TITLE: Word Problems Associated With the Use of Functional Strategies Among Grade 4

Students

Abstract

This article discusses the characteristics of word problems that are associated with students' use

of functional strategies and their ability to represent the generalization of functions. In the context

of a broader research project designed to explore and foster functional thinking among elementary

school students, twenty-five grade 4 (9- to 10-year-old) students were asked to identify functional

relationships in five problems involving specific or indeterminate quantities. Their responses to a

number of questions involving the generalization of the relationships in the problems were

analyzed and associated to the characteristics of the problems. The type of representation of

generalization used (verbal, generic, or symbolic) were also identified. Our findings indicate that

grade 4 students showed potential for functional thinking prior to receiving instruction on variables

and their notation. Such thinking was most effectively prompted when they worked with word

problems that explicitly involved an additive function. When students generalized functional

relationships, they represented them verbally or with generic examples. None of the students used

symbolic representation. The originality of this study lies in the description of the specific

characteristics of word problems that are associated with functional thinking; this information will

prove useful to both teachers and curriculum designers. Identifying these characteristics could help

build and propose tasks that encourage students to use more than one and more sophisticated

strategies.

Keywords: Early algebra, Functional thinking, Generalization, Representations, Word problems

Introduction

One of the goals pursued in today's mathematics education research is to establish the connection between arithmetic and algebraic thinking in the early grades (Warren, Trigueros and Ursini 2016). According to the literature, understanding patterns, relationships, and functions in different contexts is critical to algebraic thinking (Blanton and Kaput 2011; Kieran, Pang, Schifter and Ng 2016; Lee, Ng and Bull 2018). The study reported in this paper was designed to explore functional thinking among grade 4 students who had no prior instruction on the use of variable notation and in early algebra (specifically, early algebra that included problems focused on everyday contexts). Functional thinking includes generalizing relationships between covariant quantities; expressing those relationships in words, symbols, tables, or graphs; and using representations to analyze functions (Blanton, Levi, Crites, Dougherty and Zbiek 2011). Functional thinking provides a fertile background to develop algebraic thinking practices such as generalizing and representing relationships between quantities (Blanton et al. 2011; Carraher and Schliemann 2007; Other, Author and Other 2016). Thus, generalization plays a fundamental role in functional thinking (Kaput 2008) and is deemed to be a key to acquire mathematical knowledge from the youngest ages (Mason 2008, 2018; Pólya 1945). The two research questions we explored in this study were:

What characteristics of word problems are associated with students' use of functional strategies?

How do students who recognize functional relationships represent the generalizations?

One of the reasons for conducting this study is related to the incorporation of algebra into primary school curricula. Curricular documents emphasize the advantages of developing algebraic thinking in the early grades (Cai and Howson 2012). According to *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics 2000), students from pre-kindergarten to grade 12 should understand patterns, relationships, and functions and be able to represent and analyze mathematical situations and structures using algebraic symbols; use models to represent and understand quantitative relationships; and analyze change in different contexts (Blanton et al. 2011). In Spain, where this study was conducted, curricular objectives for all students finishing elementary school include the ability to describe and analyze changing situations; recognize patterns, regularities, and mathematical laws in numerical, geometric, and functional contexts; and evaluate their predictive utility. Curricular content at these grades also addresses the development of numerical understanding and algebraic symbolization (Ministerio de Educación, Cultura y Deporte 2014).

Another reason to research functional thinking in elementary school is that it contributes to the construction of a sound learning base for later, more sophisticated studies in algebra (Others, Author and Other 2018). Once students identify relationships in functional situations, representing them through different means (e.g., tables, graphs, drawings, and equations or algebraic expressions) may help them structure and broaden their understanding of functions (Blanton and Kaput 2011; Author and other 2008). Understanding functions and gradually using algebraic symbolism may help prevent or mitigate the difficulties observed in later grades in students' use of mathematical symbols to express relationships between quantities (Bednarz 2001; Carraher and Schliemann 2007). Students from the earliest grades can generalize and reason with functional

relationships in numerical contexts and even use variable notation (Author and Other 2008; Blanton and Kaput 2011; Other, Author and Others 2015).

Literature Review

In connection with the first research question, the strategy students use when working on tasks plays a significant role when generalizing functional relationships (Merino, Cañadas and Molina 2013; Moss and Beatty 2006). In addition, there are different factors that influence the selection of algebraic generalization strategies used by students (Lannin, Barker and Townsend 2006; Mulligan and Mitchelmore 2009). Prior research provides evidence that grade 4 students can use covariational strategies (Stephens, Isler, Marum, Blanton, Knuth and Gardiner, 2012; Warren 2005; Warren and Cooper 2008). In earlier grades, students use additive strategies such as counting (Blanton and Kaput 2004; Cañadas and Fuentes 2015; Merino et al. 2013) or multiplicative strategies such as duplication (Other, Author and Other 2016). Furthermore, which strategies are used may be conditioned by certain characteristics of the tasks proposed, such as the situations that are described in the word problems (Earnest 2014; Larsson and Pettersson 2015), the magnitude of the values involved (Lannin et al. 2006; Other et al. 2016; Pinto, Cañadas, Moreno and Castro, 2016), and the way in which the task is presented (Pinto and Cañadas, 2018; Stephens, Blanton, Knuth, Isler and Gardiner 2015; Stylianides 2016).

