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Representations of the generalization of a functional relationship and the relation with interviewer's mediation

Abstract: The ability to generalize and represent the generalization exhibited by eight fourth-grade students was analyzed in this descriptive study, designed around a semi-structured interview involving a task based on the linear functional relationship *y=x+2*. The relation of the interviewer's mediation on students' representations of generalization was determined on the grounds of students' interaction with her. Four forms to represent the generalization of a functional relationship were defined. The findings confirm the importance of mediation in helping students strengthen their ability to recognize, represent and generalize functional relationships.

Keywords: representations of generalization, mediation, functional relationship, early algebra.

Elementary school students' algebraic skills and how are manifested have been the subject of extensive research in recent decades. The identification of algebra-associated capacities in children from very early ages (Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015) has driven efforts to build on that potential from the earliest grades. Under this principle some studies have shown functions introduced in a problem-solving context to be a valid approach to further an understanding of algebra even in the earliest grades (Blanton & Kaput, 2011; Blanton, Levi, Crites, & Dougherty, 2011). One of the targets of such research is students' ability to generalize and how it is expressed (Warren, Trigueros, & Ursini, 2016).

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The importance of generalization and its representation as part of algebraic thinking is undisputed in the mathematics education research community. Researchers (Kaput, 2008; Radford, 2018) deem that algebraic symbolism is not the sole way that students represent such thinking, which may also adopt the form of natural language or even gestures. While the indeterminate quantities associated to algebraic thinking may be represented by alphanumeric symbols, algebraic thinking may also be expressed with the above semiotic systems. The present study explores students' recognition and representations of a functional relationship in the course of semi-structured interviews as well as their interaction with the interviewer. The research questions addressed include: What representations of generalization of a functional relationship manifest fourth-grade students when performing a task in a functional context? How does the interviewer's mediation help students generalize? In addition to supplementing the findings of the research cited above, the purpose is to furnish information that may be useful for teaching early algebra, which has been included in school curricula in countries such as Australia, Canada, China, Japan, Korea, and Portugal (Merino, Cañadas, & Molina, 2013), as well as Spain (Ministerio de Educación, Cultura y Deporte, 2014).

Functional thinking

This study was conducted in the context of early algebra that pursues the development of algebraic reasoning from early ages. The focus is on functional thinking. In that approach algebra is introduced through functions (Cañadas & Molina, 2016). In the functional relationships involved, the value adopted by the independent variable is paired to a single value of the dependent variable (Blanton, 2008).

One of the factors addressed in research on functional thinking is the generalization of relationships (Blanton et al., 2011; Smith, 2008) by describing the general rule whereby quantities co-vary, and represent it (Blanton et al., 2011; Hitt & González-Martín, 2016). This study explores how functional relationships are generalized and represented, in connection with functional thinking.

Recent studies have been interested on functional thinking and the diversity of student behavior involved (e.g., versatility in the use of representation systems, progress from natural to symbolic language, expression of generalization in different registers) when generalizing situations involving functional relationships. Some of these studies have shown that elementary school students are able to distinguish and describe situations in which two quantities co-vary, defining the rules that govern the relationships between them (Carraher, Schliemann, & Brizuela, 2003; McEldoon & Rittle-Johnson, 2010; Stephens, Ellis, Blanton, & Brizuela, 2017) and use of different types of representations (such as tables, figures, or graphs) to explore, describe, and symbolize functional relationships (Blanton & Kaput, 2004; Blanton et al., 2015; Carraher, Martínez, & Schliemann, 2008; Warren, 2006). Blanton et al. (2015) in a study with first graders evidenced the students' potential to represent functional relationships even with algebraic symbology, emphasizing that they can learn to think about functions in a sophisticated and generalized manner from very early ages. These authors suggested levels of sophistication for generalizing relationships from a level in which students do not describe the underlying relationship, to distinguish the function as object, recognizing the conditions in the generality of the functional relationship. In studies in third grade, it is highlighted the importance of monitoring students' recognition of the variables involved in functional relationships and the representation of the relationship by means of conventional representations, for example using the algebraic expression, instead of a recursive form in which is not stablished a relationship between the variables (Carraher et al., 2008).

The above research constitutes invaluable background for the study of elementary school students' functional thinking when characterizing and identifying features associated with the generalization of functional relationships and representing those generalizations. This complementary study aims not to describe students' learning over time. Rather, the emphasis is on what they are able to do and how they structure and represent generalizations in the absence of explicit instruction beyond the interviewer's mediation. The objective is to characterize the representations of generalization that fourth-grade students are capable to manifest when performing a task involving a functional relationship.

Generalization

The importance of generalizing in mathematics lies in the very nature of the discipline, in formulating propositions about numbers and forms (Mason, Graham, Pimm, & Gowar, 1985). Mason, Graham, and Johnston-Wilder (2005) contended that algebra is rooted in the ability to generalize from specific cases and that even pre-elementary school children exhibit that ability. From that perspective, inductive reasoning performs an important role to generalizing, it guides to detect regularities based on specific cases to be extended to a general pattern (Castro, Cañadas, & Molina, 2010).

