and also one student was only able to answer one task correctly.

Finally, two of the thirteen tasks were considered to be the most accessible for the students, since they were solved correctly by nearly everyone; twenty-seven and twenty-six pupils respectively answered activities 7b and 10a correctly. Below we describe in more detail the students' answers to each task.

*Analysis of Task 1*

Half of the group chose the correct shape for the reduced house, giving the following arguments: *"it's the most similar", "it's the same only smaller", "nothing was taken away and nothing was made bigger", "by looking at all of them and comparing"* (this last case was by visual discrimination).

The other half of the group chose an incorrect house, most of them choosing letter B, where the width is not reduced by half as it is in the others. In general, students who chose B commented that it was the most similar to the original. We observed that these students did not look at all parts of the house, in conversation with them they mentioned only that reduction is something smaller than a given figure, but at no point did they say anything to indicate that all the parts that make up the figure had to have the same size.

It was observed that several children only focused on one of the dimensions, apparently the focusing process is strongly rooted in their thinking and this becomes a psychological obstacle that hinders the student from correctly solving ratio and proportion situations.

Arguments given were classified into two categories: one corresponds to students who choose the reduced house based on their definition of reduction: *"it's the same only smaller"* or *"nothing was taken away and nothing was made bigger"*. The second category involves the argument *"it's the most similar"*, and it was used both by students who selected house C and those who selected incorrect answers. This indicates that "similar" does not reflect the idea of proportion in all cases. It also shows the need to go further in the qualitative area of proportionality and that children should progress from the stage of meaning to that of designation within Education, that is, once they have understood the real meaning of what is being addressed, they can rename it or use the name which is applied in mathematical language.

*Results from Task 2*

A series of rectangles were shown, where the dimensions of each were half the size of the preceding. Students were asked to complete a second comparable series; they needed to realize that the rectangles they drew had to be reduced by half, both in length and in width. We wanted to see whether students visually recognized the relationship (ratio) between the dimensions of the rectangles (the lengths to the lengths and the widths to the widths, which according to Freudenthal (1983) would be external ratios). We also wanted to check whether they could draw the missing rectangles based on their assessment. It was likewise important to see if they used the operator ½, and how they did so.

Explicit values were not involved in this activity, so it was placed in the qualitative sphere and in the transition from qualitative to quantitative.

Most students observed that the series to be completed was similar to the one given, the rectangles had only changed in position. Despite this, very few (eight of the twenty-nine children) reduced the figures by half in both dimensions, doing so without the use of any measuring instrument, but by approximating as they drew. This is what the task called for, making no mention of using any instrument. As mentioned above, the difficulty was that most of the group made the reduction in only one of the rectangle's dimensions.

In general, students realized that the rectangles to be drawn should be smaller than the preceding ones, but they did not discover by what proportion they were being reduced, that is,

they did not realize the pattern used in reducing the rectangles that made up the series. Furthermore, it was easier for them to complete a figure where one part is already given, as was the case in the tasks where they were provided with drawings shown on a grid, than for them to make a drawing by previously recognizing the pattern required.

There was qualitative-type comparison as seen in verbal statements like "*they should get smaller and smaller*", etc. They relied on the visual and the intuitive, since no measuring with any instrument was involved, but the flaw seen in most students was focusing on a single dimension, the same psychological obstacle found in Task 1.

*Results from Task 3*

This task was intended to address the notion of enlargement and use of the natural operator (x2), through a drawing and through possible recognition of the ratio involved. It was one of the least accessible tasks for the students, very few were able to do the activity correctly. Here they were required to enlarge a drawing linearly to twice its size, supported by a grid. Strategies used were as follows: after doing an initial count, add the total number of squares to itself to produce the doubled amount, in one dimension or in both; multiplying the number of squares from one or two sides by two. Two errors were most common, one was to double only one side of the figure, whether by adding or multiplying by two (see Figure 12). The second most common error was to add on two little squares. These students use the so-called incorrect "additive strategy" that Hart mentions (1988). This is another psychological obstacle which students present.
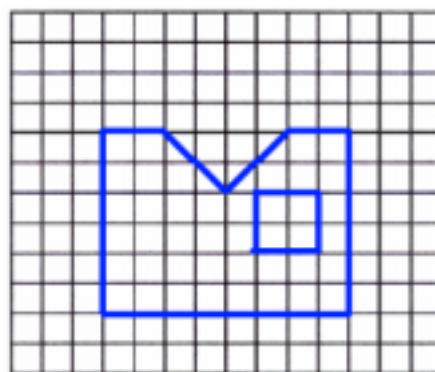


*Figure 12*. **Víctor's solution to Task 3**

In addition to the drawing, Víctor explained his answer by saying: "I did it by doing each square twice". It can be observed that the enlargement was applied in only one dimension, in this case the width[2]. This solution is considered incorrect. One can also see that the grid was a means of support for counting the squares, regardless of whether the increased quantity was correct.