In terms of the second research question, teaching has a significant impact on students' understandings of functions, especially regarding their understandings of variable notation. Some studies (e.g., MacGregor and Stacey 1995) have reported that students understood functional relationships but could not express them clearly. Research reveals significant differences between control groups and groups of students who received early algebra instruction (Other et al. 2015;

Pang and Kim 2018). Of course, algebraic thinking may be present in the early grades even though the use of letter-symbolic algebra is not (Kieran 2006) and students may also verbalize generalizations (Mason 1996; Radford 2006, 2011). However, Kaput (2008) contends that increasingly systematic generalization in a conventional system of symbols is a key factor in algebraic thinking. One approach to instruction on the use of variable notation may begin with intuitive representations, followed by the gradual adoption of conventional representations, including letters to represent variables, along with tools to represent and understand mathematical relationships such as the use of tables and Cartesian coordinate graphs (Carraher, Martínez and Schliemann 2008). In Spain, grade 3 students have been observed to understand variability in connection with word problems involving functions (Author and other 2019a). Nonetheless, in Author and other's (2019a) study, when students expressed how two quantities co-varied they did so in non-conventional ways, which differed depending on their own prior knowledge. More specifically, their use of letters depended on the meanings they associated with them; for example, some associated the value of a letter with its position in the alphabet.

Studies comparing elementary school students of different ages identify grade 4 as a pivotal period for students to identify relationships between variables and show that the role of teaching is instrumental. In the absence of early algebra instruction, the ability to generate symbolic rules differs between grade 3 and 4 students (McEldoon and Rittle-Johnson 2010), while certain task characteristics such as the type of underlying functional relationship affect their performance (Other, Author, Others 2017). Differences between grade 4 students who have received early algebra instruction or not are greatest in tasks involving far generalization and the use of variable notation (Carraher and Schliemann 2007). In studies on pattern generalization in

which students had not received any specific prior instruction, grade 4 students engaged in far

generalization (Stacey 1989; i.e., tasks that require calculating the value of f(n) for a "large" n)

and expressed the general rule verbally more effectively than grade 3 students, although very few

grade 4 students expressed the general rule algebraically (Zapatera 2018). In studies in which

students received instruction on pattern generalization in terms of positional language and on

extending children's language and thinking to describe and predict patterns, they were able to

express the relationship between two data sets using algebraic symbols such as $n + 1$ (Warren

2005). With no specific instruction in early algebra, the grade-to-grade findings from one

longitudinal analysis also revealed differences between grades 3 and 4 and an indirect relationship

between arithmetic acumen and the tasks involved in establishing relationships between quantities

(Lee et al. 2018). These authors interpreted that indirect relationship to mean that children who are

good at arithmetic tend to over-rely on those skills to solve relational tasks.

Grade 4 has also stood out as relevant in the area of representations of generalizations. In

a problem-solving study with grades 2 to 6, Radford (2018) noted that students used alphanumeric

symbols as well as non-conventional semiotic systems. In his study, students did not use letters

before grade 4. At this grade, they produced alphanumeric formulas, however they did not consider

expressions such as "$x + 3$" as an answer and tended to calculate results. The operations had to be

reconceptualized for students to readily understand the expressions in which they were used. In

later grades students were able to express the general formula with less difficulty and even use

brackets within expressions appropriately. In another study with no instruction in early algebra

where grade 4 students inductively generalized functional relationships they proved able to

represent generalizations verbally and generically and even symbolically if aided by the

interviewer (Other, Author and Other 2019). The identification by other authors of similar achievements among much younger children reinforces the importance of the role of instruction in the use of variable notation (Author and Other 2008; Blanton and Kaput 2011; Other et al. 2015).

The literature reviewed confirms the role of teaching in grade 4 students' ability to represent functional relationships, especially the use of symbols to represent variables. In spite of the fact that research in the area of functional thinking is by now quite broad, no conclusive evidence has been found regarding the role of characteristics of tasks that may have favored students' identification of the relationship between two variables. The research reported in this paper is important because it shows that, prior to instruction in variable notation, grade 4 students already have the potential to exhibit functional thinking. In this study, the proposed problems were characterized to identify some of the factors that influenced students' strategies. In addition, we also describe the kinds of representations they use when they use a functional strategy. Therefore, in this study our goal is to explore what characteristics of word problems are most associated with students' use of functional strategies and with representing of generalizations in the absence of prior early algebra instruction. Identifying these characteristics could help build and propose tasks that encourage students to use various strategies and promote more sophisticated strategies.

Background and Framework for this Study

This study was carried out in the context of early algebra, which promotes the development of algebraic thinking from the start of children's schooling and of generalizing, justifying, and representing (Blanton and Kaput 2011). Kaput (2008) characterizes algebraic thinking as a complex symbolization process whose purpose is that of generalizing and reasoning with those generalizations. The approach to early algebra that we propose in this study is focused

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on functional thinking (Carraher and Schliemann 2007). We understand functional thinking as "thinking in terms of about relationships" (Rico 2007, p. 56). The functional perspective entails reasoning with variables and establishing connections between quantities that covary simultaneously (Blanton et al. 2011). Variables and their relationships can be expressed in ways other than symbolic notation, such as words, tables, or the algebraic or parametric use of numbers (Blanton 2008; Other et al. 2015).

In this study we understand functions as relationships in which the value of each independent variable matches a single value of the dependent variable (Larson and Hostetler 2008) and we define functional relationships as the rules generating one-to-one correspondences between elements in set B and elements in set A (Vinner and Dreyfus 1989), where A and B are sets of natural numbers. In functional thinking, generalization is necessary to identify and analyze relationships between variables (Smith 2003). Generalizing consists of the transition from the identification of regularity in specific cases to other broader cases that follow a same pattern (Polya 1945). In the process of generalization, characteristics and properties of objects that have been abstracted are generalized and extended to a set of objects of a certain class (Krutetskii 1976).

