Moving beyond descriptive models: Research issues for design and implementation

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Abstract

In this paper, we draw on a models and modeling perspective to describe the design of a sequence of tasks, known as a model development sequence, that has been used to research the teaching and learning of mathematics. A central research goal of a models and modeling perspective is the development of principles for the design of sequences of modeling tasks and for the teaching of such sequences. We extend our earlier research by elaborating how a model development sequence can be used to support students in developing models that are not only descriptive but also have explanatory power when connected to existing mathematical models. In so doing, we elaborate language issues about representations and context as well as the implementation strategies used by the teacher.

Keywords. Explanatory models; model development sequences; task design; teaching modeling.

Más allá de modelos descriptivos: Cuestiones de investigación sobre diseño e implementación

Resumen

En este artículo nos situamos en la perspectiva teórica 'models and modeling' para describir el diseño de una secuencia de tareas, conocida como secuencia de desarrollo de modelos, utilizada para investigar la enseñanza y aprendizaje de las matemáticas. Un objetivo central de la investigación desde esta perspectiva es desarrollar principios para el diseño de secuencias de tareas de modelización y para su enseñanza. En este trabajo nos proponemos ampliar nuestras investigaciones anteriores elaborando la forma en la que una secuencia de desarrollo de modelos puede servir para apoyar a los estudiantes al desarrollar modelos que no sean sólo descriptivos, sino también explicativos, especialmente cuando se conectan a los modelos matemáticos existentes. Para ello, elaboramos cuestiones de lenguaje sobre representaciones y contexto, así como estrategias de implementación del profesor.

Palabras clave. Modelos explicativos; secuencias de desarrollo de modelos; diseño de tareas; modelos de enseñanza.

1. Introduction and background

As evident from the special issues in ZDM (2006*(2-3)*, 2018*(1-2)*) and the published work from the biannual conferences of the International Community of Teachers of Mathematical Modelling and Applications, there are a plethora of perspectives on the meaning and role of mathematical modeling in the field of mathematics education. Generally, mathematical modeling is understood as connecting the realm of the real world and the realm of mathematics (Niss, Blum & Galbraith, 2007) with different emphases such as solving real world problems (Pollack, 1979), teaching and learning mathematics (Barquero, Bosch & Gascón, 2013), or the social and critical aspects of modeling (Rosa & Orey, 2015). Regardless of the perspective taken on mathematical modeling, there are many challenges for researchers, teachers, and students in the teaching and learning of mathematical modeling. In this paper, we draw on a models and modeling perspective on the teaching and learning of mathematics (Lesh & Doerr, 2003) to address three research areas in need of attention.

The first area of research is the study of how teachers can engage students in meaningful activities that move beyond descriptive models that apply already learned mathematics to real world phenomena (Doerr, Ärlebäck & Misfeldt, 2017). In discussing purposes of modeling, Niss (2015) contrasted descriptive and prescriptive modeling, where the latter focuses on designing, prescribing, organizing or structuring some aspect of an extra-mathematical domain. Other researchers like Hestenes (2010) have stressed that models can serve explanatory purposes. To develop models with explanatory power, the modeler needs to transcend the particulars of a given situation, to connect one's developing model with other already known models in related areas or disciplines. In this paper, we discuss the strengths and potential of a models and modeling perspective to the teaching and learning of mathematics for engaging learners in developing models that provide not only descriptive but also explanatory power about real world situations.

A second research area in need of attention is examining how teachers can support students in developing suitable language and representations to express both their mathematical ideas and their ideas about a particular real world situation. A models and modeling perspective emphasizes that students' mathematical ideas co-develop with their understanding of real world phenomena. Teachers need to support students in developing and using mathematical representations and the related language when learning mathematical content (Temple & Doerr, 2012). Students also need to develop the disciplinary language to describe and explain various real world phenomena. As we will illustrate below, a model development sequence on the rate of change of light intensity with respect to distance from a light source provides rich opportunities for student learning, but also challenges both students and teachers in developing fluency in using language and representations to describe and explain the underlying mathematical structure and physical phenomena of light intensity.

