Representational variation among elementary school students. A study within a functional approach to early algebra

Eder Pinto, Bárbara Brizuela, María Cañadas

Representational variation among elementary school students. A study within a functional approach to early algebra

Eder Pinto[1], Bárbara M. Brizuela[2], and María C. Cañadas[1]

[1]University of Granada, Spain; pinto.eder@gmail.com; mconsu@ugr.es

[2]Tufts University, Medford, United States; Barbara.brizuela@tufts.edu

This paper describes the differences in the types of representations used by eight third-grade (8 to 9-years-old) and eight fifth-grade (10 to 11-years-old) students when working with problems that involve different linear functions. We present an analysis of students' written and oral answers during a Classroom Teaching Experiment (CTE) and semi-structured interviews from a functional approach to early algebra. The study examines how students' representations varied when working with different types of linear functions ($y = x + a$; $y = ax$; $y = ax + b$), when solving for specific values, and when generalizing. The findings show that students in both grades primarily used the representation present in the problem. The type of linear function involved appears to have had no effect on either group's use of one representation or another.

Keywords: Representation, generalization, linear functions, functional thinking.

Introduction

Functional thinking—which has been found to facilitate the introduction of algebra in the early grades—focuses on the relationships between two or more covarying quantities (Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015); these relationships can be expressed through different representations. Representations, which form an integral part of how students think about functions, "can denote and describe material objects, physical properties, actions, and relations, or things that are far more abstract" (Goldin & Shteingold, 2001, p. 4). Therefore, representations help to structure and expand students' thinking (Brizuela & Earnest, 2008). Broadly, our interest is focused on types of representations used by elementary students when working with covarying quantities.

Some researchers have described how elementary school students working with problems that involve single linear functions use, represent, and understand the relationships involved in a given problem (e.g., Brizuela & Earnest, 2008). The originality of this study lies in the exploration of the types of representations used by elementary school students when working with problems involving different types of linear functions. More specifically, this paper analyzes the answers given by eight third- and eight fifth-grade students participating in a CTE, and their answers in semi-structured interviews, after the CTE. The research question is: how do students' representations vary when working with different types of linear functions? Based on this research question, we define two specific aims to describe the type of representations used by these students when: (a) solving problems which involved three types of functions: $y=x+a$; $y=ax$; and $y=ax+b$; and (b) answering questions regarding specific values and when asked to generalize the relationship between variables.

Background

The relationship between generalization and representation is central: both are intrinsic to algebraic thinking and consequently to functional thinking. According to Kaput, Blanton, and Moreno (2008), representations—a socio-cultural vehicle used to generalize—enable students to build and complete the ideas that help them reason about general statements and compress multiple instances into the unitary form of a single statement that symbolizes the multiplicity. Thus, generalization is the "act of creating that symbolic object" (p. 20).

Several studies highlighted the role of representations in generalizing the relationship between two variables. According to some findings, 8 to 10-years-old students, instructed to use algebraic notation, represented the linear relationship $y=x+a$ by mainly using algebraic notation rather than natural language (Carraher, Schliemann, & Schwartz, 2008), while other studies reported that fifth graders (10 to 11-years-old) spontaneously used algebraic notation to generalize a problem that involve $y=mx+b$ (Pinto & Cañadas, 2018). Another study distinguished between the representations used by students when working with specific values and when generalizing, concluding that students who do not use algebraic notation when they represent a generalization do "not yet have a representational means to compress multiple instances into a unitary form that could symbolize these instances" (Blanton et al., 2015, p. 542). Further research is therefore needed on how the representations used by students vary when working with different linear functions, and how they differ depending on whether they are working with specific values or generalizations.

The types of representations that can be used by elementary school students to solve problems involving linear functions include: (a) natural language – oral; (b) natural language – written; (c) pictorial; (d) numerical; (e) algebraic notation, (f) tabular; and (g) graphic (Carraher et al., 2008). Considering the suggestions of early algebra literature, we stress the role of natural language because it is considered as a useful scaffold to understand symbolic representations (Kaput, 1987; Radford, 2003), and helps to broaden students' understanding about functions, improving their abilities to solve problems (MacGregor & Stacey, 1995).

Method

This study forms part of a broader project that explores functional thinking among elementary school students in Spain.

Students

Two groups were intentionally selected for the first year of the study: 24 third-grade (8 to 9-years-old) and 24 fifth-grade (10 to 11-years-old) students. The students had not worked previously with such problems. Then, we interviewed eight students in each group to obtain a deeper understanding of how they responded to problems involving relationships between two variables.

Data collection: CTE and interviews

The data were collected during the CTE and individual interviews. The CTE and interviews objectives were to: (a) explore how students relate the variables involved in a problem involving a linear function; (b) introduce different types of representations to express functional relationships; and (c) explore students' generalization when working with functional thinking tasks.