Strategies are understood as procedures that allow us to find a solution to a problem, arrive at conclusions from a body of concepts, and establish relationships (Rico, Castro, Castro, Coriat, Marín and Puig 1997). Strategies for generalizing functional relationships are understood to include the approaches students use to identify the relationship between elements and to extend that relationship into a more general structure. Bills and Rowland (1999) distinguish between *empirical generalization* that entails building on a small number of cases and *structural generalization* that involves no examples or only examples treated as generic representatives of

the property. Radford (2010, 2013), on the other hand, differentiates between *algebraic*

generalization in which students construct an expression from a deductive process that allows

them to obtain any specific value and *arithmetic generalization* in which students identify a

numerical pattern but that is a local commonality ob- served on some figures, without being able

to use this information to provide an expression of whatever term of the sequence. Finding the

formula by trial and error also constitutes arithmetic generalization. When identifying a functional

relationship, students identify the quantities involved and their relationship, recognizing

covariation between two simultaneously varying quantities (Blanton and Kaput 2011) or the

correspondence between pairs of dependent and independent variables (Blanton 2008; Smith

2008). In problems that include patterns and linear functions, several authors have defined

functional strategies as those procedures students use when they establish a relationship between

two variables—the position of the figure in the pattern and the number of elements in the figure—

through a mathematical expression of the form $f(n) = an + b$ ($b \neq 0$), where the difference between

consecutive terms and b is constant throughout the pattern (Stacey 1989; Zapatera 2018).

To understand students' generalizations, it's also important to understand the ways in

which they express and represent their generalizations, in a way that all elements can be

represented in a single expression (Radford 2010). When students expresses the generalization of

the functional relationship, they may represent it through spoken language, graphs, symbolic

representations, or a combination of different representations (Carraher et al. 2008; Smith 2008).

Cañadas, Castro, and Castro (2008) use the term textual or verbal generalization when students

use natural language. In a study on representing the generalization of functional relationships,

Other et al. (2019) distinguish between verbal, generic, and symbolic representations of

generalizations, drawing from research by Radford (2010) and Mason and Pimm (1984). In their

study, when representing a generalization verbally, students recognized the functional relationship

and expressed it verbally in general terms, alluding to indeterminate quantities. When they

represented the generalization generically, students recognized the functional relationship in

general cases and expressed it with an example involving specific numbers used as generic

examples. Finally, when using a symbolic representation, students recognized the functional

relationship and expressed it using algebraic symbols.

Methodology

Participants

Twenty-five grade 4 students aged 9 to 10 years old enrolled in a school in the South of

Spain, participated in this study. In addition, two individual interviews were carried out with each

of six of these 25 students. We selected these six students to represent low, intermediate, and high-

performance levels relative to the classroom as a whole during the first classroom session problem.

We also compared our selected students to their teacher's opinion of their performance in

mathematics and made sure to include at least one student in each level of performance (low,

intermediate, high). Students' communication skills were also taken into account to ensure the

fluency of their answers during the interviews. Students' anonymity was ensured by assigning each

a code, S_i, where i= 1 ... 25.
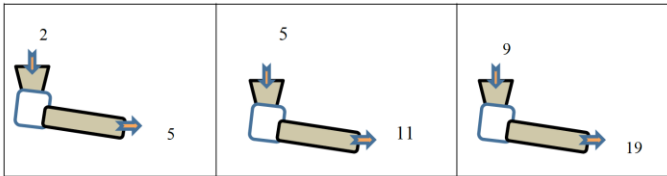
Instructional Sequence

Five 60-minute data gathering sessions were conducted by two researchers under the

classroom teacher's supervision. Students' normal classroom arrangement in groups of three or

four was used for these sessions. The sessions were characterized by different stages: (a) general introduction by the researcher-teacher; (b) individual or small group-based problem solving with worksheets; and (c) general discussion where students could express their ideas, ask classmates to explain their thinking, or make suggestions to revise the answers or generalizations proposed.

Two semi-structured interviews were also conducted with each of six students, one after the first and the other after the last classroom session. Table 1 shows the word problems presented to students in the classroom sessions and interviews.

Table 1 Background information for word problems (WP)

Session/ Interview	Name	Background
Session 1 and Interview 1	WP1. Function machine	Today we're going to play a game. This machine changes numbers. It's your job to figure out how it works.
Session 2	WP2. Amusement park in [Name of city]	[Name of city] has a new amusement park. To get in you have to buy a pass for 1 euro that you can use as often as you like. The park has lots of rides. All rides cost 2 euros.
Session 3	WP3. Amusement park in [Name of town]	The town of [Name of town] now also has an amusement park. To get in you have to buy a pass for 3 euros that you can use as often as you like. The park has lots of rides. All rides cost 1 euro.

Table 1 Background information for word problems (WP)

Session/ Interview	Name	Background
Session 4 and Session 5	WP4. Tables and chairs for a birthday party	Isabel is getting ready for her birthday party. She puts two boxes containing little presents for her guests on each table. She arranges a few tables in a row as in the drawing.
Interview 2	WP5. Balloons for a birthday party	All the guests at Isabel's birthday party get the same number of balloons and they put one balloon on the door to show there's a birthday party going on inside.

The problems proposed were designed following an inductive process (Cañadas and Castro 2007), i.e., they were divided into three sections in which the problems first involved small quantities, followed by large and then indeterminate quantities represented with natural language (i.e., any, many), symbols (e.g., question marks, drawings) or letters. An example of the sections of the problems in Session 2 is provided below (see Fig. 1).

Section 1. Small quantity	How much does it cost for a pass into the park and one ride? How did you find your answer? [Analogously, students were asked to calculate the cost for 4, 20, 11, 35, and 100 rides.]
Section 2. Large quantity	How much does it cost for a pass into the park and one million rides? How did you find your answer?
Section 3. Indeterminate quantities	• One classmate knows he bought a pass and the number of rides he took. Explain to him how to calculate how much he spent. • If a child wants to go on B rides, how can you figure out how much they have to spend?

Fig. 1 Problem in Session 2.

Data Collection

Data were collected from two sources: written worksheets and video recordings. The video recordings included those from classroom sessions (with a fixed camera located at the rear of the classroom and with a mobile camera recording students as they worked in small groups) and from individual interviews.

To classify the word problems, we considered the following characteristics: (a) the variables involved, (b) whether the functional relationship was additive or multiplicative, and (c) information about the functional relationship.

