Childrens understanding and use of inversion in arithmetic[1]

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Abstract

In this presentation, I consider the origins and the extent of children's under-standing of the inverse relation between addition and subtraction. I argue that this understanding might have its origins in children's informal experiences with physical matter but I also show that it is possible to improve children's grasp of inversion through teaching. I also show that his teaching has beneficial effects on children's solutions to sophisticated word problems in which the arithmetical operation that is need for the solution is not immediately obvious.

Key words

Inverse relation between addition & subtraction, additive composition of number, mathematical reasoning.

1   Introduction

Soon after children have learned to count, they begin to be taught about addition and subtraction at school, and a little later about multiplication and division. These four arithmetical operations are at the centre of children's formal experience with mathe-matics during their first few years at school. The operations are in some ways separate, but there are connections between them and it seems very likely, and almost uncon-troversial, that it is as important for children to learn about these connections as about the individual operations themselves.

One clear connection is inversion. This is the principle that each arithmetical operation has its opposite: the opposite or inverse relation to addition is subtraction, and vice versa. The opposite or inverse relation to multiplication is division, again vice versa. One familiar way of illustrating the inversion of addition and subtraction is with prob-lems in which the same quantity is added and subtracted – $a+b-b$. Here the addition and subtraction cancel each other out, which removes the need for any computation to solve the problem. If you add and subtract the same amount you restore the status quo. If you add more than you subtract, you increase the quantity. If you subtract more than you add, you increase it. On the whole children in their first years at school do quite well in problems of this sort (Bryant, Christie & Rendu, 1999) and their success

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suggests that they do have some knowledge of the inverse relation between addition and subtraction.

The understanding of inversion must be a basic part of learning about number. It is simply impossible to understand the additive composition of number without also knowing about the principle of inversion. Additive composition is the principle that numbers are constructed of, and can be broken down into, other numbers: 8 can be constructed by adding 3 to 5 and it follows that subtracting 5 from 8 gives you 3. How a number is composed is logically connected to how it can be decomposed, and inversion is the key to why this is so (Nunes & Bryant, 1996).

Another plausible suggestion is that inversion underlies the understanding of the exchanges in written addition and subtraction of multi-digit numbers algorithms. Bryant and Nunes (2009), Fuson (1990), Gilmore (2006) and Nunes and Bryant (1996) have argued that understanding carrying and borrowing requires understanding the inverse relation between addition and subtraction. Fuson, for example, argued that, when children are adding 7 tens and 6 tens, in order to understand the ten-for-one to the left exchange, they need to realize that they are taking 100 away from the tens place and adding 100 to the hundreds place; so the value of the total is not changed. A similar reasoning is required in a subtraction such as 2107 − 72; the conservation of the minuend cannot be understood unless one understands that taking away 100 from the hundreds place and adding 100 in the form of 10 tens to the tens place does not change the quantity.

A third reason for an effect of the understanding the inverse addition–subtraction relation on computation prowess is that one can use additions to solve subtraction sums if the numbers are close to each other. The efficiency of this computation strategy seems obvious in problems such as 71 − 69, which could lead to calculation errors if the children were trying to use the written algorithm. Counting up from 69 is a quick and easy approach to this subtraction. Torbeyns, Smedt, Stassens, Ghesquière, and Verschaffel (2009) reviewed the literature on the use of addition to solve subtractions, which they called indirect addition, and found that this is considered by many researchers as a useful computational approach (Fuson, 1986; Beishuizen, 1997 & Brissiaud, 1994).

In conclusion, it is a good move on the child's part to use the inverse relation between addition and subtraction in order to simplify some difficult computations. It may be necessary for understanding multi-digit addition and subtraction with regrouping. Although it may confer an advantage in some problems, it is not so easily adopted by children even after teaching. Comparative research in which teaching relies exclusively on mental resources versus teaching that uses external tools could shed light on why school children did not adopt the method so easily in certain studies. But before we consider the question of teaching, let us turn first to the origins of children's understanding of the inverse relation between addition and subtraction.