A third research area in need of attention is the implementation and teaching of modeling in classrooms. Despite many positive developments in research on the effectiveness of various approaches to modeling on student learning and material and support for teachers, widespread classroom implementation of modeling has progressed slowly (Blum, 2015). Although teaching mathematical modeling appears to differ in some significant ways from traditional approaches to teaching mathematics (Doerr, 2007; Doerr & Lesh, 2011), the teaching practices associated with mathematical modeling have received somewhat limited attention from researchers (Lingefjärd & Meier, 2010; Maass, 2011; Wake, 2011). The diversity and complexity of the multiple cycles of the development of students' models poses substantial knowledge demands on the teacher as teaching "becomes more open and less predictable" (Blum & Borromeo Ferri, 2009, p. 47). Responding to the openness of modeling tasks can be especially challenging for teachers in traditional classrooms (Maass, 2011), since such openness requires strategies to support students in making progress with the task without directly showing them how to resolve their difficulties (Lingefjärd & Meier, 2010). The teacher needs strategies to interpret the often unanticipated classroom events, select tasks to further the development of students' models, and engage students in the self-evaluation of their models without doing the task for them (Doerr, 2007). Characterizing such strategies and how teachers develop and learn them is a key research area.

In this paper, we address these three research areas from a models and modeling perspective on the teaching and learning of mathematics and provide insight from the design of, and our analysis of data from, a model development sequence focusing on how light intensity changes with respect to distance from a light source.

2. Theoretical framework

The models and modeling perspective is based on the design of activities that motivate students to develop the mathematics needed to make sense of meaningful situations. In this perspective, models are defined as "conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other systems." (Lesh & Doerr, 2003, p. 10). Students develop models as they engage in multiple activities of making sense of a particular context.

Much work done within the models and modeling perspective draws on *model eliciting activities* (MEAs) developed by Lesh and colleagues (Lesh, Hoover, Hole, Kelly & Post, 2000). MEAs have been used to investigate the development of students' models in a wide range of settings and contexts (Ärlebäck, Doerr & O'Neil, 2013). Solutions to MEAs go beyond what is traditionally required of ordinary textbook problems in that the solutions generally involve creating a process that can be shared with others and re-used in structurally similar situations. A single MEA, however, is seldom enough for a student to develop a generalized model that can be used and re-used in a range of contexts (Doerr & English, 2003; Lesh, Cramer, Doerr, Post & Zawojewski, 2003). Students need multiple and contextually diverse opportunities to explore and apply relevant mathematical constructs being learnt. A *model development sequence* (MDS) is a framework that can be used for the design and implementation of such instructional sequences (Lesh et al., 2003; Ärlebäck et al., 2013).

2.1. Model development sequences (MDS)

A model development sequence (MDS) begins with a *model eliciting activity* (MEA) that elicits students' initial models about a problem situation. The MEA is followed by one or more *model exploration activities* (MXAs) and *model application activities* (MAAs), as shown in Figure 1. MXAs focus on the underlying mathematical structure of the elicited model, on the strengths of various representations, and on deepening students' understandings of ways of using and interpreting representations. MAAs engage students in applying their model to new situations, often resulting in further adaptations to their models, and refining their language for interpreting, describing and explaining the context. Throughout the MDS, students are engaged in multiple cycles of descriptions, interpretations, conjectures and explanations that are iteratively refined while interacting with other students and participating in teacher-led class discussions (Doerr & English, 2003). The central mathematical goal of the MDS in this paper is describing, interpreting and explaining the behavior of the non-linear phenomenon of light intensity as it changes with respect to distance.

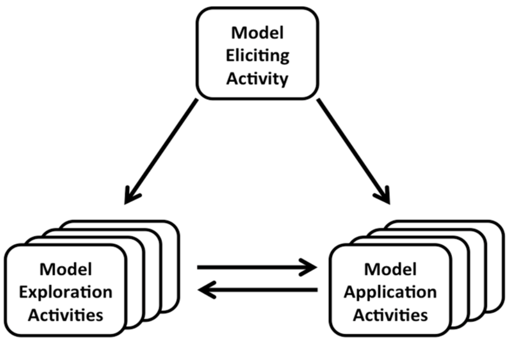


Figure 1. The general structure of a MDS

3. Aim and research goal

Our goal in this paper is to discuss research issues for design and implementation of a sequence of modeling tasks that goes beyond the use of mathematics to describe a real world phenomenon to also provide explanatory power for understanding why the phenomenon behaves as it does. To this end, we elaborate the design and implementation of a MDS aimed at supporting beginning university students in moving beyond a descriptive model in order to develop an explanatory model of the changes in light intensity with respect to the distance from a point source. We also focus on the role and function of the language students need to develop to understand the underlying mathematical structure of their models and to understand the physical context of light intensity. We examine the teaching strategies that supported the students in developing their models and the challenges faced by the teacher.