Table 2 Word problem (WP) characteristics

Problem characteristic	WP1	WP2	WP3	WP4	WP5
Context	Function machine	Amusement park	Amusement park	Birthday: tables and boxes	Birthday: Balloons
Function	2x + 1	2x + 1	x + 3	2x or x + x	3x + 1
Variable	Input unit Output unit	Number of rides Number of euros	Number of rides Number of euros	Number of tables Number of boxes	Number of guests Number of balloons
Information about the functional relationship	Three pairs of (a, f(a)) values	Explicit in word problem	Explicit in word problem	Drawing with an example	Three pairs of (a, f(a)) values and verbal information

Data Analysis

In this paper we focus our analysis on students' written answers (on worksheets) and transcripts of class sessions and interviews. The association between characteristics word problems and the use of strategies was investigated by analyzing individual students' written responses before they were discussed by the group as a whole. Earlier studies have analyzed the effect of whole group justification and discussion on representations of generalizations (Author and other 2019b).

We analyze each student's responses according to two categories: strategies used and representations of the proposed functional relationship. Then, we connect this information with the characteristics of the problems and identify those that seem to be associated with students' functional thinking. To ensure reliability of our analysis, after the first author carried out initial

coding, his decisions were discussed with each of the two other authors until we arrived at a consensus.

Students' strategies.

To answer the first research question (*What characteristics of word problems are associated with students' use of functional strategies?*), we first determined the strategies the students used in each section of the word problems. Table 3 shows the strategies we identified, based on earlier research.

Table 3 Strategies used when solving word problems

Strategy	Description	Example
Functional	Students who generalize understand the data in the various sections of the word problem as different values for the same variable, using letters, natural language, and/or tables. They identify the quantities involved and their relationship, recognizing covariation between two simultaneously varying quantities (Blanton and Kaput 2011) or the correspondence between pairs of dependent and independent variables (Blanton 2008; Smith 2008). They follow the same process they used in the various sections of the problem to find the solution and generalize the relationship when answering the section involving an indeterminate quantity.	S_{21} (in WP3): The final price depends on the rides, you pay three euros and one euro for the rides you want to go on.
Arithmetic	Students use elementary arithmetic operations to find the solution, without realizing that the relationship between data and solution is the same across the various sections. The arithmetic strategy is based on the correct choice of operations (Vergnaud 2009).	S_9 (in WP1): I knew it because I knew that from two to five I was adding. Then I thought that the difference between two and five was three. And between five and eleven, six.

Table 3 Strategies used when solving word problems

Strategy	Description	Example
Manipulative/ visual-graphic	The strategy is based on information drawn directly from visual representations (Krutetskii 1976) or manipulatives. These two approaches were combined here in a single category, for students were found not to be using arithmetic operations but counting, reorganizing manipulatives, or similar.	And between nine and nineteen, nine. And then I went, from three to seven, four. And from ten to twenty-one, eleven. S_{23} (in WP5): That he had to give his classmates three, three, three, three, three and three. The same amount to each, one was left over, but we put it on the door [while physically distributing the balloons across his drawings].
Other	The strategy used was unclear or the student did not respond.	

For each student, we identified the strategy (i.e., manipulative/visual, arithmetic, functional, or other) that they used to provide an answer to the most complex section of the problem (i.e., small, large, and indeterminate quantities). We considered the indeterminate section of the problem (see Table 2) to be the most complex, followed by large quantities, and finally with small quantities being the least complex.

Students' representations.

In order to answer the second research question (*How do students who recognize functional relationships represent the generalizations?*), when students used a functional strategy, we analyzed the external representation each student used in each word problem. Note that in this

study no student used more than one representation for generalization. External representations were defined as "assertions in natural language, algebraic formulas, graphs or geometric figures, among others, [constituting] the medium whereby individuals exteriorize their mental images and representations to make them accessible to others" (Rico, Castro, Castro, Coriat, Marín and Puig 1997, p. 101). The three categories used in the analysis and described in Table 4 are based on a proposal by Other et al. (2019).

Table 4 Representations of generalizations used when solving word problems

Representation	Description	Example
Verbal	Students generalize the relationship in natural language with allusions to indeterminate quantities, but without using algebraic expressions.	S_{16} (in WP3): From what it says about rides, you have to add the rides with the pass plus three euros.
Generic	Students describe the relationship with generic examples, in which the specific quantities used as examples are meant to represent several values at the same time.	S_{12} (in WP1): An example, for example if I have two and after two comes three, you have to add three and you get five (when asked to explain to a friend how the machine works).
Symbolic	Students generalize using algebraic expressions (letters to represent any value, equations, and so on).	None of the students in this study used this type of representation. As an example, we expected that in the WP5 they would indicate that $3x + 1$ allows you to know the number of balloons knowing the number of guests.

Results

In response to our first research question (*What characteristics of word problems are associated with students' use of functional strategies?*), Table 5 presents our findings regarding

the strategy used by the 25 students in this study (i.e., functional, arithmetic, and manipulative/graphic-visual). None of the students used several strategies in the same section of the problem. In addition, we also indicate the scope of the quantities in which the strategy was used (i.e., small, large, and indeterminate quantities).

Table 5 Strategies, scope, and representation of functional relationships

Word problem	Manipulative/ Visual	Arithmetic		Functional	Other	N[a]
		Quantities				
	Small	Small	Large	Indeterminate		
Classroom Sessions						
WP1[b]	0	4 (18%)	0	6 (27%)	12 (55%)	22
WP2	0	8 (44%)	3 (17%)	3 (17%)	4 (22%)	18
WP3	0	1 (5%)	6 (32%)	11 (58%)	1 (5%)	19
WP4	0	1 (4%)	12 (52%)	10 (43%)	0	23
Final Interview						
WP5	1 (17%)	2 (33%)	1 (17%)	2 (33%)	0	6

[a]N for the classroom sessions is not always 25 due to student absences. N for the individual interview was only 6 due to two students being absent. [b]WP1 was presented both in the first classroom session and in the initial interview.

To respond to our second research question (*What characteristics of word problems are associated with students' use of functional strategies?*), Table 6 presents our findings regarding the kind of representation used when students generalized the functional relationship. Note that all students who used a functional strategy did so with indeterminate quantities.

