Mathematical Impossibility in History and in the Classroom[1]

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Abstract

Theorems stating that something is impossible are notoriously difficult to under-stand for many students and amateur mathematicians. In this talk I shall discuss how the role of such impossibility statements has changed during the history of mathematics. I shall argue that impossibility statements have changed status from a kind of meta-statement to true mathematical theorems. I shall also argue that this story is worth telling in the classroom because it will clarify the nature of impossibility theorems and thus of mathematics. In particular it will show to the students how mathematics is able to investigate the limits of its own activity with its own methods.

Key words

Philosophy, Mathematics Education, History of Mathematics.

1   Impossibility, the noble quest

To accomplish the impossible is the most ambitious quest one can have in life. This is the central message one will get if one searches the web for quotations on "impossibility. Here is a brief list of such quotes accessible from (thinkexist.com, 2011):

"The impossible – what nobody can do until somebody does"

"Start by doing what's necessary; then do what's possible; and suddenly you are doing the impossible." (St Francis of Assisi)

"The impossible is often the untried."

" Impossible is not a scientific term."

"The Wright brothers flew right through the smoke screen of impossibility."

"Every noble work is at first impossible" (Thomas Carlyle)

"Impossible only means that you havent found the solution yet."

"I love those who yearn for the impossible." (Goethe)

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"Its kind of fun to do the impossible." (Walt Disney)

"Nothing is impossible... It is often merely for an excuse that we say that things are impossible" (Duc de La Rochefoucauld)

"Impossible is a word to be found only in the dictionary of fools." (Napoléon)

I shall not vouch for the accuracy of the quotes. I only cite them to give an impression of a consistent popular view of the impossible. With such a view in mind it is quite natural that many amateur mathematicians have tried to square the circle or trisect the angle. And it is also clear that the attitude of professional mathematicians must seem extraordinarily arrogant to them. Not only do the professional mathematician claim that the problems are impossible, they also claim that they know in advance that the solution presented by the amateur must be wrong and therefore hardly bother to look at it!

What the amateur usually has not understood is

1. that mathematical impossibility theorems do not claim that a problem is impossible to solve in general, but only that it is impossible to solve with a particular type of methods and within a particular well defined framework. In fact mathematicians often have the audacity to claim that clearly unsolvable problems such as the equation $x^2 + 1 = 0$ have solutions after all if the domain of enquiry is extended far enough. In this way they behave just like the people quoted above.

2. that mathematicians do not just claim that because they have not been able to find a solution it must be impossible but that they have a proof of the impossibility.

These misunderstandings are not new. Already in 1778 Condorcet wrote that "a mass of people, many more than one thinks, renounce their useful job in order to abandon themselves to the research of these problems" (Condorcet 1778). The problems he referred to was the quadrature of the circle, the duplication of the cube and the trisection of the angle, and the occasion was the decision made in 1775 by the Académie des Sciences to stop reviewing solutions of these problems.

The hope was that this step combined with the enlightening work of Montucla (1754) would dissuade amateur mathematicians from wasting their time solving the problems. It is well known that the effort did not work. Circle squarers continued their futile work for centuries. Still, at least in Denmark the last decades have experienced a great decline in the number of circle squarers who address their purported solutions to the universities. Is this a delayed result of the enlightenment that Condorcet and his fellow philosophers opted for? I am afraid not. Without having made a statistical investigation I am convinced that the diminished interest in the classical problems is not due to more knowledge about the problems. On the contrary, it seems rather to be the result of ignorance. Fewer children learn a sufficient amount of geometry in school to ever encounter the problems, and thus they are not tempted to try to solve them.

So in this case ignorance has had a more positive influence on the problem than enlightenment. Still, I think that there are good reasons to prefer enlightenment even if it might create more circle squarers. In fact, I think that impossibility theorems have a place in the classroom at least at high school level. To be sure such theorems do

not teach the students to solve more mathematical problems, but they will teach them that one cannot solve all problems with a given method. This is in itself an important lesson that can be of help to those who learn mathematics in order to use it in their later profession.