4. Methodology, setting, task design, data and analysis

4.1. Methodology and setting

This study used design-based research as an approach to study teaching and learning in the classroom setting (Cobb et al., 2003). The two authors and the teacher collaboratively used the models and modeling perspective and model development sequences (Lesh et al., 2003) to design a six-week summer course for students preparing to enter their university studies in engineering in the United States. We have previously reported on the design of this summer course (Ärlebäck, Doerr & O'Neil, 2013; Ärlebäck & Doerr, 2018) as well as on its effectiveness (Doerr, Arleback & Stanic, 2014). We report our analysis of a MDS used in the summer that was intended to support students in developing an explanatory model of light intensity. The entire summer course was organized around modeling tasks, and in the activities leading up to the MDS reported on in this paper, students regularly worked in small groups and had developed and become proficient with several related concepts and skills. In particular, students could (a) describe and analyze (using average rate of change) how position varies with time in the context of motion; (b) distinguish between linear and exponential functions; (c) analyze the average rates of change for an exponential function and (d) could transform an exponential function in order to stretch its graph and shift it vertically and horizontally (Ärlebäck, Doerr & O'Neil, 2013). The teacher had four years of experience teaching secondary and college students, and this was her third year teaching the summer course. There were 35 students in two sections of this third iteration of the course, all of whom had volunteered to participate in the study. Eleven of the students were female and 24 were male. All students had completed four years of study of high school mathematics; 21 students had studied calculus in high school and 14 had not studied any calculus. All students had taken a prior course in physics in their secondary education.

4.2. The design of model development sequences for explanatory models

The overall aim of the MDS was for students to develop a model of light intensity with explanatory power. The teacher needed to engage students in developing language and representations about their understandings of the phenomena of light intensity and light dispersion. In order to move beyond a descriptive model of changing light intensity, the students would need to explore a new mathematical structure (an inverse square proportional relation between distance from a point source and light intensity) and to connect that structure to another model, namely that of the spherical geometry. The central task for the students was to develop a model of the intensity of light with respect to the distance from a light source that not only described the situation, but also held explanatory power in terms of the physics of light intensity and the spherical geometry.

The MDS began with an MEA designed to elicit the students' initial models about (a) how intensity varies depending on the distance from a point source, and (b) how light disperses from a point source. Students were presented with the one-dimensional scenario of an approaching car and were asked to sketch graphs showing how the intensity of the car's headlights varied depending on the distance to the car and to describe how light disperses from a point source. This task revealed students' initial models about the changing intensity of light as a function of the distance from a light source. They were also asked to draw some representative rays of light leaving a point source. These images would ultimately sustain an explanatory model of the behavior of light intensity. The MEA was thus designed to provide a foundation for the teacher to engage students in activities that moved beyond simply finding a function of best fit for real light intensity data to understanding why light intensity behaved in a certain way.

The second task in the MDS was an MXA designed to engage students in comparing and contrasting their initial models of light intensity and dispersion in a small group setting and in a whole class setting. Students used representations (primarily graphs) and developed language to interpret those representations to express their models of how light intensity varies with distance from the source. Based on previous implementations of the summer course, we anticipated that the students' models would be descriptive and that many students would characterize light intensity as linearly dependent on distance, despite the fact that all of the students had taken a prior course in physics where they had studied the inverse square law that applies in this situation.

In the third task, an MAA, the students revised and adapted their initial models by collecting and analyzing 15 data measurements of light intensity at one cm intervals from a light source. From a design perspective, the MAA engaged students in self-evaluating the goodness of their linear and non-linear models in describing and explaining their real world data.

The fourth task, a second MXA, was designed to introduce, explore and connect the geometry of the sphere to the context of light intensity in order for the students' models to have explanatory power. This MXA connected back to the students' initial ideas about how light disperses from a point source in terms of light rays in order to support students in exploring and understanding an area model for how light intensity varies with distance from a light source. The teacher introduced a new representation of light intensity as a function of distance. This representation consisted of four 2D images with light intensity represented by number of dots per square inch at given distances (see Figure 4). Students were asked to determine the intensity at other distances from the light source. Students generated descriptive models of the data, but struggled with making a connection from this representation to the sphere surface area.

