ANALYZING THE PROVING ACTIVITY OF
A GROUP OF THREE STUDENTS

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We present an analysis and outline an evaluation of the proving activity of a group of three university level students when solving a geometrical problem whose solution required the formulation of a conjecture and its justification within a specific theoretical system. To carry out the analysis, we used the model presented by Boero, Douek, Morselli and Pedemonte (2010) that centers on the arguments and rational behavior. Our analysis indicates that the student's proving activity is close to the one we used as a reference.

Key words: evaluation, proving activity, rational behavior, type of argument

INTRODUCTION

Producing and reading proofs are complex mathematical practices because they require being able to articulate many and diverse, not necessarily routine, actions; therefore mastering them is not an easy or direct process. Understanding this has recently impelled research and reflection of a didactic character of the different methodological approaches used to teach proof and proving in the tertiary level, and to subsequent innovations which could lead to satisfactory results in students' learning (see Selden, 2010).

In this regard, since 2004, we have been engaged in consolidating and implementing a curricular innovation whose primary scenario is the Euclidian geometry course that takes place in the second semester of a pre-service teacher program at Universidad Pedagógica Nacional (Colombia). The innovation aims to deliberately support students' learning to prove and seeks to have students conform an ample idea of what proving activity consists of (Perry, Samper, Camargo, Echeverry, & Molina, 2008). Presently, we are interested in finding different types of evidence that will permit us to evaluate, in a long term, the effectiveness of the curriculum design and development achieved with the innovation. We start by undertaking the evaluation of the students' performance in specific tasks that were assigned in the third semester geometry course.

The purpose of this paper is to analyze and evaluate some excerpts of the proving activity displayed by a group of three students when they solve a given problem without the teacher's intervention. In the task, a conjecture must be found, based on a dynamic geometry exploration, and justified deductively. In order to analyze the students' proving activity, we shall use the integrated model presented in Boero et al. (2010). To evaluate the analyzed activity, we shall consider a list of key actions, that we designed in the light of the model, which we consider conform a successful

performance; naturally, the actions are coherent with the learning goals set in the innovation and the learning experiences promoted by it. Thus, we first contextualize the study indicating what we mean by proving activity and by learning to prove, and we mention three key characteristics of the methodological approach to teaching used in the course. Secondly, we expose our interpretation of Boero and his colleagues' model which will guide our analysis. Thirdly, we describe aspects of the experimental design; we include information about the course the students belonged to, state the problem we proposed, present some components of a successful performance to which we compare the students' proving activity, and give details of the data treatment for the analysis. Fourthly, we analyze the evidence that provides elements to evaluate their proving activity. Finally, we expose the evaluation.

STUDY CONTEXTUALIZATION

The purpose of our innovation is to support learning to prove. Thus we promote student participation in *proving activity* that is carried out as a means to develop the geometric course content. For us, *proving activity* includes two processes, not necessarily independent or separate. The first process consists of actions that support the production of a conjecture; these actions generally begin with the computer-based exploration of a geometric situation to seek regularities, followed by the formulation of conjectures and the respective verification that the geometric fact enounced is true. Hereafter, the actions of the second process are concentrated on the search and organization of ideas that will become a *proof*. This last term refers to an argument of deductive nature based on a reference theoretical system in which the proven statement can be a theorem (Mariotti, 1997). *Learning to prove* is a process through which students gradually become more able to participate in proving activity in a genuine (i.e., voluntarily assuming their role in achieving the enterprise set in the course), autonomous (i.e., activating their resources to justify their own interventions and to understand those given by other members of the class community), and relevant form (i.e., making related contributions that are useful even if erroneous).

Three characteristics of our methodological approach to teach proof and proving are the key roles of: (i) the student *geometrical problem solutions* as a means to provide elements that contribute to the development of the course content; (ii) the *interaction* between teacher and students or among students to develop the course content and to support individual learning; (iii) the use of a *software of dynamic geometry* (e.g., Cabri) in the feasibility of an autonomous, genuine and relevant student participation; this resource provides them with an environment in which actions such as empirical exploration, communication and validation of statements are propitiated.

ANALYTICAL TOOL

Boero's et al. (2010) integration of the Toulmin argumentation model and the Habermas rational behavior model highlights some of the elements that must be articulated to face the complexity of proof; we therefore find it useful for our purpose. In what follows, we present our interpretation of their proposal.