Generalizing consists of the transition from the identification of regularity in specific cases to other broader cases that follow a same pattern (Polya, 1989) in a way that all the elements can be represented in a single expression (Radford, 2010). In the same line, generalizing as a process it may include identifying commonalities in elements, reasoning beyond the cases at hand, or obtaining a broader spectrum of results from particular cases. On the other hand generalization is regarded not only as a process but as a product or result of those processes (Stephens et al., 2017). When these descriptions are transferred to a situation with an underlying functional relationship between two quantities, generalizing entails identifying and representing the rule that relates the quantities from particular to more general cases. The relationship is generalized when it is represented by means of spoken language, graphs or other symbolic representations or combinations of them (Carraher et al., 2008). It is certainly the various types of representations for generalization and their inter-relationships constitute an essential element inherently associated with algebraic thinking and with generalization (Blanton, 2008; Kaput, 2008; Radford, 2018). In this research is differentiated that the emphasis is on the representation of the generalization of the functional relationship.

To characterize algebraic generalization, Radford (2001) defined three levels or layers that accommodate both symbolic and pre-symbolic factors. The first level, factual generalization, would be the "generalization of numerical actions in the form of an operational scheme that remains bound to the numerical level" (p. 82), in which particular cases can be successfully addressed. In the second level, contextual generalization, students would observe a pattern and explain it for any term within the sequence, with no need to use specific cases. The language used is studied as the reflection of a level of generality through indeterminate expressions such as 'the following term' or 'for any number'. The third level, symbolic generalization, would transcend the specific object or number in a given context; students speak in general terms. Their representation would embrace all the elements of the succession or pattern. Radford (2010) also distinguished a non-algebraic type of generalization: arithmetic generalization. In it, students would identify the common features of specific terms or elements but would be unable to apply them to other cases (even particular cases) or formulate an expression that represents all the terms in the sequence. Mason and Pimm (1984), in turn, differentiated between conceptions associated with generalization that could inform categories to represent generalization. First, they defined 'specific' to denote cases involving particular numbers and distinguished 'generic' as expressions that represent or encompass generality that is nonetheless explained with specific examples. Lastly, they used the term 'general' to refer to a reply that represents all the terms involved as well as indeterminate expressions.

Further to the postulates set out in the above articles, the participants in this study were encouraged to generalize by working with a situation that involved a functional relationship. The study was designed to gather information about their understanding of the functional situation proposed based on the tools or resources used to build, analyze and represent functions (Blanton, 2008). Based on the characterization of generalization, in the data analysis section, categories of representation of generalization were established on the grounds of how the functional relationship was recognized and represented (see Data analysis below).

Mediation

Classroom work and interaction provide teachers with insight into how students reason and learn. Both the tasks proposed and the communication mediated by educational actions are consequently essential to furthering reasoning. Teachers' mediation has a considerable impact on students' ability to generalize. Such educational action arise from and are necessary to achieve the teacher's motives associated with the intended class activities (Mata-Pereira & da Ponte, 2017).

The research reviewed to ascertain the role of teachers' mediation in generalization varied in nature. A number of these studies suggested that such teachers' actions can impact students' generalizations. Mata-Pereira and da Ponte (2017) and Ponte, Mata-Pereira, and Quaresma (2013) envisaged three types of educational action geared to furthering mathematical reasoning, deeming generalization to be one of its crucial processes. In the first type of action, informing-suggesting, teachers inform, suggest, argue, or validate students' performance. In the second, supporting-guiding, they guide students through tasks with questions or observations and invite them to furnish information. In the third, challenging, students are encouraged to move beyond prior knowledge, formulate new types of representation, interpret, interconnect, or put forward reasoning or assessments. Warren (2006) also described teaching actions that help students identify and express generalization both verbally and symbolically. In her study about functional thinking, the functional relationships were established between a pattern and the position of its elements. She reported that after teacher actions such as using specific materials, patterns with an obvious relationship between position and pattern, and asking explicit questions linking position and pattern, 20 % of fifth-graders were able to describe regularity verbally and symbolically. With educational mediation, students were also able to generalize and express the rules and equations involved.

Not all the literature reviewed contained explicit descriptions of teachers' actions, for some studies focused on other research issues. Given, however, that they potentially involved practices that might help students generalize, the actions involved were transferred to teachers' actions. For example, in a study on cooperative learning, Soller (2001) observed that students tended to learn more effectively in groups where they could interactively discuss and explain their reasoning while reflecting on their knowledge. Although the present study is unrelated to cooperative learning, it reinterprets the value of teachers' actions (including asking, explaining, justifying, motivating, accepting, rephrasing, suggesting, rejecting, discussing, and redirecting) and participant interaction. In this study the interviewer engaged in these actions as students performed the assigned task and described what they were thinking. In keeping with Soller's (2001) contributions, the interviewer gathered information about students' understanding of generalization and their skills, redirecting the process as necessary by posing specific questions. Other authors (Blanton & Kaput, 2004; Carraher et al., 2008) reported that educational mediation and actions such as paraphrasing, organizing, or redirecting students' responses or procedures played an important role in helping students acquire information and exhibit their functional thinking. In Soller's (2001) study and in research conducted by Mata-Pereira and da Ponte (2017), mediation was found to reinforce or redirect actions designed to enhance learning or the expression of the knowledge or skills mastered.