The fifth activity of the MDS, and the third MXA, was designed to connect the inverse square model based on the 2D images to the spherical geometry. A new representation was designed to support students in connecting their representations of light dispersion to the surface area of a sphere at different distances from the source (see Figure 7). This new representation, a larger 4-piece cardboard set of a light-emitting candle was used by students to physically enact how light rays emitted from a point source disperse and look at different distances from the source. The MDS ended (after about 6 hours work) with the students summarizing their analysis of their collected data and their understandings of their representations of light dispersion in a written report.

4.3. Data and analysis

The data sources included videotapes of all class sessions, written field notes and memos, class materials such as worksheets and a record of board work, the teacher's lesson plans and annotations made by the teacher during the lesson. Following each lesson, there was an audio-taped debriefing session with the teacher, which captured the teacher's reflections on the lesson and any changes to the plans for subsequent lessons. The MDS took place over three lessons; each lesson lasted one hour and 50 minutes. The analysis of the data took place in two phases. Consistent with the iterative approach of design-based research, the first phase of analysis took place during the three days of teaching. In this phase, our analytic approach was a collaborative examination of the teacher's actions in and interpretations of classroom events. The research team met with the teacher and discussed the tasks in the MDS, the progress of the class as a whole, and our observations about students' thinking about their mathematical representations for expressing their ideas. Analytic memos were written by members of the research team to document their emerging understandings of the teaching practices and observations about student learning. In particular, we attended to how the teacher supported the students in developing mathematical language about their representations and contextual language for explaining the behavior of light intensity, and to the strategies used by the teacher to address the challenges arising from the openness of the modeling tasks.

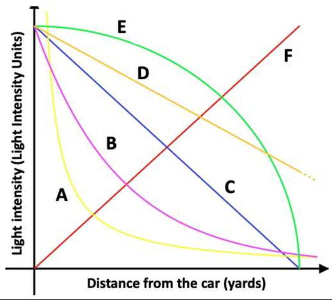
In the second phase of the analysis, we examined the classroom videos and written student work using grounded theory (Strauss & Corbin, 1998). Codes were developed to categorize the students' reasoning and answers on each of the questions in the activities of the MDS, focusing on capturing the students' models of how the light intensity varies with distance from the light source and how light disperses from a point source. The students' final lab reports were read and coded, focusing on how students' interpretations and descriptions of how the intensity varied with the distance from the light source and included an explanatory model. Our analyses in particular focused on how the students moved from a descriptive model of changes in light intensity to a model that had explanatory power when connected to the geometry of the sphere.

5. Results

As we report on how the students' initial ideas and descriptive models developed towards an explanatory model of light intensity as the MDS unfolded, we highlight the development of the students' capacities to interpret, describe and explain representations and the context of light dispersion from a point source, as well as the challenges for the teacher that surfaced and what strategies she used to address these.

5.1. The MEA. Students' initial models of light intensity and light dispersion

The students' initial models of the relation between light intensity and the distance from the light source were revealed in the MEA and are shown in Figure 2. Although all students had taken a course in physics nearly all of them (n=28, 83%) drew a linear relation between the intensity of a car's headlights and the distance from the car. All but one of these linear graphs (C and D in Figure 2) correctly show the light intensity decreasing, but incorrectly show it as decreasing at a constant rate. This is likely due to students assuming that the speed of the approaching car is constant, and confusing the constancy of speed with the constancy of the decrease in light intensity. The four students who drew graph A, with its asymptotic behavior at the y-axis, may have been drawing on their formal physics knowledge of the inverse square law for light intensity.



Graph	Number of students
A	4
B	1
C	22
D	5
E	1
F	1

Figure 2. Students' initial models of intensity vs. distance from light source (*n*=34)

When asked to "Draw some representative light rays leaving the light source" the students drew figures of light dispersing as cones-like rays, parallel rays or waves (see Figure 3). Students drawing a cone-like model (3A) have a potential rationale for explaining *why* the light intensity decreases when distance increases. In contrast, the parallel model (3B) implies a constant light intensity, independent of distance. One student drew the light dispersing as waves (3C). All students concluded that the light intensity would decrease as distance increase. The students expressing the model of light dispersion shown in Figure 3B did not express a conflict with their descriptive models of light intensity decreasing with respect to distance (as shown in Figure 2).