Table 6 Students' representations of functional relationships

Word problem	Representations		N^a
	Generic	Verbal	
	Classroom Sessions		
WP1[b]	1	5	22
	(5%)	(23%)	
WP2	1	2	18
	(6 %)	(11%)	
WP3	8	3	19
	(42%)	(16%)	
WP4	3	7	23
	(13%)	(30%)	
	Final Interview		
WP5	1	1	6
	(17%)	(17%)	

[a] N for the classroom sessions is not always 25 due to student absences. N for the individual interview was only 6 due to two students being absent. [b]WP1 was presented both in the first classroom session and in the initial interview.

The strategies identified and the features of the word problems that were most frequently associated with the use of functional strategies are described in the sub-sections below, followed by a discussion of how generalization was represented and an overall analysis of student performance across word problems.

Strategies

Eight students (S_{05}, $S_{06,}$ S_{07}, $S_{08,}$ S_{09}, $S_{10,}$ S_{11}, S_{14}) did not use a functional strategy in any of the problems in this study (see Table 7). At some point during the study, each of the students used the arithmetic strategy with large quantities.

Table 7 Strategies used among students who did not use a functional strategy

Student	WP1	WP2	WP3	WP4	WP5
S_{05}	Arithmetic (Small quant.)	A (S)	A (Large quant.)	A (L)	N/A

Table 7 Strategies used among students who did not use a functional strategy

Student	WP1	WP2	WP3	WP4	WP5
S_{06}	A (S)	Other	A (L)	A (L)	N/A
S_{07}	N/A	Other	N/A	A (L)	N/A
S_{08}	Other	A (S)	A (L)	A (L)	N/A
S_{09}	Other	N/A	A (S)	A (L)	A (S)
S_{10}	A (S)	N/A	N/A	A (L)	N/A
S_{11}	N/A	A (L)	N/A	N/A	N/A
S_{14}	Other	A (S)	NR	A (L)	N/A

Note. The information in brackets indicates the most complex scope in which the strategy was used. N/A refers to students being absent or not participating in these sessions.

When they did not generalize, students exhibited difficulty in understanding what the questions involving indeterminate quantities (expressed as many, any, a question mark, or a blotch) meant. They tended to answer with a specific value and use expressions that were not general, such as in the following example (S_{09}'s response in WP3).

I: How do we know how much no-matter-how-many rides (the blotch) can cost?

S_{09}: I said you go on four rides and then I added five. And I said you go on five rides.

I: Why?

S_{09}: I saw that when there was a blotch it meant four. So, I put four and added one. And that gave five so I said there were five rides.

The other 17 students did use a functional strategy at some point in the study (see Table 8).
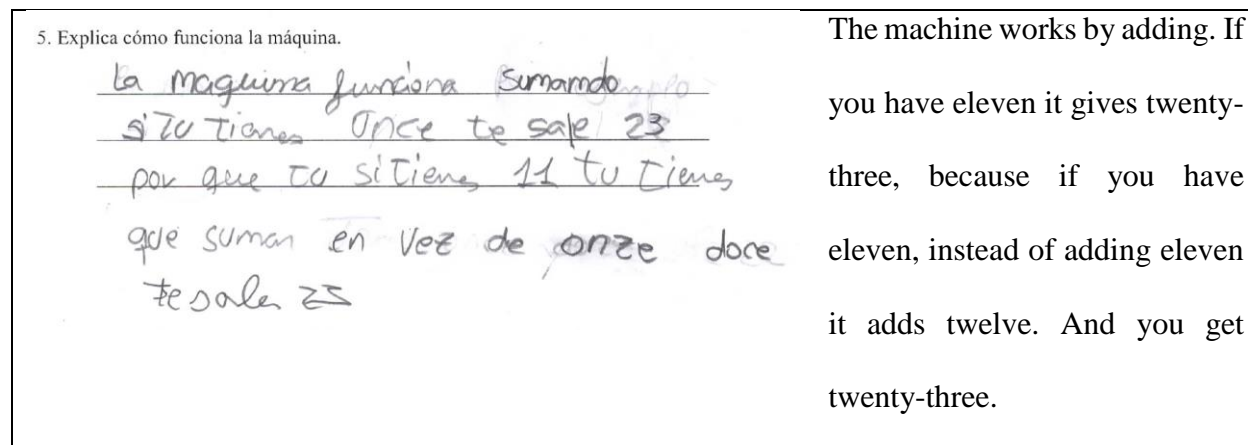
Table 8 Students who did use a functional strategy at some point in the study

Student	WP1	WP2	WP3	WP4	WP5
S_{01}	N/A	N/A	F (Generic repr.)	F (V)	N/A
S_{02}	Arithmetic (Small quant.)	N/A	F (V)	F (V)	N/A
S_{03}	Functional (Verbal repr.)	F (V)	F (G)	F (G)	N/A
S_{04}	F (V)	N/A	N/A	F (V)	N/A
S_{12}	F (V)	A (Large quant.)	F (G)	F (G)	A (L)
S_{13}	F (G)	F (G)	F (G)	A (L)	N/A
S_{15}	Other	N/A	A (L)	A (L)	F (G)
S_{16}	Other	A (S)	F (V)	A (L)	A (S)
S_{17}	Other	A (S)	F (G)	A (L)	N/A
S_{19}	Other	A (S)	F (G)	N/A	N/A
S_{21}	Other	A (L)	F (V)	F (V)	F (V)
S_{22}	Other	Other	N/A	F (V)	N/A
S_{23}	F (V)	F (V)	F (G)	F (G)	M (S)
S_{24}	F (V)	N/A	N/A	F (V)	N/A
S_{25}	Other	Other	F (G)	F (V)	N/A

Note. For students who used an arithmetic strategy, we indicate the most complex scope in which they used this strategy. For students who used a functional strategy, we indicate the representation they used, since they all dealt with indeterminate quantities. N/A refers to students being absent or not participating in these sessions.