In general, the text was not properly interpreted, the term enlargement predominated (although in most cases students had an erroneous understanding of this word), and apparently they did not notice that each side of the vest should be enlarged by twice its size. After conversing with some students, several things were observed. One group of students did not have a clear notion of proportion, from the point of view of drawing to scale, since in their answers they only mention that enlarging something is "to make it bigger". In some vest drawings produced, the quantity of squares added indicates no understanding of proportion. Apparently many students confused the expression in the text with adding two squares to each side. Another group of students were characterized by doubling the number of squares in only one dimension, and the third group of children showed a clear notion of proportion from the perspective of drawing to scale, since they drew a vest which had twice the number of squares for each dimension.

*Analysis of Task 4*

It appears that students find it more difficult to draw a figure, in this case enlarging it from a sample figure, than to complete one where they have been provided one part of the figure already drawn. This activity also made use of the grid as a support for enlarging a figure, with one part of the figure drawn. It was observed that all students could correctly complete the lower part of the ship, but they did not triple (linearly) the next two parts that make up the ship, thereby allowing us to observe the closure property that Piaget mentions. It was difficult for them to work with the natural operator (x3) applied to the drawing. Most of the group (twenty-four of twenty-nine) did not perceive that the part that they completed was three times larger than the original figure, and they fell into the error of doubling instead of tripling the original figure linearly, for the two remaining components of the ship.

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[2] When working with tasks supported by a grid, measurement is expressed by counting the small segments that correspond to the sides of the squares that make up the grid. The children say that they count the squares and proceed to describe their actions from that.

*Analysis of Task 5*

Here we inquired into whether the student could complete a figure which is the reduction of an original, such that proportionality, or the shape of the original figure, is maintained. We also analyzed how students were helped by both the grid and a part of the figure already drawn, when asked to reduce the segments of a given figure by half.

Half of the group performed this activity correctly, most of them looked at the part that was drawn and completed it correctly, but students also observed that the part of the truck that was already done had half the number of little squares that the original had, both in length and in width, this being stated in the students' written explanations.

It appears that students were more familiar with working with numbers like 4 and 6, whether in doubling or reducing by half, because in this case they captured the fact that the drawn portion of the truck corresponded to half of four squares for the width, and half of six squares in the length. This manner of proceding was also observed in Task 3, where most students doubled the width of the vest, which corresponded to four squares. In this task, both the grid and the part already drawn were means that helped students to solve the task successfully.

In none of the above tasks, numbered 3 to 5, was there any explicit recognition of the ratios involved, since students did not work with fractions or expressions that might indicate the relation between the lengths in the figures, as in the cases of the vest, the ship and the truck. Likewise, no such recognition was seen in the series of rectangles. It may be said, therefore, that not being able to recognize relations between magnitudes by comparing them using a quotient, in qualitative fashion, is a psychological obstacle; thus, when working in the quantitative area students are not able to establish these relations, also referred to as ratios.

This first block of tasks addressed the two important notions of *reduction* and *enlargement,* seen from the idea of the photocopier and from drawings to scale. We emphasize that this was our starting point because these ideas are precursors to proportionality, as indicated by the NCTM (2003) standards, where it is stated that work with proportion can begin with small children through construction of figures that are smaller and larger with respect to another figure. Furthermore, as indicated in the introduction to this article (Piaget &

Inhelder, 1978a and b), the child acquires qualitative identity before quantitative conservation, this being reaffirmed in the thesis by Streefland (1984a and b).

*Analysis of Task 6 (parts a and b)*

These tasks attempted to see what relations the student might establish between two magnitudes, whether between two dimensions of the same type (comparing length to length) or between different dimensions of the same figure (length and width); in other words, to see what was easier for them, to work with external or internal ratios, as Freudhental refers to them. This was done by completing a figure where the length and width values of one rectangle are known, as well as the segment which represents the width or length of another rectangle which was to be proportional. We also wanted to find out whether the operator was used in establishing the ratios found between the rectangle dimensions (length to length or width to width).