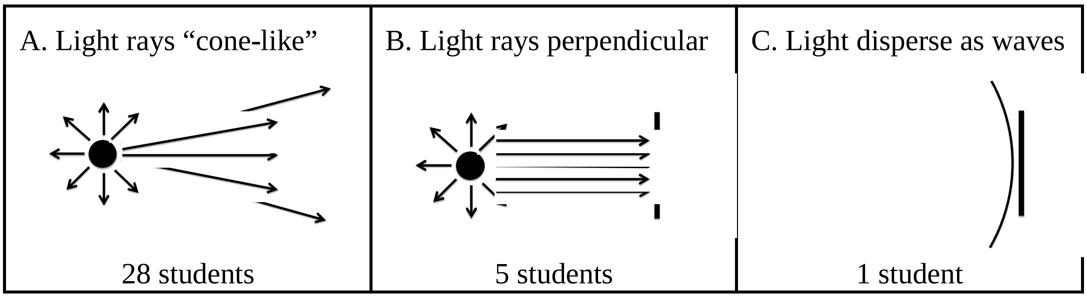


Figure 3. Students' initial models of light rays leaving a light source (total *n*=34)

5.2. The first MXA. Exploring student ideas

In the first MXA, the teacher wanted to explore the representations of changing light intensity and light dispersion that were elicited in the MEA. She asked the students about the meaning of their representations on the changes in light intensity: "Imagine the tail lights of a car moving at a constant speed away from you. Is the light intensity (1) fading at a constant rate, (2) fading slowly at first and then quickly, (3) fading quickly at first and then slowly, and (4) unsure." The teacher polled the students and displayed for them the results shown in Table 1. She routinely used the option of "unsure" to encourage students who see difficulties or ambiguities in a question to continue thinking (i.e., to keep self-evaluating their models), without being forced to choose a particular response.

Although all of the students had had a prior course in physics in secondary school, where the relation between light intensity and distance was studied, only 6 (17%) of the students correctly identified the rate at which the light intensity fades: quickly at first and then slowly. The majority of the students concluded that either the light faded at a constant rate (49% of the responses) or slowly at first and then quickly (20% of the responses). Several students (14%) expressed their uncertainty.

Rather than resolve the differences for the students, the teacher commented that she wanted to know from them "why did you choose the answer you chose?" To accomplish this, she arranged students in groups to discuss their answers. Each of these teaching strategies –eliciting their ideas with the initial question, asking them to engage in peer discussion, and listening to their reasoning– served to encourage the students to self-evaluate their responses to the question on changing light intensity and to develop their language about the underlying mathematical structure and the context of the phenomena.

Table 1. *Student responses to the rate at which light intensity changes*

Responses	*Number and percent response*
Fading at a constant rate	17 (49%)
Fading slowly then quickly	7 (20%)
Fading quickly then slowly	6 (17%)
Unsure	5 (14%)

After a few minutes of peer discussion in small groups and teacher listening, the teacher pulled the class together for discussion. Students of multiple groups were soon actively engaged in arguing whether or not the light was fading at a constant rate. Many of those who thought the rate was constant were arguing that it had to be constant because the speed of light is constant (”Isn't the speed of light constant?”) or because the car's speed was constant (“The car is moving away at a constant speed so I think the intensity decreases at a constant speed”). After a student refocused the discussion (“...Yes, but the speed of light is the travelling speed of light. We're talking light intensity which is what you see!”), many students offered ideas, explanations and experiences such as the far visibility of plane guiding lights in airports and relative motion in different reference frames (Ärlebäck & Doerr, 2015).