When using a functional strategy, students understood the data in the various sections of the word problem as different values for the same variable. S_{12}, for instance, answered WP1 as shown in Fig. 2.

5. Explica cómo funciona la máquina. La maquina funciona Sumando si tu Tienes Once te sale 23 por que tu si tienes 11 tu tienes que suman en Vez de onze doce te sale 25	The machine works by adding. If you have eleven it gives twenty-three, because if you have eleven, instead of adding eleven it adds twelve. And you get twenty-three.

Fig. 2 S_{12}'s written answer to WP1 in classroom Session 1.
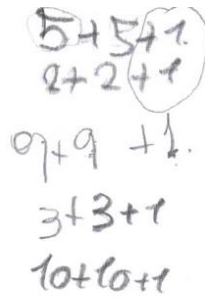
During the interview, S_{12} explained her strategy and pointed out that she added the next number. She explained her answer in the interview as follows:

I: OK! You mean that if I enter a number the result will be… What do I have to do to that number?

S_{12}: Add it to the next number.

When using a functional strategy, students perceived the regularity in the various sections of the problem and understood the functional rule as the relationship between the information provided in the problem (the independent variable) and their answers (the dependent variable). For instance, S_{23} perceived the functional relationship $x + (x + 1)$ in the section of the problem that included specific cases and was able to explain that the machine "works by adding the same number and then one more" because he recognized that pattern in his answers (see Fig. 3).

Fig. 3 S_{23}'s answer to WP1 in the interview.

Student S_{21} provided another example of students' interpretation of the functional relationship as the dependence between one variable and another. After initially using manipulative and arithmetic strategies, he used a functional strategy to express the general relationship when queried about indeterminate quantities. The following extract illustrates S_{21}'s response to WP5.

S$_{21}$:   Multiply and give three balloons to each. Multiply times three.

I: And when you multiply times three if there are many, what do you do then?

S$_{21}$:   Many times three… equals… many balloons.

I: You only multiply times three? What about the door?

S$_{21}$:   Plus one equals… many balloons plus one.

I: So, let's summarize, what do you do when there are many guests? Many...

S$_{21}$:   Multiply times three. And then I add one more for the door.

In addition, in WP4 student S_{21} established the functional relationship and answered both directly (calculating the number of boxes from the number of tables) and inversely (calculating the number of tables from the number of boxes).

I: When there were twenty tables, how many boxes did she need?

S_{21}: twenty boxes… forty.

I: And how did you find that answer?

S_{21}: Using half.

I: Half?

S_{21}: Of forty, yes. If you multiply twenty times two, you get forty.

I: And half?

S_{21}: Half of forty is twenty.

Students who used arithmetic strategies showed no signs of recognizing regularity in the operations they performed in the successive sections and in some problems were not able to answer all sections. In WP2, S_{16}, for instance, used the operation *1 + 2 = 3* to answer the question "How much does it cost for a pass into the park and one ride?", adding the price of the pass (1 Euro) and one ride (2 Euros), while for his answer to the question involving 11 rides he used the operation *2 x 11 = 22* since he only took into account the price of the rides (see Fig. 4).

1. How much does it cost for a pass into the park and one ride?	4. How much does it cost for a pass into the park and 11 rides?
1. ¿Cuánto tienes que pagar por hacerte socio del parque y comprar 1 viaje? 1€ 3€ ¿Cómo lo sabes?	4. ¿Cuánto tienes que pagar por hacerte socio del parque y comprar 11 viajes? 22€ ¿Cómo lo sabes?
3€ [euros], because the pass is 1 [euro] and the ride is 2 [euros], so 3[euros] altogether	22€ [euros]Because 11 rides. You have to add 2 euros for each ride

Fig. 4 S_{16}'s answers to WP2.

When using arithmetic strategies, students' use of repeated addition instead of multiplication constituted an obstacle to subsequently apply that strategy to large quantities, such as in the following example. In WP5 S_{16} replied using repeated addition when asked about the number of balloons needed if 15 guests attended the party and explained his answer as follows:

I: Eighteen? How did you find the answer?

S_{16}: By adding.

I: How? What did you add?

S_{16}: Three, plus three, plus three, plus three…three plus three, six and three more, nine

Student S_{23} used a manipulative strategy in WP5, where he based his solution for the various sections without using the underlying arithmetic operation. "Well I handed them out… I hand them out… like you explained, one on the door and the ones the three guests had to have." This strategy enabled him to solve the problem for small quantities but not to find the answer for larger quantities or to generalize for an indeterminate quantity.

Discussion

The inference drawn from these findings is that the strategy students used was associated to the extent to which the strategy was used. Our results show that students who solved the section with indeterminate quantities consistently did so using the functional strategy. When they used manipulative strategies, they only solved the section that involved small quantities. With

arithmetic strategies students were able to solve the sections of the problems that included large quantities, but were unable to generalize beyond numerical examples.

Characteristics associated with the use of functional strategies

WP3 was the word problem in which the functional strategy was most frequently used (58% of students). WP3 involved an additive functional relationship ($y = x + 3$) and explicitly described the rule. Given that the sole difference between WP3 and WP2 was the linear function involved ($y = x + 3$ for WP3 versus $y = 2x + 1$ for WP2), employing less complex functions may have facilitated students' use of functional strategies. Four students ($S_{12}, S_{16}, S_{17},$ and S_{19}) who used arithmetic strategies when working on WP2 ($y = 2x + 1$) were only able to solve the sections that included small or large quantities, whereas they used functional strategies for the sections that included indeterminate quantities and an additive relationship in WP2 ($y = x + 3$). Only three students ($S_{03}, S_{13},$ and S_{23}) used a functional strategy in WP2, which is 17% lower than the percentage of students who used a functional strategy in WP3.

Of the four problems that included multiplicative relationships (WP1, WP2, WP4, and WP5), WP4 is the one in which most students used a functional strategy (43%). The function in WP4 was $y = 2x$, while the functions in WP1, WP2, and WP5 included a constant or a larger coefficient ($y = 2x + 1$ in WP1 and WP2 and $y = 3x + 1$ in WP5).