Impossibility theorems have even more important lessons to teach those students who want or need to learn something about the nature of mathematics. In particular, it is important for the students to understand how mathematics has been able to deal with its own limits using its own methods. This is rather unique to mathematics. In most other areas of life a solution of a problem and a statement of its impossibility are two very different types of statement and they call for different methods. For example take the problem of flight alluded to in one of the impossibility quotes above. Here a solution was an engineering accomplishment namely the construction of an airplane. An impossibility argument might have relied on fundamental physics or philosophy but could of course never have been an engineering construction.

Another lesson to be learned from impossibility theorems in mathematics is the utmost precision required to make this type of statements meaningful and true. For example it is not enough to state that the classical problems are impossible, One must state that the problems are only impossible if one requires that they be constructed by ruler and compass, and one even has to make quite precise what one is allowed to do with the ruler and the compass. One can only draw a straight line between two given points and a circle with given center and radius. And one can consider all intersection points arising in this way as new given points. One is not allowed to make a so-called neusis construction although such a construction is in a sense made by a ruler.

2   Are impossibility theorems something special?

While many amateur mathematicians have tried to disprove the theorem stating the impossibility of the quadrature of the circle there have been few who have tried to disprove positively formulated mathematical theorems such as Pythagoras' theorem. This alone suggests that impossibility theorems play a special role. Is it only the mathematician's know-all attitude displayed in impossibility theorems that provoke the amateur or are there other differences between impossibility theorems and other theorems?

If we consider mathematics as a collection of theorems it is hardly possible to distinguish impossibility statements from other mathematical statements. An impossibility statement usually says that something does not exist. Such a statement has the form:

$$\neg \exists x : p(x)$$

According to the usual rules of logic this statement is equivalent to the universal statement:

$$\forall x : \neg p(x)$$

For example Fermat's last theorem, which is usually stated as the impossibility of solving the equation $x^n + y^n = z^n$ in natural numbers when n is larger than 2, can

just as well be formulated as the universal statement:

$$\forall x, y, z, n \in N : n > 2 \Rightarrow x^n + y^n \neq z^n.$$

In this way there seem to be no logical difference between impossibility theorems and other theorems.

But it is a fact that certain theorems are usually formulated as impossibility theorems and are recognized as such by amateur and professional mathematicians alike. How can that be? In order to answer that question it is important to distinguish two different ways of looking at mathematics: It can be considered as a theorem proving enterprise and also as a problem solving enterprise. As we have seen the distinction between impossibility statements and universal positive statements is not clear from a theorem proving point of view. However, if mathematics is considered as a problem solving enterprise there is a clear distinction between finding a solution of a problem and proving that a solution is impossible. And all famous impossibility theorems do indeed state the impossibility of solving a problem that might at first sight seem solvable.

This distinction between solutions of problems and impossibility statements seems to be partly responsible for the fact that many amateur mathematicians do not realize that impossibility statements can be proved just as other mathematical theorems. And a view of the history of impossibility theorems will reveal that in earlier periods even first rate mathematicians have considered some types of impossibility theorems as a kind of meta-theorems that are not amenable to proof. This aspect of the history of impossibility theorems seem to me to be one of the major reasons why the history of impossibility theorems can help shed light on the nature of these theorems in a class room.

3   Impossibility statements as meta statements

The classical construction problems mentioned above were formulated quite early in the history of Greek mathematics (Katz 2009). They were all solved by various means but no construction with the Euclidean tools of ruler and compass were found. It has been discussed when and how strictly the Greeks formulated a preference for ruler and compass constructions but with the late Greek philosopher mathematician Pappus (about 340 AD) a strictly normative requirement of simplicity of constructions was formulated (Pappus, see in particular book III chapter VII and book VI chapter XXXVI). According to Pappus a problem is plane if it can be solved by ruler and compass, and it will be a serious methodological mistake to solve such a problem using other means. According to Pappus the trisection of an angle and the duplication of a cube are solid problems. This means they can be solved by intersection of conic sections, but they cannot be solved with ruler and compass. He provided proofs of the positive parts of the statements, i.e. that the problems can be constructed by intersections of conic sections, but the impossibility of solving the problems with ruler and compass remained just a postulate. He mentioned that previous attempts of constructing the two problems by plane means had failed, but it is also clear that when he claimed that the problems were not plane, he meant more than just this empirical fact. He made it clear that the

problems were somehow in principle unsolvable by plane means and even poked fun of an unnamed colleague who had tried to solve the problems by ruler and compass. And yet he never even indicated that he considered this impossibility as a fact that called for a mathematical proof.