2   Inversion: identity and quantity

Children's understanding of inversion may have some roots in the fact that some aspects of inversion are familiar parts of their lives very early on. These aspects of inversion are probably not quantitative. Suppose that a child gets some mud on his shirt, which someone then washes off, and thus the shirt is restored to its former state. This is a form of inversion: mud was added and then this mud was subtracted, thus restoring the status quo. But it is possible to understand this without having to think about quantity, because the same material –mud – is added and then subtracted. For this reason, we call this kind of inversion *the inversion of identity.* We contrast it with *the inversion of quantity.* To understand this kind of inversion, you must grasp the fact that adding and subtracting the same quantity to and from some initial amount restores the status quo even it different items are added and subtracted. For example if I have 5 tennis balls and someone gives me two more and also subtracts two different balls from my initial set, I still have 5 balls even though the addend and the subtrahend were not the same items.

We set out to look at 5- and 6–year-old children's understanding and use of both kinds of inversion in a simple experiment in which each child was given a set of inversion problems. In each trial we started with an initial quantity which was a tower of bricks stuck together. We showed this to the child without allowing him to count the number of bricks in it, and we also covered part of the tower with a cover to prevent the child counting all the bricks in the tower. Then, with the child looking on, we added some bricks to the tower and we also subtracted some bricks from it. Sometimes we added and subtracted the same quantity (e.g.+2-2). At other times we added more than we subtracted (e.g. +2-1) or vice versa (+1-2). After the addition and subtraction, we asked the child whether there were now more or less bricks in the tower than at the start of the problem or whether the final quantity was the same as the initial quantity. So, this was a straightforward test of how well children understand that adding and subtracting the same amount restores the status quo, adding more than you subtract increases the overall amount and subtracting more than you add decreases it.

We also used this task to compare children's understanding of the inversion of identity and of the inversion of quantity. We gave children these problems in two different conditions. We called one *the identity condition* and the other the quantity condition. In *the identity condition* we added bricks to and subtracted them from the **same** end of the tower. So, when, for example we added two bricks and then subtracted two bricks, we added and subtracted the same, identical, bricks. In the *quantity condition,* we added bricks to one end and subtracted them from the other end, and so the bricks that we added and the bricks that we subtracted were entirely different bricks.

Our hypothesis was that children learn about the inversion of identity first and extend the idea to quantity later. This led to the predictions that

- the identity condition would be easier than the quantity condition, but some children would solve the quantity problems as well as the identity problems
- the older children would do better than the younger ones with the quantity problems

- the children would learn about identity before quantity and so there would be several children who understand identify inversion but not quantity inversion, but there would be no children who understand quantity inversion but not identity inversion.

All three predictions turned out to be correct. The overall success of the children was far greater with the identity than with the quantity problems. The 6-year old children were more successful than the 5-year old children in the quantity condition but no better than them in the identity condition. Finally, when we looked at children who made a significantly better than chance number of correct choices in the two conditions, we found that 20 out of the 64 children produced significantly above chance scores in the difficult quantity condition, and all of these 20 children also produced significantly above chance scores in the easier identity condition. So there are children who can solve the identity problems well, but cannot solve the quantity problems, but no children who can solve the quantity problems and yet fail with the identity ones. This is clear evidence that understanding the inversion of identity proceeds, and may lead to, the understanding of the inversion of quantity.

3   Inversion: transparent and non-transparent

Having established something about the origins of children's discovery of quantitative inversion, I would now like to consider to what extent children are able to use their growing knowledge of the principle of quantitative inversion in their mathematical activities. It's not much use knowing about this principle unless you can use it flexibly. If you simply wait for a+b–b sums to take advantage if the principle if invariance, then you are going to have a long wait.

In the next study that I want to present, we looked at the possibility that 8-, 9- and 10-year-old children sometimes use decomposition to convert problems into ones that can be solved through inversion. Consider the problem $52 + 29 - 30$. This would be quite a hard sum for anyone trying to solve it by computing first the addition and then the subtraction. However, by decomposition of the 30 into 29+1 the sum can be transformed into 52+29–29–1 and then into 52–1. We could find no evidence that children were ever taught to use decomposition to create an inversion problem, but we were interested in the possibility that some children might be able to invent this kind of solution themselves.