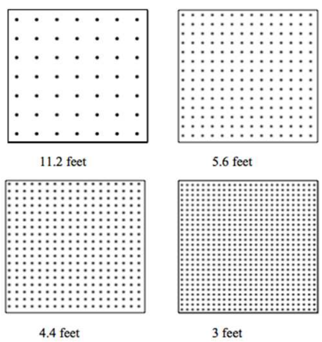
From a teaching perspective, modeling tasks that draw on students' thinking present a challenge to the teacher since it is not possible to fully anticipate what all the student ideas might be, what they would mean, and how they would relate to the central question about the rate of change of intensity of light. The discussion engaged students in expressing and developing language about the context (articulating the constancy of the car's speed and distinguishing between the speed of light and the intensity of light). The discussion was ended by the teacher, but not by drawing a conclusion for the students. Instead, the teacher continued to engage the students in self-evaluating their emerging models of light intensity by initiating the next task (”we are going to sort this out”) – an MAA of collecting light intensity data that would enable the students to resolve the question by applying their existing linear and non-linear models to actual data.

5.3. The MAA. Describing changing light intensity

In the MAA, the students worked in groups and collected 15 measurements of light intensity at one cm intervals from a light source. Using their calculators to graph their data, they quickly found that the light intensity was not decreasing linearly. In this way, they resolved for themselves the open issue from the MEA and the first MXA about how the light intensity was changing with respect to the distance from the light source. Based on prior implementations of the MAA, we knew that all students would be able to find a function that provided a reasonably good descriptive fit for their data, but that many of those functions would be exponential decay functions which had been studied earlier in the course. We anticipated that only a very few of the students would come up with inverse square functions. However, the inverse square function can provide the basis for an explanatory model of *why* light intensity changes as it does. This posed a new dilemma to the teacher: how to connect the students' descriptive models of light intensity to an explanatory model, drawing on the underlying mathematical structure of the spherical geometry, without directly presenting it to the students? In the next sections, we present the two MXAs developed by the researchers with the teacher to draw on students' images of light intensity and light dispersion, elicited in the MEA, to move beyond descriptive models to an explanatory model connected to the spherical geometry.

5.4. The second MXA. Explanatory representations of light intensity

The second MXA explicitly focused on students' images of light intensity. Following the data collection and the resulting graphs of the MAA, the teacher posed a question intended to further develop students' representations of light intensity. The students were asked to interpret a "dot" representation of intensity at various distances from a light bulb and to find the intensity at 2 feet and 6 feet from the bulb (see Figure 4). The students had difficulty understanding and using this dot representation of light intensity. The teacher then introduced the table representation shown on the right in Figure 4. The students recognized that an equation fitting this data would be useful in finding the intensity at two unknown distances; as one student commented, "we need an equation, but we don't know what it would be."



Distance (feet)	*Intensity (LIU)*
2	?
3	169
4.4	81
5.6	49
6	?
11.2	12.25

Figure 4. A dot and table representation of light intensity

At this juncture, the teacher polled the students to find out which parent graph they thought would best correspond to the table of data, thus revealing students' ideas about a possible symbolic representation, shown in Table 2.

Table 2. *Student responses to what parent graph corresponded to the dot data*

Parent graph	Number and percent response
$y = 1/x^2$	12 (34%)
$an\ exponential\ function$	9 (26%)
$y = 1/x$	7 (20%)
$y = 1/\sqrt{x}$	7 (20%)

The teacher asked the students to resolve the question of finding an appropriate equation for the data, another instance of the teacher's use of the self-evaluation strategy. Using their graphing calculators and working with partners, the students rejected $y = 1/\sqrt{x}$ and $y = 1/x$ as parent graphs. In one of the two classes, two pairs of students came up with two functions: $y = 1400/x^2$, $y = 715(0.58)^x + 12$, both of which fit the given data reasonably well (see Figure 5). This students' response had not been anticipated by the teacher in her planning and left her uncertain as to how to proceed.

Unlike in the first MXA, where the teacher knew that collecting and graphing data would enable the students to self-evaluate and revise their ideas about the linear or non-linear change in light intensity, the teacher was unsure how to engage the students in a critique of these two functions. Both functions provided a reasonably good description (or fit) of the data. The teacher juxtaposed the projection of the graph of each function and the data, shown in Figure 5, and turned the question over to the students, and instead of asking about best fit, the teacher asked “which [function] makes more sense?” In asking about making “more sense”, the teacher was intending to support the students in expressing how their emerging models are connected to the context (and physics) of light dispersion. However, many students were more focused on the best fit.