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Methodology

This descriptive study, which forms part of a research project on the algebraic capacities of elementary school students in functional contexts, pursued two objectives. The first was to describe the representations of generalization manifested by these students when performing a task that involved a functional relationship. The second was to analyze the relationship of mediation with their ability to represent the generalization as they worked through the task. Generalizing, as noted above, entails establishing a general rule applicable to all the elements of a statement, from the particular to the general.

Semi-structured interviews were conducted with eight fourth-graders (9- and 10-year- olds), intentionally selected to represent low, medium, and high levels of academic performance in mathematics, as assessed by their teacher.

The year before, these same students had participated in research designed to explore functional thinking (e.g., Molina, Ambrose, & Rio, 2018; Pinto, Cañadas, Moreno, & Castro, 2016). That was their first experience with functions and the use of symbolic-algebraic and tabular representations. Since the students were consequently familiar with the research team and had had some prior experience with tasks involving generalization, they may have felt more comfortable about expressing their thoughts.

In that prior study 25 students had participated in four (approximately 90-minute) classroom sessions with no previous instruction into functional thinking. They were handed one worksheet per session and allowed to work either individually or in small groups of three or four. After they completed the worksheets, the answers were discussed by the group as a whole. The tasks were arranged according to an inductive design, with particular instances leading to the general case in situations involving a functional relationship. All the tasks entailed determining the value of the dependent variable given the value defined for the independent variable, although in some of the questions the reasoning was reversed. Letters were introduced to represent indeterminate quantities.

For example, in the first session students were told that 'María and Raúl are siblings who live in La Zubia. We know that María is 5 years older than Raúl'. The functional relationship between María's and Raúl's ages is $f(x)= x + 5$. The students were asked to calculate values for one variable given values of the other and then to complete a table with the information, specifying the operation performed to determine each result. In one question the use of a letter was introduced to designate Raúl's age, from which the students were asked to find María's.

Interview design

The interview, structured around a situation involving a functional relationship, was designed to help students transition to a general formulation from particular cases. The situation described was: 'My family and I love to ski. While we ski, we park the car in the parking lot. The parking lot has a €2 admission fee and charges €1 per full hour the car is parked.' In the respective functional relationship, the number of hours (the independent variable) that the car is in the parking lot is associated with the cost in number of euros (the dependent variable).

In the interview students were asked to determine the number of euros that had to be paid in the parking lot according to the time the car was there, following the structure: first seven questions involved specific number of hours (such as 3, 5, 10, 20, 120, 500). In question eight was introduced an indeterminate quantity of hours as 'the number of hours, whatever many'. Final question involved another indeterminate quantity represented by a letter as the number of hours the car was parked.

The first seven questions involved particular cases. The expectation was that after working with specific quantities, students would identify and represent numerically the underlying functional relationship. The first five questions involved natural numbers under 100, and the sixth and seventh numbers greater than 100. The last two questions were designed to guide students to formulate the functional relationship generally, first verbally or using generic examples, and then with algebraic symbols. At the end, proposing algebraic symbology was understood to mean using letters to represent indeterminate quantities with which to succinctly express functional relationships (Stephens et al., 2017).

The interview was semi-structured. The interviewer supplemented the questions of the interview with others aiming to make students' thinking explicit and to explore their generalization skills.

Data analysis

Video recordings taken of the interviews were subsequently transcribed and coded. The transcripts were used to analyze both the interviewer's mediation (essentially coaching by asking questions) and the evidence of representation of generalization in the students' replies. The categories used in the analysis were informed by the studies cited in the theoretical discussion and an initial review of the data collected. Two units of analysis were defined. The interviewer's mediation given by each interviewer action to further generalization, different of posing the questions of the interview, and students' full response to each interviewer action, specifically to identify evidences of representation of generalization. Two groups of categories were defined to analyze the information: the representations of generalization of the functional relationship exhibited by the students, and the interviewer's mediation.

Representation of generalization of functional relationships

The representations of generalization were categorized on the grounds of the characteristics defined by Mason and Pimm (1984) and Radford (2001, 2010). Here students were acknowledged to have represented generalization at a given category when they consistently represented the functional relationship according the cases from the particular to the general, involved in questions. The following four categories of representation of generalization were defined, based on how the students recognized and represented the functional relationship underlying the task.

Numerical (N): working with different specific amounts, the student recognized the functional relationship underlying the task and expressed it by referring to those amounts. The student consistently answered the questions using the operation inherent in the functional relationship. This category was divided into two, depending on whether the student referred to numbers lower (N1) or higher (N2) than 100. Although at a numerical representation of generalization it could be understood that the student has not generalized, he has perceived a generality in a set of particular cases. Numerical category is associated with that Radford's (2001) factual generalization and Mason and Pimm (1984) conception of specific. In the following example student A1 recognized the relationship between a specific number of hours and the cost in euros.

> Interviewer [I]: Exactly: you leave the car in the parking lot for 20 hours. How much would you have to pay?
>
> A1: Well 20, if you have to pay every hour, and we have 20 euros, well 20 euros plus the 2 euros admission fee, 22, 22 euros.