We presented 8- to 10-year old pupils with a series of three-term sums which consisted of an addition followed by a subtraction . In some sums:

Each child was given 9 different types of sum as follows:

1. Control: a+a–b, e.g. 13+13-9

2. Inversion: a+b–b, e.g. 18+16–16

3. Inversion (involving multiples of 10): a+b–b, e.g. 17+30–30

4. Minus 1 decomposition: a+b–(b+1), e.g. 19+12–13

5. Minus 2 decomposition: a+b-(b+2), e.g. 21+12-14

6. Minus 3 decomposition: a+b-(b+3), e.g. 21+6-9

7. Minus 1 decomposition (involving multiples of 10): a+b-(b+1), e.g. 18+30-31

8. Minus 2 decomposition (involving multiples of 10): a+b-(b+2), e.g. 19+10-12

9. Minus 3 decomposition (involving multiples of 10): a+b-(b+3), e.g. 26+20-23

The control problems were designed as sums to which it would be very difficult and probably impossible for the children to apply the principle of inversion. So, the children had to compute to do the sum. In the two inversion problems, exactly the same quantity was added and subtracted, and these were therefore paradigm a+b-b inversion problems. In the remaining problems (problems 4–9) inversion was possible if the child decomposed the subtrahend. In some cases (problems 4 & 7) the subtrahend differed from the preceding addend by 1, in others (problems 5 & 8) by 2 and in others (problems 6 & 9) by 3.

Table 1 gives the mean number of correct answers for the control problems and the two straight inversion problems (problems 2 and 3). This shows that the children were able to use the inversion principle and this was was very helpful to them to do so since they managed to answer the inversion problems so much better than the control problems.

Table 1
Mean and standard deviation of number of sums correct (out of a possible 4)
in the control and the straight inversion problems

Age (years)		Control	Inversion	Inversion (10s)
8	Mean	1.57	3.04	3.13
9	Mean	1.33	3.84	3.95
10	Mean	2.20	3.60	3.80

Table 2 gives their mean correct scores in the remaining problems, in which they had to decompose the subtrahend in order to take advantage of the latent inversion in the sum. The pattern of the results show that most of the scores are noticeably higher than the score for the Control problems for the relevant age group as shown in the previous table (Table 1). The main exception is the performance of the 8–year-olds when the difference in the quantity of the addend and the subtrahend was 3. This was, most probably, because they found it hard to decompose the subtrahend to transform the sum into an inversion sum when the addend–subtrahend difference was so great. When the addend–subtrahend was 2 or 1, they clearly benefited from the inversion that they were able to uncover because their scores were significantly better in these problems than in the control problems.

Table 2
Mean and standard deviation of number of decomposition sums correct (out of a possible 4)when the difference between the quantity of the addend and the subtrahend was 1 (problems 4 and 7), 2 (problems 5 and 8) and 3 (problems 6 and 9)

	Age (years)		Difference of 1	Difference of 2	Difference of 3
Standard: problems 4-6	8	Mean	1.91	1.83	1.43
	9	Mean	2.58	2.42	2.21
	10	Mean	3.15	2.90	3.05
With multiples of 10: problems 7-9	8	Mean	2.48	2.00	1.65
	9	Mean	2.89	2.84	2.79
	10	Mean	3.10	3.10	3.16

These results demonstrate that many 8- to 10-year-old children seem to be able to recognise the possibility of transforming a complex sum into an inversion problem and thus of making the problem easily soluble. They can actively create inversion.

In other work, which I will not describe in any detail here because it is already published, Camilla Gilmore and I (Gilmore & Bryant, 2008) showed that many 8-year-old children can create and use inversion in another context. To put it in a nutshell, we presented on a screen 5-term addition and subtraction sums and we found that the majority, but not all, of the children did much better with problems with a latent inversion structure like 15+11-8-3 +? than with control problems like 13+11-5-4+?. The positive results are indeed impressive. The children who successfully constructed a 4-term inversion problem (14+11-11=?) out of 15+11-8-3 +? needed no hint to do so. They plainly saw the power and the usefulness of inversion in arithmetic.

4   Inversion as part of children's relational calculus

Up to now, I have stressed the importance of understanding inversion in making numerical calculations of one type or another. Now, I wish to turn to what Terezinha Nunes and I (in press) have called "relational calculus", which is about working out relations between quantities. The solution to many arithmetical problems rests on an understanding of the underlying relations between the quantities that the problem concerns. Sometimes, this underlying set of relations is not transparent. This certainly applies to some problems whose solution depends on the understanding and use of inverse relations.