A four-session CTE was designed for each grade during the last term of 2014/2015 period, with each session lasting approximately 60 minutes. Each CTE session was divided into three parts. First, we introduced the context of the problem, highlighted the representations given and asked the students questions about specific values to ascertain whether they had understood it. Second, the students were given individual worksheets with questions about specific values and generalization related to the problem. Third, the researchers led a classroom discussion around the responses to some of the questions on the worksheets. During the 2015/2016 period, eight students from each grade were interviewed in two 30-minute interviews. The interviewees were deliberately selected to include children who had performed differently during the CTE and it provided a way to more closely detail individual students' representations from the CTEs. Table 1 shows the general context of problems posed, types of functions, and types of representations introduced during the CTE and interviews.

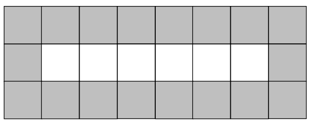
Timing	Problem posed	
CTE	Third **Session 1**. María and Raúl are brother and sister. María is the elder. We know that María is 5 years older than Raúl ($y=x+5$) *(natural language – written and table*). **Sessions 2 and 3**. Carlos earns 3 euros for each T-shirt he sells ($y=3x$) *(natural language – written, table, and graphic*). **Session 4**. Different corridors are composed of white and grey tiles. All the tiles are square and of the same size and are to be laid in the following pattern: ($y=2x+6$) *(natural language – written, and pictorial*) (Küchemann, 1981). 	Fifth **Session 1**. Carlos earns 3 euros for each T-shirt he sells ($y=3x$) (*natural language – written and table*). **Session 2**. Carla earns 3 euros for each T-shirt she sells, while Daniel earns double that amount for each T-shirt and has saved 15 euros ($y=3x$; $y=2x+15$) (*natural language – written and table).* **Session 3**. A grandmother tells her grandson that she has some money to give him and proposes two deals ($y=2x$; $y=3x–7$) (natural language - written) (Adapted from Brizuela & Earnest, 2008). **Session 4.** Different corridors are composed of white and grey tiles. All the tiles are square and of the same size and are to be laid in the following pattern: ($y=2x+6$) *(natural language – written and pictorial*) (Adapted from Küchemann, 1981).
Interviews	Fourth **Interview 1.** It costs 2 euros to enter a car park and 1 euro per hour to park there ($y=x+2$) (natural language- oral, algebraic notation). **Interview 2**. Elsa conducts a train. Three passengers get on at each stop ($y=3x+1$) (natural language – oral, algebraic notation).	Sixth **Interview 1**. A geometric pattern with a different number and arrangement of points ($y=4x+1$) (natural language – oral, algebraic notation, and pictorial). **Interview 2**. Two telephone rates have different costs ($y=10x$; $y=5x+60$) (natural language – oral and algebraic notation).

Table 1: Problems posed in CTE and interviews

The problems proposed involved different linear functions ($y=x+a$, $y=ax$, and $y=ax+b$); some of them were selected from previous studies and others were designed by the research team, considering different types of linear functions. The questions concerning each problem were sequenced according to the inductive reasoning model of Cañadas and Castro (2007), which structures questions from specific values to generalization. For instance, the fourth task for third and fifth grade included questions regarding:

- Specific values. For example, “How many grey tiles do they need for a corridor with 10 white tiles?”, and
- Generalization. For example, “The workers always lay the white tiles first and then the grey tiles. How can they calculate how many grey tiles they need in a corridor where they’ve already laid the white ones?”

The data analyzed in this paper were the students’ CTE and interview worksheets and the transcriptions of the video-recorded interviews.

Data and analysis categories.

The first step was identifying the types of representations used by the students. Figure 1 gives an example of each type of representations[1] used by students when they answered in different tasks.

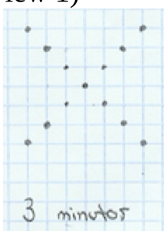
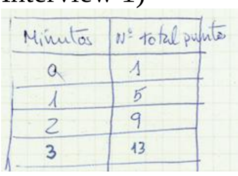
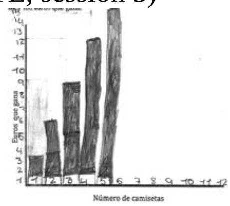
Natural language - oral (Fourth grade, Interview 2) Interviewer: How would you find the number of passengers for any number of stops? Student: (…) *multiplying that number times three plus one.*	**Natural language – written** (Third grade, CTE, session 1) We found a picture of Raúl’s birthday party and all you can see are the candles on the cake. How could you find María’s age? *I add five to Raúl’s age.*	**Pictorial** (Sixth grade, Interview 1) 	**Numerical** (Third grade, CTE, session 1) When Raúl is 15, how old will María be? *Raúl’s 15 and María’s 20.* How did you find your answer? *15+5=20*
Algebraic notation (Fifth grade, CTE, session 4) How many grey tiles do they need for a floor with five white tiles? *They need 16 grey tiles. Formula: (X* x *2)+6=16. X = number of grey tiles*	**Tabular** (Fourth grade, Interview 1) 	**Graphic** (Third grade, CTE, session 3) 	

Figure 1: Examples of types of representation in students’ answers

The representations used by the students to solve problems were first identified and then analyzed considering: the type of linear function in each problem ($y=a+x$, $y=ax$, and $y=ax+b$), and the students’ answers when working with specific values and when generalizing.