This indicates that Pappus considered these impossibility statements as a kind of meta–mathematical statements: a statement about the mathematical problem solving enterprise but a statement that in itself is not a mathematical theorem. Mathematics is still full of such statements. For example we may state about a proof that it is elegant or about a theorem that it is important and no–one will dream of asking us to prove our statements.

One can point to a similar situation in the history of the solvability of polynomial equations by radicals. Here Lagrange (1770/71) made a great effort of analyzing the method of solving equations of degree 2,3 and 4 in order to generalize the method to obtain a solution of equations of higher degree. His analyses of the previous methods were penetrating and yet they did not lead him to a method of solving the quintic. He still decided to publish his results because he hoped that his successors might put them to use in the solution of the quintic if such a solution existed (Lagrange, 1770/71, pp. 355, 357, 403). It is interesting that he explicitly mentioned the possibility that the quintic might be unsolvable by radicals, and equally interesting that he did not suggest that his methods might be of help in proving this impossibility. A few years later (1799) Ruffini attempted just that (Ruffini, 1915) but Lagrange did not bother to respond to his attempts.

4   The lack of importance of impossibility results

This indicates that even when Lagrange was made aware of the possibility of proving the impossibility he did not consider it particularly interesting. In fact there are other historical instances where impossibility results have been overlooked or even explicitly denounced as unimportant: One example is connected to Fermat's formulation of impossibility theorems in number theory. Today he is most famous for his last theorem, but he actually formulated and in one case proved other impossibility results such as the impossibility of forming a right angled triangle with integer sides and an area that is a square number. As pointed out by Goldstein (1995, 136) Fermat's contemporaries did not think highly of this type of theorem. To them mathematics was about solving problems not about finding problems that cannot be solved. For example Wallis wrote about Fermat: "I do not see why he mentions them [negative propositions] as things of a surprising difficulty. It is easy to think of innumerable negative determinations of this sort" (Wallis, 1657, quoted in Goldstein, 1995)

A similar view can be detected in Gauss's dealing with the construction of regular n–gons (Gauss, 1801). He proved in detail how to construct a regular n–gon by ruler and compass if n is of the form $2^k p_1 p_2 \ldots p_i$ where k is a natural number or zero and $p_1, p_2, \ldots, p_i$ are different Fermat primes, i.e. primes of the form $2^{2^j} + 1$. He also claimed that he could prove that the regular n–gon was impossible to construct with ruler and compass if n is not of this form. However, he did not include his proof in the

book. Today we consider this impossibility theorem as at least as interesting as the positive constructive part of Gauss' theorem, but apparently Gauss considered it less important, or perhaps he just thought that his contemporaries would consider it less important.

In (Lützen, 2009) I have shown that Wantzels proof from 1837 of the impossibility of constructing the duplication of the cube and the trisection of the angle by ruler and compass was almost overlooked for a century even though it settled these two very famous classical problems. Again this seems to indicate that such impossibility theorems were still considered as less important than positive theorems even as late as the beginning of the 19$^{\text{th}}$ century.

5   A surge of impossibility results

However, the period around 1830 saw a surge in impossibility theorems. In addition to Wantzel's impossibility proof the most famous is Abel's proof of the impossibility of solving the quintic by radicals (Abel, 1824) and Liouville's proofs that one cannot find certain integrals in finite terms or integrate certain differential equations by quadrature (i.e. in expressed in terms of indefinite integrals) (Lützen, 1990). Wantzel himself wrote several other papers on impossibility results. For example he established that it is impossible to avoid the use of complex numbers when one expresses the roots of a cubic equation with three real roots in terms of radicals (Wantzel, 1843). Moreover Fermat's last theorem was proved around 1830 for $n = 5$ and 7 by Legendre, Dirichlet and Lamé (Katz, 2009).