Consider start-unknown (?+b=c; ?-b=c) and change-unknown (a+?-b; a-?=b) problems. When the story in start-unknown problems is about an addition (e.g. I had some sweets and my friend Mary gave me 6 more sweets. I counted up how many I now had and it came to 11. How many sweets did I have before Mary gave me any?), the solution depends on subtraction, and when it is about a subtraction, the solution is an addition. In a change-unknown problem, when the story is about an addition, the solution is a subtraction. However, when the story is about a subtraction, the solution

is also a subtraction. So, in these start– and change–unknown problems, children have to reason in quite a sophisticated way about the underlying structure of the quantitative relations in the story, in order to decide whether to add or subtract, and it seems highly likely that their understanding of the relation between addition and subtraction play an important role in this reasoning.

We examined this hypothesis in an intervention study in which we taught some 7– and 8–year–old children about the inversion principle over two sessions and others for the same amount of time about numerical procedures to do with counting and computation. When we taught the inversion group about the inverse relation between addition and subtraction, we included start–unknown but not change–unknown problems. We gave all the children a pre–test with both start– and change–unknown problems just before the first intervention and an identical immediate post–test just after the second of the two intervention sessions. We also gave the children a delayed post–test, with start– and change–unknown problems 8 weeks after the end of the intervention.

The results, which I shall present in detail in my oral presentation, were mainly positive. The children who were taught about inversion did better in the change–unknown problems in the post–tests than the children who had been taught about numerical procedures, even though the inversion intervention did not include any change–unknown problems. It seems that the experience of being taught about the inverse relation between addition was a radical help to the children when they had to work out that an addition was the right solution to a change–unknown story about a subtraction, and vice versa.

5   Conclusions

We conclude that the growth of children's understanding of inversion is an important and interesting part of their mathematical growth. This understanding is probably based on informal experiences with what we call identity inversion, which children eventually extend to quantitative inversion. This achievement has a momentous impact on their understanding of the additive composition of number, on the use of decomposition in arithmetical calculations and also on their ability to carry out multi–digit subtractions. It also helps them solve problems, such as the start–and change–unknown problems, which depend on their understanding the underlying quantitative relations in the problem itself. It is possible to teach children about inversion, as our intervention study showed, and we hope that children will be taught more than they are at present about inverse relation between addition and subtraction, and also between multiplication and division (though at present we know very little about children's understanding of this last aspect of inversion).

References

Beishuizen, M. (1997). *Development of mathematical strategies and procedures up to 100.* In M. Beishuizen, K. P. E. Gravemeijer, & E. C. D. M. van Lieshout (Eds.), The role of contexts and models in the development of mathematical strategies and procedures. Utrecht, The Netherlands: Beta. (pp. 127–162).

Brissiaud, R. (1994). Teaching and development: Solving "missing addend" problems using subtraction. In B. Schneuwly & M. Brossard (Eds.): Learning and development: contributions from Vygotsky. *European Journal of Psychology of Education,* 9, 343–365.

Bryant, P.; Christie, C.; Rendu, A. (1999). Childrens understanding of the relation between addition and subtraction: Inversion, identity and decomposition. *Journal of Experimental Child Psychology,* 74, 194–212

Fuson, K. (1986). Teaching children to subtract by counting up. *Journal for Research in Mathematics Education,* 17, 172–189.

Fuson, K. (1990). Conceptual structures for multiunit numbers: Implications for learning and teaching multidigit addition, subtraction, and place value. *Cognition and Instruction,* 7, 343–403.

Gilmore, C. (2006). Investigating children's understanding of inversion using the missing number paradigm. *Cognitive Development,* 21, 301–316.

Gilmore, C.; Bryant, P. (2008). Can children construct inverse relations in arithmetic? Evidence for individual differences in the development of conceptual understanding and computational skill. *British Journal of Developmental Psychology,* 26, 301–316.

Nunes, T.; Bryant, P. (1996). *Children doing mathematics.* Oxford: Blackwell.

Nunes, T.; Bryant, P.; Hallett, D.; Bell, D.; Evans, D. (2009). Teaching children about the inverse relation between addition and subtraction. *Mathematical Thinking and Learning,* 11, 61–78.

Nunes, T.; Bryant, P.; Evans, D.; Bell, D. (In press) *Teaching children how to include the inversion principle in their reasoning about quantitative relations.* Educational Studies in Mathematics.

Piaget, J. (1950). *The Psychology of Intelligence.* London: Routledge.

Torbeyns, J.; Smedt, B.; Stassens, N.; Ghesquière, P.; Verschaffel, L. (2009). Solving subtraction problems by means of indirect addition. *Mathematical Thinking and Learning.* 11, 79–91.