Generic (G): the student recognized the functional relationship in general cases and expressed it with an example involving specific numbers used as generic examples. The specific example was drawn not from the questions of the interview, but from the student's own thought process. This category of representation of generalization is based on the characterization of generic example proposed by Mason and Pimm (1984). To exemplify, student A2, exhibited this representation of generalization.

> I: Imagine you don't know exactly how many hours we left the car in the parking lot. I'm going to write "the number of hours", however many. How have you been figuring out how much you need to pay?
>
> A2: For example 50, one for each, that makes 50, then 50 plus 2 that I had to pay when we drove in, that comes to 52.

Verbal (V): the student recognized the functional relationship and expressed it verbally in general terms, alluding to indeterminate quantities. Verbal category is related with Radford's (2001) contextual generalization, and the idea of general (Mason and Pimm, 1984) expressed in form of expressions of indeterminacy. In the following example, student A3 identified the relationship between the number of hours in the parking lot and the number of euros to be paid and used the phrase 'whatever that is' to refer to the former.

> I: Here's just any number, how would you find the number of euros?
>
> A3: Don't I have the time there?
>
> I: Yes, but the problem is we don't know the number of hours we left the car.
>
> A3: Well the time that's clocked.
>
> I: The time that's clocked: and what do you do with the time that's clocked?

A3: Well, you add 2 to whatever that is [referring to the number of hours]

Symbolic (S): the student recognized the functional relationship and expressed it with algebraic symbols. This category included students who while not representing the functional relationship symbolically on their own, understood the expression when proposed by the interviewer. Symbolic category is based on what Radford (2001) also calls symbolic generalization and with the conception of general given by Mason and Pimm (1984). In the following excerpt, the student wrote down the symbolic expression representing the total amount of euros after the function of the letter *z* had been explained to him.

I: OK and if we say *z* hours, how would you write it? Can you think of a way?

A6: I don't know what *z* hours is.

I: It's like *x*, the other number we don't know. That's always the way with letters, we don't know exactly what number they are, because they can be any number.

A6: Well maybe I could write *z*+2, as the number of hours we're there plus 2 for admission.

I: And if I say it was *n* hours? How would you find it then?

A6: Well *n*+2.

Mediation

Mediation is described and analyzed in terms of the interviewer's actions. It was classified in three groups according to the educational actions described by Mata-Pereira and da Ponte (2017): informing-suggesting, supporting-guiding, and challenging. The system of classification was further broken down into the interviewer's actions to favor mathematical reasoning towards the generalization and the representation of it. Such actions were inspired in the contributions of Soller (2001), Mata-Pereira and da Ponte (2017) and Ponte et al. (2013). The specific categories of interviewer's actions defined here are: reaffirming, process suggesting, correcting, changing course, repeating information and clarifying, as follows.

Informing-suggesting (Mata-Pereira & da Ponte, 2017)

Reaffirming (M1): the interviewer approves the procedure used by the student and on occasion asks them to explain it. Even when the reply is not correct, students are encouraged to persist in or explain their reasoning about the task. In the following fragment, the interviewer verified and confirmed the functional relationship established by student A1 when working with a specific number of hours. This action informs and provides a possible invitation to continue and strengthen the same reasoning.

> A1: Well 20, if you have to pay every hour, and we have 20 euros, well 20 euros plus the 2 euros admission fee, 22, 22 euros.
>
> I: Okay, fine, fine. Very good, you can't be fooled easily. [M1]

Process suggesting (M2): the interviewer suggests or insinuates a process, or rather the search of a procedure to find the answer, sometimes alluding to procedures used previously and encouraging discussion and information gathering. In the following segment the student was prompted to reason about the procedure he had been following and consequently explain how he established the functional relationship.

> A6: 1002 euros.

I: How do you know that, A6? [M2]

A6: Well, because every hour costs one euro, then you tell me for example 1000 so you add 2.

Challenging (Mata-Pereira & da Ponte, 2017)

Correcting (M3): designed to identify and correct mistakes. By asking students to confirm the results, the interviewer prompts recognition of their mistakes. The mediation below followed an error made by student A1 when working with specific numbers. Mediation helped the student check and correct his response.

I: There's something I don't quite understand here: your answer says that one hour costs one euro, but if you're there for just one hour, how much would you have to pay? [M3]

A1: 3 euros.

Changing course (M4): the purpose is to guide and support students in their progress and obtain answers other than those initially found. The interviewer paraphrases, re-formulates her observations or questions, or redirects the information acquisition process. This type of mediation is exemplified in the following fragment. When the letter *x* was introduced to student A6 he explained the functional relationship verbally and with specific examples. The interviewer changed course to prompt the student to explain the use of the letter when she introduced the expression *x+2*. The student ultimately exhibited symbolic generalization (see the results section below).

I: Could we write it like this? If I tell you how many euros it will cost and I write it like that, x + 2, what do you think? Is that possible? Is it different from what you said? [M4]

A6: Let's see, yes, it's possible because x is a number, whatever number, because if you add 2 you get what you have to pay.

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I: And could you write it like that? x + 2?

A6: Yes.

Supporting-guiding (Mata-Pereira & da Ponte, 2017)

Repeating information (M5): the interviewer repeats data or refers to previous results. In the following excerpt, after student A3 verbally generalized when introduced to the letter *x*, the interviewer repeated the meaning of the letter.