However as I pointed out above Wantzel's proofs were not really appreciated at the time and the same holds true for Liouville's results that were only taken up again more than half a century later.

6   The difficulty of the parallel postulate

The story of the emergence of non–Euclidean geometry is also the story of impossibility, namely the impossibility of proving the parallel postulate from the other axioms of geometry. However, the story shows how difficult it was to realize that such an impossibility could be proved. In fact the proof presented itself in a somewhat backward way and at first it was not generally accepted as a proof at all. In fact when Gauss, Lobachevsky and Bolyai developed their non–Euclidean geometry they had no proof that the parallel postulate was not a consequence of the other postulates and therefore did not really know that their new geometry was consistent at all.

The road to the proof of the independence of the parallel postulate was opened in 1868 by Beltrami who used Gauss' theory of surfaces to show that a surface of con–stant negative curvature did indeed possess a non–Euclidean geometry if geodesics are playing the role of straight lines. He realized such a surface as the an open circular disc equipped with a suitable metric. The surface of constant negative curvature is now considered a model of non–Euclidean geometry and it is used to argue for the relative

consistency of non–Euclidean geometry. The argument goes as follows: an inconsistency in non–Euclidean geometry would turn up in the model as an inconsistency in Euclidean geometry in which the surface of constant negative curvature lives. Thus if Euclidean geometry is consistent non–Euclidean geometry is consistent as well. This way of putting the consistency argument was explicitly put forward by Poincaré in 1902 in *La Science et l'Hypothèse*. But when Beltrami first presented what he called a real substrate for non–Euclidean geometry it was not immediately realized that it implied the impossibility of proving the parallel postulate. As documented by Voelke (2005) Beltrami himself took some time to draw this conclusion and some of his less prominent contemporaries interpreted it very differently. They took Beltrami's model as evidence that Euclidean geometry was indeed the only correct geometry. To them Beltrami's model showed that Gauss, Lobachevsky and Bolyai had not found a new geometry. They had only developed a geometry of geodesics on a surface in Euclidean space.

This story show how difficult it was for Beltrami and his contemporaries to appreciate the model as a method for proving independence of an axiom. I cannot tell if this is a suitable story to tell to high school students, but it is useful knowledge to their teachers, because it exemplifies the problems the students may have in understanding the meaning of impossibility proofs.

7   Impossibility theorems become main stream

Today impossibility theorems have obtained a central place in mathematics. Indeed many of the most celebrated mathematical results are impossibility theorems. This happened around 1900. Already Abel (1839) had emphasized that his predecessors had made a mistake by posing the problem: Find the solution by radicals of the quintic. Instead Abel suggested that the right question to pose is: Is the quintic solvable by radicals? Only if this question could be answered in the positive could one then go on to ask the question of finding the solution. By changing the problem in this way Abel claimed that all mathematical problems would have answers (Abel, 1839).

Three quarters of a century later Hilbert in his famous lecture on mathematical problems (Hilbert 1900) rephrased this idea. According to Hilbert one must count an impossibility proof as a kind of solution to a problem. In this way all mathematical problems could be solved either by a proof of impossibility or by exhibiting a solution. In mathematics there is no Ignorabimus, as Hilbert famously claimed. He remarked that this decidability postulate was not proved, but he based his claim on general philosophical grounds. In this way impossibility results obtained their full citizenship in mathematics. By the irony of fate the next major impossibility theorem showed that Hilbert was mistaken. Theorems by Turing and Gödel showed that in a sufficiently rich mathematical system there are in fact problems that cannot be solved or proven to be impossible.

8   Different kinds of impossibility results

We have seen that some impossibility results have been considered as meta-statements about mathematics rather than as true mathematical statements. At first this may sound strange. After all the first impossibility theorems are ancient. Probably the first proof of impossibility is the famous proof of the incommensurability of the side and the diagonal in a square. It shows that it is impossible to find a line segment that measures both the side and the diagonal a whole number of times.