Type of argument: Accepting that proving activity involves arguments of different nature, describing it requires focusing on the different types of arguments (i.e. inductive, deductive, abductive) that students formulate during the problem solving process. Every argument, according to Toulmin, has three basic components[1]: a statement whose validity is argued by someone (claim), premises that motivate concluding the claim (data) and the statement considered as valid that connects data to claim (warrant). More precisely, the analysis is centered on how the three components are connected, that is, what the structure of the argument is, because our methodological approach induces it and requires it.

Teleological aspect: Considering proving activity as a special case of problem solving, an important part of it is focusing on the goal which must be reached, so that the different actions carried out have a clear purpose. Also included in this aspect are the formulation of a plan to reach the goal, the determination of the strategies that can contribute to following it and reaching the goal, and the control of the latter.

Epistemic aspect: Considering proof as an object that must satisfy epistemic requirements established by the community of mathematical discourse in which it is being constructed or presented, when describing it and evaluating it, the focus is on if there is or not conscious validation of the statements, taking into account shared premises and legitimate forms of reasoning.

Communicative aspect: Considering proving activity as a sociocultural practice, it is natural to take into account the care students have in the way they communicate their arguments, and how conscious they are of the elements, associated to proof, that affect communication.

EXPERIMENTAL DESIGN

The problem proposed to the students

In a one and a half hour class session, the students, in groups of three, worked collaboratively on the following problem. As usual, they were asked to hand in a group document that reports: details of the Cabri construction and exploration, the conjecture formulated as the result of empirical exploration, and its proof.

> With Cabri, construct a circle with center C and a fixed point P in its interior. For which chord AB of the circle, that contains point P, is the product $AP \times PB$ maximum?

The teacher informed the students that they had to work without his intervention. The teacher and the members of the research group acted as non participative observers in some of the student groups, with the intention of registering in video the solution process and intervening, only if necessary, to favor the exposition of ideas by students and thus obtain as complete information of the process as possible.

The students and the course content

The students were registered in the third course of the geometry trend of the pre-service program. They had participated of our methodological approach since their first semester. With respect to the geometrical content covered up to the moment of the proposed task, congruency of triangles, parallelism of lines, and quadrilaterals were given a thorough treatment. With respect to similarity of triangles, the definition, the criteria to determine it, and theorems such as Ceva and Menelao were established. The students had experience in proving properties that are deduced from the similarity of two triangles, and in using the similarity to prove other geometrical properties. The existence of chords, diameters and secant lines was discussed from a theoretical point of view. The *Theorem of the interior point of a circle*, which establishes that a line which contains a point in the circle's interior, intersects it in two points, was proved. The special relations between angles and circles had not been studied. Precisely, with the proposed problem these were expected to be introduced. In this article, we concentrate on the analysis of the activity of just one group (henceforth NAF) that, for the purpose of this research, had no special characteristics with respect to the other groups.

Components of a successful performance

The students perform relevant intentional actions towards the final goal or the recognizable sub-goals throughout the solution process (teleological aspect), such as: modeling the situation in Cabri appropriately; exploring by dragging and measuring; detecting the regularity; producing different and relevant types of arguments (inductive, abductive, deductive) in the different phases of the solution process; enriching the figure with an auxiliary construction, if necessary, to favor a search process of key ideas for the proof. Specifically, to solve the proposed problem, the students construct another chord containing point P to verify that the result obtained with the first chord is also true for the second one or as a mechanism to prove the thesis. They use the two chords to determine two triangles, visualize or conjecture their similarity, establish a path to obtain the equality of the two products involved, and justify such invariance within the theory of similar triangles. They also carry out empirical explorations with Cabri to identify the corresponding angles that are congruent, and to discover the theorem that establishes that inscribed angles that subtend the same arc are congruent, that can be used provisionally as a justification since it is not yet part of their theoretical system.