I: And how would you write that using the x, can you think of a way?

A3: [shakes his head no]

I: If x is the number of hours how would you do it? [M5]

Clarifying (M6): the interviewer clarifies her mediation or introduces new information during the interview. In the last question the interviewer frequently resorted to mediation type M6 to explain the use of letters to represent indeterminate quantities. This type of mediation is illustrated, for example, in the following description of symbolic generalization.

A6: I don't know what z hours is.

I: It's like x, the other number we don't know. That's always the way with letters, we don't know exactly what numbers they are, because they can be any number. [M6]

Results

The most prominent findings are set out and discussed in this section. Table 1 lists the number of students exhibiting each representation of generalization by question. The first value denotes the number who did so without coaching when the respective questions are posed and the second, in parenthesis, the total that exhibited the category. For example, in question eight the value 2(5) under the verbal representation column means that five students in total exhibited verbal representation of generalization, two of them without prompting, that means that the other three did so in relation with the interviewer's mediation. In all categories of representation of generalization, when the relationship of these with the interviewer's mediation is considered, the total of students representing generalization increases as can be appreciated in the parenthesis of Table 1.

[Table 1]

All the students recognized the functional relationship underlying the task. The representation of generalization observed in the responses was in keeping with expectations in all but the last question, where it was less sophisticated than expected.

Table 2 lists the types of mediation related with students recognizing and representing generalization for the first time during the interview. Be it said that was not the only mediation related to each representation of generalization and to each student, as will be appreciated in next sections. For example, in some cases a student may have manifested a specific representation of generalization without strict relation with interviewer's mediation and at other time manifests the same representation but related with mediation.

[Table 2]

Mediation was observed be related to the first-time recognition of generalization, particularly on the generic, verbal and symbolic representations. In numerical representation, the main mediation associated was M3 (correcting), whereas M2 (process suggesting) and M4 (changing course) were connected with generic and verbal representations. The symbolic representation of generalization was associated with mediation types M2, M4, and M5 (repeating information).

Representations of generalization and relation with mediation

The most significant findings for each representation of generalization are discussed below, including the way students expressed generalization, the relationship with the interviewer's mediation on their ability to do so, and the specific questions involved.

Numerical

All students exhibited this representation of generalization in the first questions (see Table 1). Five students generalized without mediation, whereas the other three needed support, making mistakes even in tasks with numbers under 100. In this case, mediation M3 was associated with the correction of their mistakes.

Seven of the eight students exhibiting numerical representation generalization for numbers under 100 required no prompting to express that category for higher numbers. The exception, student A4, failed to correctly identify the operations involved in answering the particular case questions. Depending on the question, A4 either counted mentally, summed over and over, or resorted recursively to earlier results to find the new ones. Using these strategies he occasionally found the right answer but failed to recognize the functional relationship, even when coached. He was more concerned with answering the questions (most incorrectly) than identifying a pattern that would have enabled him to find the result efficiently. In the following excerpt he can be seen to change his answer to one even farther removed from the underlying relationship.

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> I: Imagine you go [skiing] for a day and you spend the whole day there. So of course you leave the car in the lot all day and it's there for 10 hours. How much would you need to pay?
>
> A4: Well..., 10 hours, 5+1 equals 6; 6+1, 7; 7+1, 8; 8+1, 9; 9+1, 10, 10+1, 11: it would be 21.

In very specific cases, with numbers such as 1000 or 1,000,000, he calculated the result in euros correctly as the number of hours plus 2. In those cases, mediation categories M3 (correcting) and M4 (changing course) helped him find the right answers.

Challenging-type mediation was helpful at this representation of generalization, particularly to correct mistakes. The findings revealed that mediation M1 (reaffirming), in which students were reassured that their answers were correct, enhanced the fluency of the interview.

The three students who expressed this category of generalization in question nine needed to resort to specific values to verbalize the functional relationship when the use of letters was proposed. Two of these students (A2 and A8) initially assigned a unique value to the letters, based either on their position in the alphabet (making *n*=13) or their meaning as Roman numerals ('Since *x* hours in Roman numerals is 10, then it would be 12'). After coaching in the form of M4 (changing course) and M2 (process suggesting), student A2 generalized generically. The use of letters was not an obstacle for A8, who when coached was able to use symbols (when one letter was replaced with another).

After setting up the table shown in Figure 1 with the interviewer's help, the third student, A5, recognized the relationship 'number of hours +2' by examining the operations shown in parentheses. She identified the functional relationship, although when including the letters *n* and *x* she explained that she would need to check them against the actual number of hours (e.g., by viewing the security camera videos).

[Figure 1]

Generic

Six students exhibited this representation of generalization, only one without prompting (see Table 2). It was observed in the last two questions, when students were asked to calculate the number of euros that 'whatever number of hours' would cost or to use a letter representing that idea.

Student A2 exhibited this representation in question 8 with no need for coaching (see Table 1), generalizing the functional relationship with specific numbers (not used in previous questions).