While WP1 involved the same functional relationship ($y = 2x + 1$) as WP2, the function machine problem, WP1 posed a greater difficulty for students than WP2, for in WP1 55% of students were unable to provide an answer, even for small quantities. Students unable to use a functional strategy in WP1 found it more difficult to generate further examples or propose other pairs of numbers that illustrated the functional relationship proposed in the examples set out in the

problem. Such difficulties did not arise in WP2, for the information provided about the functional relationship was enough for students to apply it to other specific cases.

WP1 and WP5 were the problems in which information was provided as pairs of values. The percentage of functional strategies for each of these problems were 27% and 33% respectively. Although WP5 provided students with information to complete the problem, only two of the six students interviewed (S_{21} and S_{15}) used functional strategies. The use of manipulatives in WP5 had unequal effects on students. Whilst it helped some students solve the sections that included small quantities and subsequently enabled them to adopt functional strategies, for others the dependence on their use prevented them from using strategies that were not manipulative.
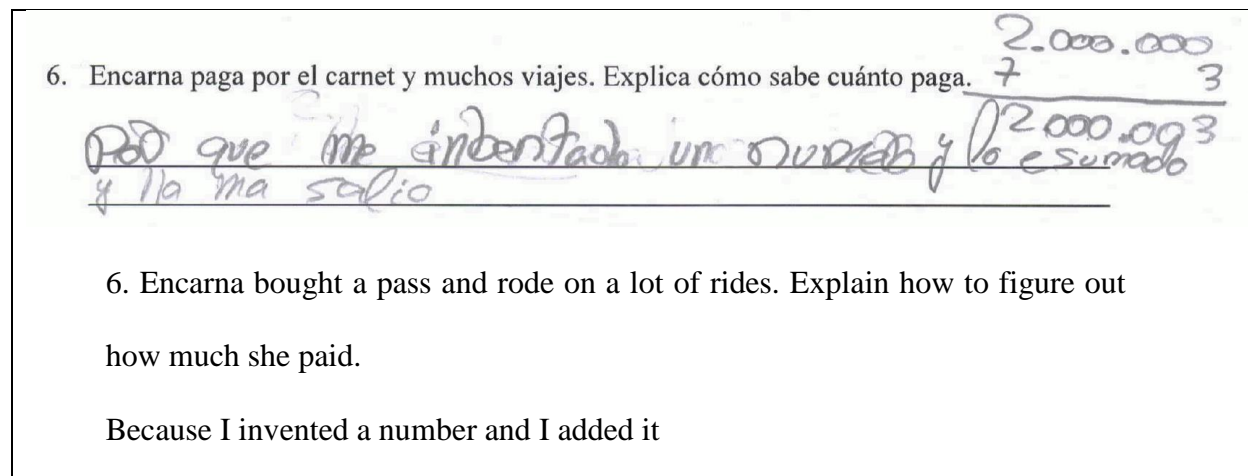
Representation of generalizations in functional strategies

Functional strategies were represented verbally or generically almost the same number of times (eighteen and fourteen, respectively; see Table 6). Students who used verbal representations defined the general functional rule. In WP4 S_{21}'s explanation was "to multiply times two [the number of] boxes on each table." When they did so generically, in contrast, they resorted to an example of their own to represent the indeterminate quantity, such as the example below, also for WP4.

> I: How do you know how many tables there are when you know the number of boxes?
>
> S_{23}: Because if there are twenty tables, you double it and you get the answer.

In some cases, students tended to use very large quantities to represent indeterminate quantities. S_{23}'s answer to WP3, shown in Fig. 5, illustrates this approach.

6. Encarna paga por el carnet y muchos viajes. Explica cómo sabe cuánto paga.

6. Encarna bought a pass and rode on a lot of rides. Explain how to figure out

how much she paid.

Because I invented a number and I added it

Fig. 5 S_{23}'s answer when asked about an indeterminate quantity in WP3.

None of the students used symbolic representations and all interpreted letters in the same ways as reported by other authors for students (albeit much younger) using notation for the first time (Author and Others 2015; Molina, Ambrose and del Río 2018). They did not understand that letters could represent indeterminate quantities but spontaneously assigned to them a very large quantity (e.g., S_{12}:*V + 3 = 1000003*) or values based on the letters' position in the alphabet (e.g., S_{15}: Well, since B is the second letter, I added two times three, because three plus three is six…), the first letter of the number (e.g., S_{12}: V for twenty; S_{21}: Q for fifteen[1]), or the letters' shape (e.g., S_{12} said R was nine and S_{21} said Z was quantities seven).

Overall student performance

Our findings show that, prior to instruction in variable notation, grade 4 students already have the potential to use functional strategies. Among the six students who were interviewed twice, we also identified a relationship between students' use of functional strategies and their academic

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[1] The word for '20' in Spanish is 'veinte' and for '15', 'quince'.

performance. None of the six students used functional strategies across all of the word problems. However, one student, S_{23}, used such strategies in all but WP5, a word problem involving a multiplicative function defined by three pairs of values (a, f(a)). S_{12} tended to represent generalizations generically more frenquently than verbally, whereas S_{23} used verbal and generic representations the same number of times. These two students were chosen for the study on the grounds of their high performance both in Session 1 and in classroom mathematics beyond this study.

Two students, S_{21} and S_{15}, used functional strategies in WP5. S_{21} represented generalizations verbally in all three problems in which he used a functional strategy. Although his performance in Session 1 was low, his teachers deemed him to be a high performer in mathematics. He spontaneously transformed expressions involving operations with letters and numbers (e.g., in WP5, he said: *2R* times three is equal to *6R*; plus one, *7R*) and even used infinity to represent the indeterminate quantity in WP5:

> S_{21}:   Well multiplying three times infinity is equal to infinity.
>
> I: Anything else? Only the guests, nothing for the door this time?
>
> S_{21}:   Yes, the door also counts, so infinite number of balloons plus one is infinite
>
>   one balloons.