For both tasks, the width was the known value of the rectangles which were to be proportional. In the case of activity 6a, one third of the other rectangle's width was represented. Solving this problem was quite difficult for the students, with only seven of the twenty-nine students reaching a correct answer. These students compared the values for the lengths of the two rectangles and realized that one of these measurements went three times into the other, some of them wrote: "3 goes into 9 3 times" and they had to find another number that went into six three times, which was the length measurement. All the pupils performed operations, but the twenty-two children who reached wrong answers established an incorrect relationship, some of them thought the length should be half the value of the rectangle's width, other pupils only looked at the numbers, without caring whether some of these corresponded to lengths and others to the widths of the figures. This can be seen in Figure 13, where Ana Belén explained that "6 does not go into 9, but 3 goes into 6 two times, so the answer is 3".
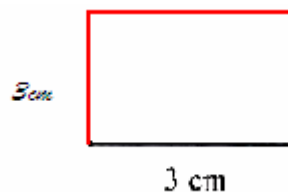


**Figure 13.** Ana Belen's solution to Task 6a

Establishing incorrect relationships, such as comparing the length of one figure with the width of the other, represents another psychology obstacle. In this case pupils were not able to understand that, while a ratio is established by comparing two magnitudes by quotient, the comparison must be between one dimension and its counterpart, for example: the length of one figure with the length of another, the width of one figure with the width of the other, or, two dimensions of the same figure, that is, length and width of a single figure, what Freudenthal (1983) refers to as external and internal ratios.

More students were successful at solving Task 6b, where the width values were also known, and they were asked to find the length of the proportional rectangle. Eleven students of the twenty-nine found the relation between the length and width of the first rectangle, realizing that six is half of twelve, and so they looked for a number which was half of five, that is, half the width of the other rectangle. But the remaining eighteen pupils followed a procedure which led them to a wrong answer. Most of them did so by dividing the value of the length of the rectangle (6cm) by (2), thus obtaining an incorrect answer. When asked why they performed this operation, they commented that they had to find half of six because six is half of twelve. We observe that they started out well by noticing that the rectangle's length was half its width, but at this point they went off track, in not being able to establish the right relationship.

When comparing results from the two tasks, it can be said that working with external ratios (Freudenthal, 1983) was somewhat easier for them than with internal ratios, although the values of the figure's dimensions also played a part. One example made use of a third of one value and the other used a half, and the pupils are more accustomed to working with halves than with thirds.

*Analysis of Task 7 (parts a and b)*

This task was divided into two parts, part 7a consisted of answering certain questions based on a situation which was familiar to students. 7b referred to filling in a table, thereby establishing a connection to the previous activity. We can conclude that filling in the table turned out to be quite accessible for the pupils, but establishing the relationship between the table and the problem situation was not so easy.

The table is a specific form of representation that must be taught, in other words, the table is an object of instruction; students were observed to complete the table with ease, showing that they had been taught to do so in prior years of schooling, just as is indicated in the official study program of the *Secretariat of Public Education* (1991a). In this case, in order to fill in the table, the children either added the quantity to itself to produce its double, or they used the multiplication table of twos. This also indicates that they are familiar with the scalar operator (x2) when using this form of representation.

Students who answered the questions correctly did so for the most part without needing to refer to the table. In their explanations, some of them commented that they doubled the quantities. For example, they said: *"for two chocolate bars you need one liter of milk; so for 4 bars you would use twice as much, that means 2 liters of milk, which is the same thing as saying 2 liters of milk need 4 chocolate bars".*

*Analysis of Task 8*

Those who gave wrong answers to the questions did so for different reasons. One of them was because they did not relate the table to the questions, another was confusion in interpreting the text (on two occasions they were asked for the number of chocolate bars needed, and the third time they were asked for the liters of milk). In the latter case, the pupil answered every case in terms of finding the chocolate bars needed, not answering the third question in terms of liters of milk, thus producing a wrong answer.

In this task the student had to determine the price of two paint buckets, one single-liter bucket and one four-liter bucket; they were given the price of a twenty-liter bucket. Only eight of the twenty-nine students found the correct answer, using different strategies. Most of them did so by finding the unitary value, dividing 300 pesos by 20 liters; having determined the cost of one liter, they used multiplication to find the price of four liters. Only one child realized that four liters was one fifth of twenty liters, so he proceeded to divide the price of 300 pesos by 5 in order to find the cost of the four-liter bucket, and later divided 60 pesos by four liters in order to find the cost of one liter.