With that in mind one might suppose that it would be rather obvious to later Greek mathematicians that one could prove other impossibility statements for example the impossibility of constructing the classical problems using only ruler and compass. And later in the 19th century after one had proven these impossibilities and the impossibility of solving the quintic by radicals it may seem strange that it seemed so difficult to accept a proof of impossibility of proving the parallel postulate.

In order to understand these difficulties I think one must distinguish different kinds of impossibility statements. The distinction I shall introduce goes according to the thing that is claimed to be impossible or non-existent. In the simplest case, for example in the case of the incommensurability or Fermat's last theorem it is an object of the theory that does not exist (such as a common unit of the side and the diagonal of a square). On the next level it is a construction or an algorithm in the theory that does not exist (for example a construction by ruler and compass or an algorithm using only radicals and rational operations). A third level deals with the non-existence of a proof in a theory (for example of the parallel postulate) and the fourth level deals with the impossibility of a proof about a theory (for example that every problem has a solution in Hilbert's sense).

Historically there is evidence that these levels are increasingly difficult to accept as treatable by mathematical means. We have seen that at a given time impossibility theorems of a particular level could be considered as amenable to mathematical proof whereas impossibility statements of higher levels were considered as meta-statements. I think that this observation may help teachers to understand the difficulties of their students in coming to terms with the nature of impossibility statements.

9   Impossibility in the classroom

There are good reasons to teach the students at least in high school about the limits of mathematics. There are different kinds of limits: one kind has to do with the limits of a mathematical model of a phenomenon in nature or society. This type of limits is not treatable with purely mathematical methods and I shall not discuss them any further here. The other type of limits is the kind I have discussed in this paper, namely the inability of solving a mathematical problem with a given mathematical method and within a given mathematical system. As mentioned above it is desirable to teach the students that there are such limits and to explain that they can be investigated with mathematical methods. However, many of the most striking impossibility theorems dealing with these limits are mathematically too difficult to prove in a high school

class. Fermat's last theorem, Gödel's theorems and even the quadrature of the circle are certainly beyond this level. The impossibility of proving the parallel postulate is perhaps within reach and so is the impossibility of the duplication of the cube and the trisection of the angle. Many years ago I wrote a book for the Danish high school in which I went through the history of the classical problems. In this book I also included a rather elementary proof that the duplication of the cube and the trisection of the angle are impossible by ruler and compass (Lützen, 1985). Some teachers worked through the proof with their high school classes and reported that it was difficult but very rewarding at least for the good students.

But if such a proof is too hard to present to the students one can also convey the message by telling the history of one or more of the famous impossibility theorems without going into detail with the proofs and give a baby example of an impossibility theorem. The following example has successfully been used by my colleague Mogens Esrom Larsen when he was faced with circle squarers:

The problem is to find a natural number whose square has a remainder 3 when divided by 4. Here the students may begin to check the squares of the natural numbers beginning with 1 in order to find their remainders modulo 4. They will probably soon discover that the remainders are apparently never 3 but seem to be 0 and 1. They may also observe that the squares of the even numbers have remainder 0 and the squares of the odd numbers have remainder 1. The question then arises if this is evidence enough. Some students may at this point get the idea of trying to prove that any even number has a remainder 0 when divided by 4 and every odd number has a remainder 1 when divided by 4. The proofs are easy:

$$(2n)^2 = 4n^2 \text{ and so } (2n)^2 \cong 0 \text{ mod } 4$$
$$(2n+1)^2 = 4n^2 + 4n + 1 \text{ and so } (2n+1)^2 \cong 1 \text{ mod } 4$$

Thus we have proved that the problem is impossible.

Although this impossibility theorem will hardly in itself seem exciting to the students it may very well convince them that other problems like the classical problems can be given similar but more complicated impossibility proofs. In this way a teacher has succeeded to convince the students that impossibility can indeed be proved in mathematics. This may convince them that it is no longer a noble quest of intelligent people to try to solve the impossible in these cases but rather a futile quest of fools. And it will have shown the students how mathematics can in some sense deal with its own limits using its own methods.

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