[1] Which were not mutually exclusive.

Results and Discussion

We present the findings for the types of representations used by each group of students, considering our research questions.

Third and fourth grades

Table 2 lists the frequency for each type of representations used by students in CTE and interviews, considering different linear functions and when working with specific values and when generalizing.

	Representation	CTE			Interview	
		y=x+a	*y=ax*	*y=ax+b*	*y=x+a*	*y=ax+b*
Specific values	N-oral				8*	7*
	N-written	2*	4*	5*		
	Pictorial					
	Numerical	4	7	6	1	7
	Algebraic	4*	5*			
	Tabular				1	2
	Graphic					
Generalization	N-oral				7*	6*
	N-written	5*	2*	2*	1	
	Pictorial					
	Numerical					
	Algebraic				6*	5*
	Tabular					
	Graphic					
* = representation introduced in the problem itself						

Table 2: Representations used by third- and fourth-graders

As the data in Table 2 shows, the students tended to use the representation introduced in the problem itself, regardless of the type of linear function involved. Specifically, oral and written natural language prevailed in this group of eight students. Mario's response during the first interview to a problem with a function of the type *y=x+b* is an example.

Interviewer: Let's suppose I don't know how many hours I'm going to park, but I tell you that I'm going to be there for *x* hours (…). How can I know how much it's going to cost me?

Mario: (…) Well, if you're there for *x* hours, you add 2. For all the hours [the car is parked] you add 2.

As exemplified in the above extract, the prevalence of oral language is an essential element in learning to recognize and understand a function (MacGregor & Stacey, 1995). The use of a numerical representation was also observed at least once in all the problems involving specific values. This makes sense, since students were being asked about specific, and not general values. Numerical representation was as frequent as natural language in students' answers to problems with functions of the form *y=ax+b*.

During the interviews one year later, some students spontaneously arranged the values of the variables in tabular form. Figure 2 shows how Susana arranged the data to explore the specific values in a problem of the type *y=mx+b* (interview 2).

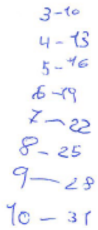


Figure 2: Susana's answer when working with specific values

As Figure 2 shows, Susana arranged the specific values in two columns separated by a dash (-). She listed the number of stops on the left and the number of people on the train on the right. This table, according to Martí (2009), shows that "data be organized (categorization, establishing correspondences) in a certain spatial layout" (p. 134). Susana's table is a novel way to represent, relate and understand the values involved in the problem. Spontaneous tabular representations were observed during the interviews in the problems involving functions such as *y=x+a* and *y=ax+b*.

The eight students used different representations depending on whether they were working with specific values or generalizing. During the CTE, the variety of representations was wider when they were exploring the relationship with specific values (written natural language, numerical, algebraic notation) than when generalizing, when they used written natural language only. One possible explanation for this difference may lie in the complexity inherent in generalizing the relationship between two variables, as suggested by other authors (e.g., Radford, 2003). Nonetheless, natural language would appear to be the way these students explain general rules. During the interviews, the students used the representation present in the problem to generalize the relationship, whereas for specific values they used numerical and tabular representations as well.

Fifth and Sixth Grades

Table 3 lists the types of representations used by the eight students in this group, when solving the problems posed.

		CTE			Interview		
	Representation	*y=x+a*	*y=ax*	*y=ax+b*	*y=x+a*	*y=ax*	*y=ax+b*
Specific values	N-oral				1	1*	1*
	N-written	1*		5*			
	Pictorial			3*	5*		
	Numerical	5	2		2	4	4
	Algebraic			1			
	Tabular				6*	3*	4*
	Graphic						
Generalization	N-oral				4*	3*	3*
	N-written	3*	4*	7*			
	Pictorial						
	Numerical		1	1	1	1	1
	Algebraic	2*	3*	2	5*	3	4*
	Tabular						

	Graphic						
Note. * = representation introduced in the problem itself							

Table 3: Representations used by fifth and sixth graders

We can see in Table 3 that these students tended to use the same type of representation as introduced in the problem. Significantly, in the tiles problem (CTE, $y=mx+b$ type function), two of the students used algebraic notation spontaneously. Camila's answer to the first question associated with that problem is reproduced in Figure 3.