The students perform general actions associated with the requirements of the mathematical discourse of proof related to the production of appropriate arguments in the different phases of the solution process (epistemic aspect): every statement that is part of the arguments must be justified; every justification must come from the theoretical system in which they are working; the representation system used provides information only based on the conventions previously established in class; a warrant can be used to obtain a conclusion only if the conditions required in its antecedent have been established before; in a deductive chain the premises change their operative status (i.e., a premise obtained as a conclusion, in one step, can be data in a posterior step); between two deductive chains that lead from the same premises to the same conclusion, the one that presents a simpler path is preferred. Specifically, they look for possible warrants to validate the congruency between at least two pairs of angles, and recognize that in the theory available to them they can only show the congruency of the vertical angles.

The students communicate their ideas carefully: formulating the conjecture as a conditional statement; reformulating the conjecture, if needed, to facilitate the construction of the proof; using the terminology established in the classroom appropriately; and using the format established in class to expose their final proof.

Data treatment for the analysis

The video of the group's work was transcribed, and the observer's figures and notes were included in the margins, so that reading the transcript permitted following the students' detailed activity comprehensively. The transcription was divided in phases, each one covering an important sub-activity of the complete process. The different types of arguments were identified, typified and outlined, and the interventions analyzed to determine signs of the other three aspects of the integrated model. Due to space limitations, we shall present emblematic episodes that well represent the activity we are evaluating.

It is necessary to make two comments. Firstly, when the data for the study was collected, using the model to analyze the activity was not part of the plan; therefore, no questions were designed to promote student allusions to the epistemic and teleological aspects. Secondly, we are not analyzing finished reports made retrospectively, but student conversation when carrying out the task. Thus the arguments are mostly a collective construction, although, occasionally, the observer's questions impel one of the group members to synthesize the discussion and thus assume the responsibility of exposing the co-constructed argument. This is why we evaluate the group's proving activity and not that of individuals.

EVIDENCE TO EVALUATE NAF'S PROVING ACTIVITY

NAF detects that the product is constant, writes the conjecture and sketches a way to prove it

Alejandro reports the result of their first exploration: "I am measuring $\overline{AP}$ and $\overline{PB}$ to multiply them and check if the maximum is when it is a diameter, or if it is in some other place […] The product remains the same always […] even if the measure of the chords change; the product will be the same rotate it wherever we rotate it."

In this fragment, we see signs of an inductive argument. The students, using Cabri, generate innumerable positions of chord $\overline{AB}$ together with the respective products $AP \times PB$ and thereof detect the invariance. The premises that provide evidence (data) to affirm that the product is constant (claim) come from the conditions found in the problem statement and of the numerous cases that are offered by dragging the chord. The warrant is the conjecture that suggests that for any chord that contains P, the product is constant. On the other hand, in Alejandro´s verbalization we find an initial plan to answer the question asked in the problem, plan that he carries out in Cabri as he talks, and that is evidence of the presence of the teleological aspect.

The students become involved in writing the conjecture as a conditional statement. To start with they mention the if-then format as the proper one to express the conjecture. With Fabian's intervention: "Shall we put given or must we construct it?" they evaluate if they can assume the existence of chord AB as given or if they must include, in their proof, statements and justifications that theoretically validate the construction of the chord. Afterwards, they agree on a first statement: "*Given a circle with center C, a fixed point P which belongs to the interior and a given chord AB which contains P, then the product AP × BP is constant.*" However, Nancy manifests inconformity: "No, look, you know what? ... It's better, given a circle with center C and a fixed point P such that P belongs to the interior of the circle, for **any** chord AB of circle C such that P belongs to AB, then the product such and such." Alejandro points out that the purpose of changing the word is to bring out the generality of the fact: "…we had said **one**, only one chord; then, it was saying that only one chord AB exists; now we are saying that for any chord that passes through point P, then it's going to be constant."

The epistemic aspect appears when they ask themselves if they must justify the existence of chord AB; this suggests that they see the difference between a chord that exists, because it is given in the statement's premises, and a chord whose existence is justified theoretically; they are obeying the class norm of justifying the existence of the geometric objects that are being used. We also see the communicative aspect when they write their conjecture; on the one hand, because they know they have to formulate it as a conditional and, on the other hand, because they note that their first formulation is incorrect since it does not express the detected generality, reason why they include the universal quantifier.