> I: Right. I'm going to jot something down here. Imagine you don't know exactly how many hours we left the car in the parking lot. I'm going to write "the number of hours," however many. How have you been figuring out how much you need to pay?
>
> A2: For example 50, one for each, that makes 50, then 50 plus 2 that I had to pay when we drove in, that comes to 52.

Process suggesting (M2) and changing course (M4) were the main types of mediation associated with this representation of generalization. The former encouraged students to reason about how they would calculate the cost in number of euros for any given number of hours. With some students changing course (M4) was exemplified by an exercise in which they were asked to make a sign showing the parking lot rates. The idea was to prompt them to use a more general explanation than they had provided until then.

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Student A1, for instance, resorted to particular cases to explain his interpretation, using generic representation of generalization. Even when he was asked a new question about parking lot rates involving the use of a letter to represent the number of hours, his answer was generic rather than symbolic.

> I: I'm going to write something down here. Imagine that we don't know how many hours we're going to leave the car in the lot. We know it's going to be quite a few and we're going to call it *n*: *n* is the number of hours. How would you figure out how much you need to pay for *n* hours?
>
> A1: What number is *n*? Well I don't know, if it's 30 hours you have to pay what the parking lot says.
>
> I: And if for example we write here "it's *n* hours". How would you know how much it will cost?
>
> A1: Like before, the number of hours you stay.
>
> I: OK... we're going to call this the number of hours. How would you find the number of euros?
>
> A1: Well one hour one euro and then adding, adding and then 2 for the admission fee and if I was there for 30 hours, well 32.

After mediation, some students moved between the verbal and generic representations of generalization. A5, for instance, failed to understand the notion of an unknown number of hours. When asked how to calculate the cost if she did know the time, she ultimately recognized the functional relationship '2 plus number of hours'.

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I: *x* hours, we don't know how many, but *x*. Then I tell you that I know there were *x* hours. How much would you need to pay?

A5: Well I don't know. I could review the security videos and see how long the car was there.

I: And when you know the number of hours you were there, how would you find the amount it would cost?

A5: Well one euro per hour plus 2 always. If it were one million hours it would be one million euros plus 2.

Verbal

Seven students represented generalization verbally, two with no specific relation with mediation. This representation of generalization was observed almost exclusively in the last two questions of the interview, where the number of hours was indeterminate. The sole exception is described below.

In the particular case questions, A5 represented verbally. The following excerpt illustrates her answer to the question on the cost for 120 hours:

I: What have you done up to now, A5?

A5: Because when you come in, you told me that when I come into the parking lot you have to pay 2 euros and then after 1 hour you have to pay another euro and another euro and another euro, and you add 2 euros to each number [referring to the number of hours].

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Despite this expression of a verbal generalization in the particular case questions, however, she later refused to accept the indeterminate 'any number of hours' and the letter representing that idea.

Two students represented generalizations verbally without specific coaching in question eight, whereas the others were supported to express their ideas or understand the questions.

Process suggesting (M2) was the type of mediation that most frequently occurred with verbal representation of generalization. According Table 2, four of seven students who represented for first time verbally were coached by M2. Students were invited to explain and reassert their reasoning and adopt a position, inasmuch as they were asked about the procedure they used or would use or prompted to organize their ideas to define what they could do.

After the interviewer repeated information (M5) and suggested a process (M2), student A3 progressed from not knowing what to answer to represent the generalization verbally, replying that she would add 2 to the number of hours clocked.

Mediation had a visible relation with the transition from generic to verbal generalization in the last two questions. Asked to generalize how much 'a certain number of hours' would cost, A6 gave a specific example (183 hours), noting that the number would then have to be increased by 2 (a generic generalization). The question was rephrased (M4), asking the student how he would explain the parking lot rates to someone else. Although initially confused, he ultimately represented generalization verbally, contending that 'Well, the number of hours you're there counts for one euro but then you have to add 2 because that's what the parking lot necessarily charges to let you in.'

Student A8, after explaining 'then I add the hours you've been there plus 2', went on to use specific numbers after M2-type coaching (asking her how she would calculate how much the 'number of hours' would cost). This student moved from a verbal to a generic categories of representation of generalization.

Interestingly, in last two questions, five of the seven students who represented generalization verbally also did it generically.

Symbolic

In their explanations, three (A3, A7, and A8) of the four students who represented generalization symbolically exhibited an understanding of the use of letters to represent the underlying functional relationship when given the algebraic expression. The fourth student (A6) used the symbolic notation suggested to write out the functional relationship on his own. All four represented generalization symbolically in the last question. The letters introduced to express indeterminate quantities were, in most cases, n, x, y, or z.

After having expressed the functional relationship verbally in the preceding question, A7 explained that it was the same as before 'because it would be those x hours, like here [pointing to the preceding results], except that instead of a number it would be x.' When asked how to write it out, the student replied 'you add the 2 euros admission fee to x and you get the answer.' While she failed to write down the expression x+2, she accepted it as valid when it was proposed by the interviewer, replying that it was the same but 'summarized in numbers and x's.'