One student, S_9, did not use functional strategies and was only able to answer the sections with small quantities. She was not able to respond to WP1 during classroom Session 1 or during the individual interview. She also exhibited low performance on Session 1 and was believed by her teachers to have a poor grasp of mathematical content.

These findings suggest a relationship between students' performance level (low, medium, high) and their use of functional strategies. In this study, the high performers used functional strategies, whereas the students with more tentative mathematical skills were unable to use functional strategies. With instruction and more class sessions, the results would very likely have differed, for other authors have reported that children younger than those studied here recognized functional relationships and used symbolic representations (Author and Other 2008; Blanton and Kaput 2011; Other et al. 2015).

Conclusions

The present findings show that prior to early algebra instruction, grade 4 students are able to generalize relationships between two quantities, i.e., to think functionally (Blanton et al. 2011). With one exception, they all generalized the relationship in at least one of the five word problems, perceiving the information in each section as different values of a given variable (Blanton 2008; Carraher and Schliemann 2007). Once the students recognized a functional relationship, they were able to represent it verbally or with generic examples, although none used algebraic notation (Mason 1996; Radford 2006, 2011). Certain word problem characteristics such as the use of additive functions (which are less complex than multiplicative functions) and the explicit mention of the functional relationship in the problem wording facilitated students' use of functional strategies (Merino et al. 2016; Moss and Beatty 2006; Pinto et al. 2016). With no prior instruction in the use of variable notation, the students in this study exhibited a lower level of symbolic representation of functional relationships than younger students analyzed by other authors (Author et al. 2015; Molina et al. 2018).

Characteristics associated with students' use of functional strategies

When using functional strategies, students perceived the regularity in the sections comprising each problem. Although they generalized primarily for indeterminate quantities, they sometimes did so as well in the sections involving large quantities (McEldoon and Ritte-Johnson 2010; Zapatera 2018). When they used arithmetic strategies, students were able to solve only small and large quantity sections, whilst when they used manipulative or visual strategies their solutions were limited to small quantity sections (Merino et al. 2013).

The use of functional strategies was more frequent in word problems that involved explicit additive functions and situations familiar to students, like the number of euros or the number of rides in an amusement park. As in other studies, the use of less complex linear functions (additive rather than multiplicative) were found to facilitate students' use of functional strategies (McEldoon and Rittle-Johnson 2010; Other et al. 2017).

A second feature that facilitated students' use of functional strategies was wording that enabled students to generate more examples of associated (a, f(a)) pairs of values. In problems where functional relationships were explicit or exemplified with a drawing from which the pattern could be identified, students were able to generate other specific cases or verify their conjectures about the general expression with other examples. That was not possible in the function machine problem, where the information provided was limited to a series of examples. Nonetheless, students who recognized how the machine worked used functional strategies only, which helped them identify the variables involved. The use of manipulatives to help students understand the situation described in the problem constituted an obstacle in some cases, inducing over-dependence on such material. Although it was helpful for small quantities, its use did not facilitate the development of functional strategies (Pinto et al. 2016).

Representations of generalizations in functional relationships

Students who used functional strategies tended to represent the general rule either verbally or with generic examples (Merino et al. 2013; Pinto et al. 2016). Unlike other grade 4 students (McEldoon and Ritte-Johnson 2010) and even younger children (Other et al. 2015), none of the students in this study used symbolic representations to express functional relationships. Students need experience in recognizing and describing functional relationships. Some students in this study perceived functional relationships but could not express them clearly, as reported in other studies (e.g., MacGregor and Stacey 1995). These findings underscore the importance of instruction or prior experience with symbolic notation for 9- to 10-year-olds (Carraher and Schliemann 2007; Radford 2018; Other et al. 2019). When asked about indeterminate quantities, they spontaneously associated the letters used with positions in the alphabet, numbers beginning with the same letter, or with large numbers (Author and other 2019a; Author et al. 2015; Molina et al. 2018).

Implications for teaching

Although the small number of students in this study and the fact that they had received no prior instruction precludes generalizing our findings, this study may nonetheless provide some implications for early algebra instruction in grade 4. Students' performance in the word problems was related to their performance in mathematics class. Given the differing results between high and low performing students, we should design problems with various levels of difficulty to accommodate such differences, whilst exercises used to perceive students' ability to use functional strategies should be designed bearing in mind the complexity of the operations involved relative to other mathematical skills. This study found that inductive problems (from small to large to

indeterminate quantities) explicitly stating an additive functional relationship may pose fewer difficulties than problems that include multiplicative relationships. Other features that facilitate the use of functional strategies are the use of familiar situations to contextualize variables (Lannin et al. 2006) and visual patterns to complement the sequence to be generalized (Moss and McNab 2011).

The finding that manipulatives may hinder children's thinking is an important one as many teachers assume that providing concrete objects may help children learn. Perhaps there is a need to add a caveat that manipulatives may support strategies but equally important that children need to relinquish their use of manipulatives to make a leap to generalization. Word problems that included the use of manipulatives revealed that the effect on performance depended not on the resource itself but on its effective use. The presence of these materials in WP5 limited student S_{23}'s strategies to the manipulative approach, preventing him from solving the sections that included large or indeterminate quantities. In contrast, the other five students who worked on this problem used arithmetic or functional strategies in WP5. Our results seem to indicate that children cannot rely on their use of manipulatives to make a leap to generalizations. Instruction plays a significant role in this regard, guiding students to use several strategies: manipulative/visual for small, arithmetic for large, and functional for indeterminate quantities. The importance of early algebra instruction was observed to be greatest for the use of symbolic representation. Students found it difficult to understand what they were being asked when indeterminate quantities expressed as 'any, many…' were involved and symbols posed a particular limitation. Students should consequently be afforded the opportunity to employ and reflect on the use of letters. Viewed as the acquisition of an external system of representation, the use of letters constitutes an element that

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facilitates students' ability to solve situations involving indeterminate quantities (Martí and Pozo

2000).

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