When they are rewriting the conjecture, Alejandro asks: "In the proof we can construct another chord, right? To have similar triangles." and explains, "…because what we need to prove is a ratio." When the observer asks him the reason for the auxiliary construction, he answers "Because we see that the product will always be the same, right? Then another chord can give us similarity or ratio between this side, the segment that we would create new and this..." Nancy amplifies Alejandro's idea: "We would have that the ratio of… or that by ratios we get that *AP* times *BP* is going to be the same for both chords. That way we would confirm that it would be for **any** chord… not only the given one but also the one we compare it with."

NAF has set the goal: proving that the product is constant for any chord. This leads them to construct two chords that contain *P* with the purpose of obtaining similar triangles, to thereof work with ratios that will lead to equal products. This goal motivates an auxiliary construction without which, as we know, it is practically impossible to prove the conjecture. We recognize the teleological aspect in the conversation because they sketch a plan to reach their goal and propose an auxiliary construction as a tool to obtain it.

NAF examines how to justify the existence of another chord

As they start writing the proof, they consider how to justify theoretically the existence of both chords that contain *P*, which they have represented in Cabri. They establish that the first chord is given and that they only have to justify the existence of the other chord. Alejandro points out that they must guarantee the existence of a line through *P*, maybe motivated by the construction done in Cabri. Nancy mentions that the line must also contain a point of the circle. Fabián says: "To create the line we need the Line Postulate[2] and it requires the two points. What shall we do?"

In the summarized interchange, the existence of the other chord containing *P* can be seen as the claim of a possible abductive argument that does not take shape because the warrant is not explicit (i.e., the chord is a subset of the line). In contrast, the existence of the line containing *P* (and not the given chord) becomes the claim of an abductive argument when Nancy indicates the necessity of having two points (data) and Fabián completes it by mentioning the Line Postulate (warrant). The goal they establish, due to this argument, is to justify the existence of the two points; it guides their next actions. Thus the teleological aspect is present. With respect to the epistemic aspect, it is worth noting that in the first case the warrant is not mentioned while in the second it is.

Trying to justify the existence of two points that determine the line whose existence they want to show, Nancy suggests using the Interior Point of a Circle Theorem, and, due to Alejandro's petition, she says: "if we have a circle and a point of it and a line, ah! no, but we need a line anyway, that is we have to construct it." When Nancy discards this possibility, Alejandro proposes a plan: "The best would be to construct

a line by the Line theorem [Postulate] and then we can say that the points intersect the circle." NAF eventually realizes that neither way is useful for their purpose.

The claim of the argument in the conversation is the existence of two points, *P* and a point on the circle; the failed warrants are the Interior Point of a Circle Theorem and the Line Postulate together with its intersection with the circle. The data required is the existence of a line that intersects the circumference in two points. The epistemic aspect is evidenced when Nancy realizes that they do not have the elements of the hypothesis of the theorem they want to use, and that, therefore, they must discard it.

NAF discards the established path to validate the existence of a pair of congruent angles
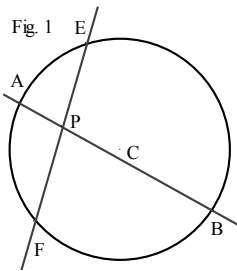
Fig. 1

The students have two chords that contain *P* (Fig. 1). Alejandro declares the congruency of ∠*FPB* and ∠*APE* because they are vertical angles. Nancy questions: "And, from there, where are we going?" As an answer, Fabián proposes the following plan: "We construct the triangles and then we talk about the angles to talk about similarity." Alejandro adds that with the Angle-Angle Criteria they would already have similarity. Nancy objects: "And, where is the other [pair] angle? Okay, we already have these two angles, and the others, where are we going to get them? We need another one." Once they have the triangles, Fabián discards parallelism as a way to reach their goal: "There are no parallels. Because if we had them, this would facilitate finding alternate interior angles, and we already have the other angle and similarity would be the result."

We find two abductive arguments. In the first one, the claim is the existence of similar triangles, the warrant the Angle-Angle Similarity Criteria, and the data required the congruency of two pairs of angles, of which one is already guaranteed. In the second one, the claim is the existence of another pair of congruent angles, the warrant is the theorem that guarantees that alternate interior angles between parallel lines are congruent, and the data required is assuring that two lines are parallel.