Symbolic representation of generalization was associated with mediation types M2 (process suggesting), M4 (changing course) and M5 (repeating information). In M2 the interviewer suggested that students should find a way to calculate the result using the letter proposed. With M4 students' replies were redirected, proposing the expression x+2, for instance, and asking whether it meant the same as what they were saying. In another instance of this type of mediation students were asked to formulate the expression for the amount to be paid after the letter representing the hours was changed from *n* to *x* (or others). In mediation M5 the interviewer rephrased the meaning of the letter or its role in the situation.

Student A3, after exhibiting verbal representation of generalization in question eight, was asked to calculate how much the parking lot would cost if the number of hours was *x* (question nine). A3 based her reply on his previous answer 'Well, I did that here. Because you add 2 to whatever the amount of time is'. In other words, the student understood that regardless of the representation used for 'whatever number of hours,' it sufficed to add 2. While claiming not to know how to express that idea, A3 recognized the expression *x+2* when written by the interviewer. When asked about changing *x* to *n*, the student explained that 'then it would be *n* plus 2.'

When introduced to the letter *x* and asked about the amount to be paid, A6 replied 'well if you have *x* hours and you add 2, all the hours you're there you add 2. For example, if you're there for 10 hours well you add 2 and if you're there for 3, you add 2'. When showed the expression *x+2*, the student accepted it as a valid representation of the amount to be paid. After some initial confusion about calculating the cost for *z* and then for *n* hours, A6 realized that the cost would be *n+2* or *z+2*.

As noted earlier, student A8 associated *x* with the Roman numeral, although the student subsequently generalized verbally. The interviewer then suggested using the letter *z* and repeated the question about calculating the cost. At that point the student replied 'the number that *z* is, plus 2.' When then asked whether the expression *z*+2 meant anything, A8 claimed that it did because *z* could be any number, plus 2.

None of the other students generalized with symbols. Three attributed no meaning whatsoever to the use of letters or refused to combine them with numerical operations. In contrast, A4 (who had difficulties expressing the idea), when asked how to calculate the result with the letter *n*, replied that it should be regarded as a number and it would be '*n* 2'. That was interpreted to mean that he was referring to the appearance of the result: for numbers with units ranging from 0 to 7, adding 2 would not change the ten's column, so he represented the result of the addition using *n* as the digit for the tens and a number for the units (*n*1, *n*2, *n*3, *n*4, *n*5,..., *n*9).

Discussion

According to the above findings, all eight students interviewed recognized and represented the functional relationship underlying the task and were able to express that understanding when justifying their answers. They all represented generalization numerically in first seven questions with particular cases, although some were challenged (M3 and M4) and corrected their mistakes or adopted another approach. Students recognized the functional relationship with less difficulty as they continued working with particular cases, once students were able to represent generalization numerically with numbers under 100, they found no difficulty in representing with larger numbers.

Although seven of the eight students generalized verbally, all but one was prompted (primarily via process suggesting) and do so. Whether students represented verbally or generically the generalization was related with the interviewer's mediation in most cases. For example, five of six students who represented for first time the generalization generically received mediation by the interviewer, and five of seven students in the case of verbal representation (see Table 2). In last two questions five students exhibited both, an indication that neither representation of generalization prevailed over the other when indeterminate quantities were involved. As Radford (2018) noted, while students may recognize variables and the functional relationship between them, their ability to verbally express those concepts is limited. Nonetheless, they were observed to draw from other

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resources (e.g., expressions for indeterminate values or generic examples), revealing their ability to think algebraically and express functional relationships without algebraic symbolism.

Symbolic representation of generalization was observed in four students who understood the use of letters and algebraic statements to express functional relationships although they did not themselves use symbolic language. With one exception, these students were challenged (M4, changing course), informed (M2, process suggesting), or supported (M5, repeating information) to prompt this representation of generalization. The other four students experienced difficulties in understanding the role of letters or even refused to use them. These findings are consistent with the results reported by Molina et al. (2018) in an earlier study on the use of letters by the same participants a year earlier: some students tended to assign fixed values to letters in keeping with their position in the alphabet or failed to regard them as indeterminate quantities. Those authors reported the same students' initial reactions as third-graders, when the use of letters was proposed to represent indeterminate amounts involved in the functional relationship $y=x+5$. The present findings confirmed the persistence of certain interpretations of letters, as would be expected given these students' scant prior experience with the use of letters in mathematical contexts. In other studies (Blanton et al., 2015; Carraher et al., 2008), participants younger than the students interviewed here proved able to use letters. Those younger students had received prior instruction, however. In the present study, despite being aided by specific coaching, only half of the interviewees grasped the notion of algebraic symbology in functional relationships.

In the last two questions, which involved general cases, students interacted more dynamically with the interviewer and used more resources (such as particular cases or the verbal expression of indeterminate values) to express their ideas and the functional relationship more generally. Other semiotic resources (Radford, 2018), such as numerical and verbal expression, were also highly useful for assessing indications of algebraic thinking in students. In another vein, some students represented generalization at a given category in one question and then required coaching in another for the same representation of generalization, a finding that could be partly attributed to the fact that the situation proposed in each question varied and posed new challenges.

When working with particular cases students required significantly less effort than when working with expressions involving indeterminate values. Some of their difficulties were found to be due to their lack of experience with algebraic symbolism and the expressive resources needed to formulate their ideas. One student represented the functional relationship symbolically and others explained the meaning of the expression representing that relationship when suggested to them. As contended by Blanton et al. (2015), students' use of different types of representation to make sense of functional relationships can be regarded as a means to guide their thinking, as well as a source of information about their understanding of such functions (Blanton, 2008).