1. How many grey tiles do they need for a floor with five white tiles?
They need 16 grey tiles. Formula: (X x *2)+6=16. X = number of grey tiles*

Figure 3: Camila's answer to the tile problem (CTE, session 4)

Camila used a representation not introduced in the problem. Although some of the questions posed in the earlier sessions involved the use of algebraic notation, here the student used it to express the general rule, although she was answering a question about a specific value. This type of representation was also observed in the interviews, specifically in four students' answers to the problem in the second interview, in which the functions were of the type $y=ax$ and $y=ax+b$. That spontaneous use of algebraic notation differed from the findings for the third- and fourth-grade students and suggests that some of the sixth graders were quicker to adopt the use of algebraic notation to express general relationships between two variables.

Most of the students in fifth and sixth grades generalized using (oral or written) natural language or algebraic notation, whereas for specific values they used a wider variety of representations (oral and written natural language, pictorial, numerical, algebraic notation and tabular). Numerical representation prevailed in these students' solutions to the specific value questions. Once again, this make sense given that were being asked about specific values.

Conclusions

This paper seeks to shed light on how the representations used by intermediate and upper elementary school students (8 to 12-years-old) vary when solving problems that involve different types of linear functions. Specifically, in both groups, when the students generalized the functional relationship, they used the same type of representation as introduced in the problem (natural language or algebraic notation). The variety of representations was broader when they worked with specific values. These findings reveal that while they were aware of different types of mathematical representations (used when working with specific values), when generalizing they only used two. As noted by other authors (e.g., Brizuela & Earnest, 2008), this highlights the importance of teaching representations in elementary school to enable students to gradually assimilate them as they are constructing meaning for different types of representations when generalizing and grasping the meaning of functional relationships.

Concerning this study' specific objectives, no major differences were observed in the representations used in one type of function or another by either third- and fourth- or fifth- and sixth-graders. That may be because both groups had participated in a CTE in which they solved problems involving different linear functions. Tabular representations appeared spontaneously in some of the third/fourth-grade students' answers, whereas algebraic notation appeared among the fifth/sixth-graders. Significant use of numerical representation was found in both groups and it was consistent with what the problem asked them. The major difference between the two groups was in the types of representations used when working with specific values and when generalizing. When generalizing, the third- and fourth-graders used a narrower variety of representations than the older group of students. This seems to suggest that in this sample, the students in the higher grades of elementary school had more resources from which to draw when expressing general rules.

Acknowledgment

This study was funded by the Spanish Ministry of Economy and Competitiveness under National R&D Project EDU2016-75771-P; the first author benefited from a PhD. grant awarded by the Chilean Government through the CONICYT, folio 72160307-2015.

References

Blanton, M. L., Brizuela, B. M., Gardiner, A., Sawrey, K., & Newman-Owens, A. (2015). A learning trajectory in 6-year-olds' thinking about generalizing functional relationships. *Journal for Research in Mathematics Education*, *46*(5), 511-558.

Brizuela, B., & Earnest, D. (2008). Multiple notational systems and algebraic understanding: The case of the "best deal" problem. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 273-302). New York, NY: LEA.

Cañadas, M. C., & Castro, E. (2007). A proposal of categorisation for analysing inductive reasoning. *PNA*, *1*(2), 67-78.

Carraher, D. W., Schliemann, A. D., & Schwartz, J. L. (2008). Early algebra is not the same as algebra early. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 235-272). New York, NY: LEA.

Goldin, G. A., & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A. Cuoco, & F. Curcio (Eds.), *The roles of representation in school mathematics. 2001 yearbook* (pp. 1-21). Reston, VA: NCTM.

Kaput, J. J. (1987). Towards a theory of symbol use in mathematics. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 159-195). Hillsdale, NJ: LEA.

Kaput, J. J., Blanton, M. L, & Moreno, L. (2008). Algebra from a symbolization point of view. In J. J. Kaput, D. W. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 19-55). New York, NY: LEA.

Küchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics: 11-16* (pp. 102-119). London, United Kingdom: John Murray.

MacGregor, M., & Stacey, K. (1995). The effect of different approaches to algebra on students' perceptions of functional relationships. *Mathematics Education Research Journal*, *7*(1), 69-85.

Martí, E. (2009). Tables as cognitive tools in primary education. In C. Andersen et al., (Eds.). *Representational systems and practices as learning tools* (pp. 133-148). Rotterdam, The Netherlands: Sense Publishers.

Pinto, E., & Cañadas, M. C. (2018). Generalization in fifth graders within a functional approach. *PNA*, *12*(3), 173-184.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic–cultural approach to students' types of generalization. *Mathematical Thinking and Learning*, *5*(1), 37-70.