We point out three issues. Firstly, we can see the control Nancy exerts over the trend of the activity they are developing, sign that she is conscious, on the one hand, of the necessity of not losing sight of what they want to justify and, on the other hand, of the class norm of justifying every statement in the context of the situation they are studying. Nancy's interventions —the first one of teleological nature and the second of epistemic nature— lead them to formulate a plan or discard a possible path. Secondly, Fabián's argument, with which he discards parallelism as useful to reach their goal, makes us think that he tacitly assumes that the congruent angles are ∠*EBA* and ∠*FAB* or ∠*BEF* and ∠*AFE,* which is not correct. Thirdly, we are surprised that, during a good part of their activity, NAF refers to similar triangles without explicitly establishing the correspondence for the similarity. Maybe they could have established it much sooner than when they actually did if they had allowed

themselves to explore the situation numerically with Cabri (angle measurements, calculating the proportion) as Fabián suggested: "The only thing we can do is measure this one with…, the ones that are going to determine, form similarity and see whether that proportionality remains in the three [angles] and then, with that we can then determine that they [the triangles] are similar." Nancy responds: "[…] we find measures and that, but how do we find it here… geometrically?", and Fabián accepts the veiled objection: "With the postulates and that."

NAF determines the proportionality of the measurements of the sides and the congruent angles

After various failures in trying to find the way to validate the congruency of two angles, without indicating exactly which pair they are referring to, Alejandro turns to the observer: "Help us. How do we relate another angle?" She responds: "Are you sure that the triangles are similar?" Nancy explains why they are similar: "[…] they are similar, not because of the angles but… because we have this angle [referring to the vertical angles] and as Alejandro showed, having *AP* times *PB* is equal to *EP* times *PF* [she writes $AP \times PB = EP \times PF$] we can make our proportion […] Then *AP* is to *PF* as *EP* is to *PB* [writes $\dfrac{AP}{PF} = \dfrac{EP}{PB}$] and that way we have it." Fabián asks surprised: "*EP* to *PB*? … This segment is to this segment … which triangles are you talking about?" Nancy responds: "Of triangles *FPB* and *APE* because as you superimpose [moves hand] let's say this one [signals $\triangle APE$] over this one [signals $\triangle FPB$] we have that this [shows $\overline{PE}$] is to this one [points to $\overline{PB}$] as this [shows $\overline{AP}$] is to this [points to $\overline{PF}$]." Alejandro adds: "In the calculator I looked at angle *PAE* […] and angle *PFB* and they are congruent."

To explain why the triangles are in fact similar (claim), Nancy recurs to the Side-Angle-Side Similarity Criteria (warrant) which she does not mention. She takes into account the empirically found fact when exploring the situation: the constant product of the measures of the segments determined by point *P* on each chord of the circle (data). Using that fact, she obtains the proportionality of the measurements (intermediate claim). Thus we see that with the purpose of justifying the similarity of triangles *FPB* and *APE*, Nancy carries out a deductive argument. It must be noted that the students are conscious that such reasoning is not the adequate one for the situation they are tackling because they use what they want to prove.

CONCLUSIONS

As we compare the description of NAF's activity with the components established for a successful performance, we recognize that NAF sets four sub-goals that lead them to the expected goal: detect regularity; formulate a conjecture in the terms used by the class community; justify the auxiliary construction of a chord; and prove the congruency of two pairs of corresponding angles. Intentional actions to reach the sub-goals are evidenced although proving that the triangles are similar is not

proposed with the precision required and desired. Only after more than an hour, does NAF discover which are the corresponding angles that guarantee the similarity, and they never find the relation between the angles and the circle which would permit justifying their congruency. Maybe with this deficiency in their proving activity, many of their actions to search for the justification could be considered as not relevant. Related to the former is the fact that NAF prefers using the theory as resource more than the empirically obtained information; this shows an undesired unbalance between exploratory actions and justification actions. Yet, the abductive arguments that arise show that NAF has enough knowledge of the theme to allow them to make connections that are not incongruent with the situation they are studying; these arguments impulse and guide their actions. Although in the whole process we evidence skill in handling the teleological, epistemic and communicative aspects, they still lack the mastery needed to perform as an expert. Or maybe we cannot expect the students to act as if they were already writing a report of successful arguments, in the course of the proving activity process.

NOTES

1 In fact, Toulmin's model presents six components of an argument: claim, data, warrant, backing, qualifier and rebuttal.

2 The Line Postulate states that given two points there exists a unique line containing them.

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