The most common error found was dividing 300 pesos by 4 liters; in other words, these students established an erroneous relationship, the same obstacle presented in Task 6a.

This task presented part of the information through text, and part through the drawing. We observed that this helped students solve the problem; they commented that that when the saw the drawing of a one-liter bucket it occurred to them to find its price first, so they could get the price of the four-liter bucket afterward. With regard to students who committed the error of dividing 300 pesos by 4 liters, we can state that establishing the relationships was difficult for them.

*Analysis of Task 9*

This problem was classified as being a missing-value or unknown-value type. Students were given the price as well as the corresponding number of bags of candy, and they were asked to determine the cost of another quantity of the same bags of candy.

Nineteen of the twenty-nine students were able to reach a correct answer. All of them used the strategy of determining the unit value and then they either added this value as many times as there were candy bags, or they used multiplication. The drawing played an important part in solving this task, since counting the bags helped them obtain the cost of the projected total; once they found the price of a single bag, the drawing was even a support for using the unitary value strategy, since they wrote the cost of a bag below each figure that represented a candy bag.

Students did not use the rule of three for this problem either, verifying that the group's teacher had not taught it, in compliance with the study program. It also appears that this rule was not taught in earlier grades.

*Analysis of Task 10 (parts a and b)*

Here we wanted to find out whether the student could invent or formulate a problem situation based on three known quantities found in a table; the column headings were *number of packets* and *number of stamps*. Two of the known values fell under the first column, while the other value was in the second column. On this occasion students were not given the unitary value. One of the strategies they used to fill in the table was to first determine the unitary value and afterward proceed to find the remaining values. Other students filled in the table by

using the multiplication table of fives, and the rest of the students used repeated additions of five. We were interested not only in how they filled in the table, but also in seeing whether they established a relationship between the table and the problem posed. In this regard, most of the group did not use the information in the table in designing their problem.

Some students created situations where they did not incorporate the variables in the table, but they invented missing-value problems using the three values that were known. Others formulated problems that were unrelated to the table headings or to the data found there. In this group there were children who used addition without repeating a value a certain number of times; in other words, the problem did recognize multiplication as an abbreviated form of addition, and these students needed to make the transition from the additive to the multiplicative, as indicated by Hart (1988).

Other students did use multiplication, recognizing it as abbreviated addition, because they commented that instead of multiplying, they also could have added.

**Discussion**

Several aspects of students' cognitive thinking were detected through the strategies they used in solving the tasks. We summarize briefly those which were already described above. Scholastic education has not fully exploited students' qualitative thinking around proportionality, as was observed when we saw students focus on one of the dimensions of the figures they were asked to reduce or enlarge. To view a drawing as a whole and not to consider each of its parts in order to select the reduction of the original shows the need to work more on the qualitative aspect of proportionality.

For some students, meager qualitative understanding precedes the quantitative; their linguistic categories consisted of terms such as "it's bigger than …", "it's smaller than …", reflecting a certain rudimentary understanding of proportion, but further categories which would show greater understanding of the idea of proportion were not observed in these subjects. In general, they showed confusion when establishing relationships between magnitudes.

We could recognize many students' familiarity with drawings to scale when one part of the drawing was already done, indicating the predominance of the "law of closure" in these pupils. The difficulty detected was in not recognizing the scalar factor (x3) in this type of task.

We observed students' ease in filling in a table, either through adding the same number a certain amount of times, or through use of multiplication, once they found the corresponding scalar operator. The problem detected in use of the table was that students did not draw data from the table in order to address the problem situation.

Students did not manifest use of different forms of representation—the table, the drawing and the numeric—when solving the so-called "missing value problems". Several tasks in the questionnaire fall into this category, although the situations were different, and only one of them (T9) was correctly solved by most of the group.

The questionnaire activities allowed them to indiscriminately use any of the three representation types mentioned, but we observed that students did not realize that using any of the three would lead them to the same result; in their solutions they would only work with the representation presented in the problem. No student on their own initiative solved any single task through the use of at least two different forms of representation.

Another aspect which appeared frequently in students' responses was the use of the unitary value strategy, in solutions for several tasks. This concurs with findings from several other researchers (see the introduction to this article), who were interested in recognizing solution strategies for ratio and proportion problems.

The same can be said of the way students' incorrectly established relationships when working on ratio and proportion activities, the lack of meaningfulness when using algorithms in this area, the need to delve more deeply into the qualitative aspect of learners' thinking, and so on.

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