Mediation encouraged students to reply and strengthen ideas and constructs associated with the recognition and representation of functional relationships. It also enabled transitions between representations of generalization. Interviewer's mediation was found to be valuable in helping students express their algebraic skills in the context of generalization. Coaching associated with informing-suggesting and challenging was particularly useful. Mediation M2 (process suggesting) was associated with contributing students to consolidate their ideas about the functional relationship and the operation involved. M4 (changing course) encouraged them to focus their intellectual effort on identifying, defining, and expressing the functional relationship. It guided their thinking to the representation of generalization of the functional relationship, and provided options to reason more effectively, while putting forward explanations about the existing relationship between the variables.

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Conclusions

The present findings, based on a study conducted in Spain, support earlier reports (Blanton et al., 2015; Blanton & Kaput, 2011; Carraher et al., 2008; Warren, 2006) to the effect that elementary school students are vested with the mathematical skills and algebraic aptitudes needed to generalize functional relationships. Performing a task involving a functional relationship students showed be capable to recognize and represent in different ways the generalization of the functional relationship.

The study highlights the role of functions, especially functional thinking, as a way to introduce students to algebraic reasoning and enable them to develop the associated skills (Blanton & Kaput, 2011), such as generalizing relationships between quantities and representing and dealing with indeterminate quantities in different situations.

Its primary contribution lies in a proposal to categorize specific representations of generalization and types of interviewer mediation in tasks involving functional relationships. Implementation of these categories afforded evidence of elementary school students' ability to generalize in functional contexts.

The findings also supplement existing knowledge. Blanton et al. (2015) characterized levels of sophistication in children's thinking in functional contexts in a teaching and learning environment. Radford (2001, 2010) defined layers of algebraic generalization in pattern tasks in terms of the recognition of commonalities and their progressively more general representation. In the present study have been identified representations of generalization based on recognition of the underlying functional relationship in a task and how it is expressed, in the absence of prior instruction. It also gives categories of mediation to favor reasoning and further students' ability to generalize, from the results reported by Mata-Pereira and da Ponte (2017), Ponte et al. (2013) and Soller (2001).

This study provides insight into how mediation can favor students representing generalizations at different categories. That in turn provides useful information for task design and suggests ways in which teachers can encourage students' active participation and help them develop functional thinking. Such findings corroborate and supplement the actions described by Warren (2006) to foster the ability to generalize. Teacher mediation can spur student progress toward that goal. For example, mediation in the form of change of course may include actions such as guiding students' answers in a new direction, paraphrasing questions, proposing different types of representations (such as illustrations), suggesting the use of manipulative materials or inventing hypothetical situations that call for explanations and arguments that demonstrate the understanding of the functional relationship.

The present findings support some of the results reported by Blanton (2008) and Blanton and Kaput (2004), who showed that children can work with and describe situations involving functional thinking as well as relationships between covariate amounts. They observed progress and sophistication in the use of mathematical representations from grade to grade. In this study transitions among representations of generalization were observed in a single grade during just one task, with students advancing from the verbal expression of indeterminate values to the understanding and use of symbols to describe and express a functional relationship.

This study suggests another research question: how would elementary school students generalize and progress in formulating representations of generalization over several sessions in a classroom environment? Exploring that issue would enhance also the understanding of the relation of mediation described here with the generalization, transferred to teacher's mediation in a context of learning.

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Table 1

Number of students representing generalization, with and without mediations

	Representation of generalization				
Question	N1	N2	G	V	S
1-5	5(8)				
6-7		7(8)		0(1)	
8			1(4)	2(5)	
9	1(3)		2(5)	2(4)	1(4)

Note: The value preceding the parentheses specifies the number of students expressing each representation without relation with mediation and the value in parentheses the total number expressing that representation. The shading indicates the representation of generalization expected in light of the type of question.

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Table 2

Types of mediation prompting the expression of each representation of generalization for first time during the interview

	Representation of generalization				
Student	N1	N2	G	V	S
A1	U	U	M4	M2, M6	
A2	U	U	U		
A3	M3, M5	U	M5, M6	M5, M2	M5, M2
A4	M4, M1	M3, M4		S	
A5	U	U	M4, M5, M2	M4, M5, M2	
A6	U	U	M2	M4	M4, M5
A7	U	U		M2	U
A8	M3	U	M2	U	M4, M2
Total*	5(8)	7(8)	1(6)	2(7)	1(4)

Note: U= exhibited uncoached; M1=reaffirming; M2=process suggesting; M3=correcting; M4=changing course; M5=repeating information; M6=clarifying. * The value preceding the parenthesis specifies the number of students expressing the representation of generalization without mediation (regardless of the question) and the value in the parenthesis the total number of students who represented in each category. A blank cell means that the student failed to represent generalization even after the interviewer's mediation.

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Figure legends

Figure 1: Student A5's summary table.

The heading of the first column translates as "Number of hours". The word "comprobar" to the right of letter N in euros column translates as "check".

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Figure 1