Several students saw the exponential function as “more accurate” and one student argued that the graph of $y = 1400/x^2$ would show up in the second quadrant and hence “wouldn’t be accurate to the data.” Still uncertain as to how to engage the students in a critique of these functions, the teacher re-polled the students as to which parent function would best model the data. This time, the students shifted to an exponential function (86%) rather than an inverse square function (14%). Re-polling the students gave the teacher some additional time to think about how to proceed; during which she quickly conferred with a member of the research team who suggested focusing students’ attention on the long-term behavior of both functions. The teacher asked the students to compare the long-term behavior of the two functions to build on their intuitions that the intensity of light should get “closer and closer to zero as we get out further and further.” This led students to reject the exponential decay function, which did not approach zero. This engaged students in developing language to connect their understanding of the graphical representation to the physical phenomenon of changing light intensity.

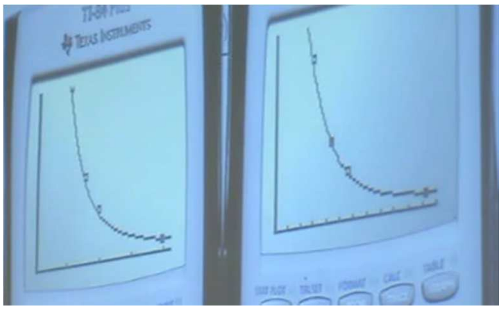
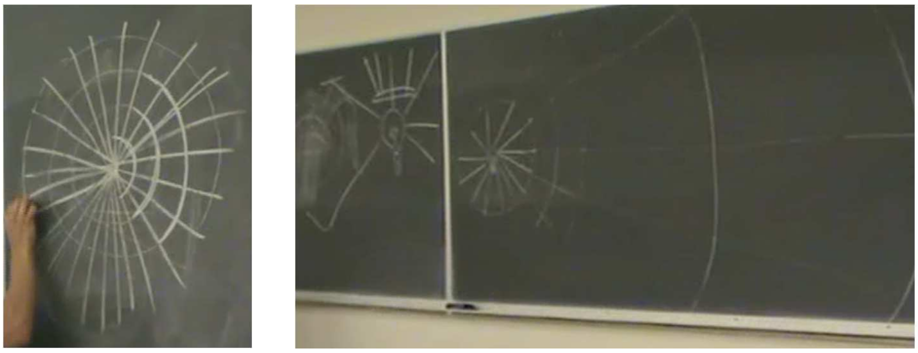


Figure 5. $\mathrm{y} = 1400/\mathrm{x}^2$ vs. $\mathrm{y} = 715(0.58)^{\mathrm{x}} + 12$

Knowing how to further the students’ own thinking, in the moment of teaching, was neither obvious nor easy from the perspective of the teacher. The teacher ended the second MXA by focusing students’ attention on the critical question of *why* an inverse square representation was reasonable. She said that the “thing I want you to think about is ‘why’? Why does this inverse square function make sense in this [physical] situation?” To answer this question, the students would need to further develop their representations of light intensity. At this point, the students had not connected the “dots” representation introduced by the teacher to their images of light dispersion. Importantly, the students’ representations of light dispersion still needed to be connected to the spherical geometry.

5.4. The third MXA. Exploring and connecting representations of light intensity

In the next lesson, the teacher drew on students’ initial models of light dispersion from the MEA and again focused the students’ attention on making sense of how light intensity is changing with respect to distance. She began by asking the students about "why it [an inverse square function] would make sense?" and "How do you think about light coming out of a light source?" Several students responded with ideas about light going in "all directions equally," "travels evenly," and "in all directions". The teacher pursued these ideas and encouraged the students to externalize their representations by asking "what image do you think of when you think of all directions equally?" An important shift in the discussion occurred as one student offered an image of rays: "near the point source, they are really close. But then they go apart. … As they [the rays] get farther from the point source, they get farther from each other. … And that is why the intensity is less." This student had developed language that moved beyond describing the decrease in intensity with increasing distance to offering a justification about why this is so, connecting the change in light intensity with the changing distribution of light rays hitting a line segment of a given length on different distances from the light source. Several other students offered an image of "spheres" moving out from the light source, connecting these two images of light dispersion to the sphere geometry.



(A) (B)

Figure 6. Students' images how light comes out of a light source

The discussion went on as the teacher built on these images, with student generated representations of enlarging 2D-representations of spheres and re-visiting the dot-based representation of intensity (see Figure 6). As the discussion continued, she recalled for students the formula of the sphere surface area. To support students in connecting their 2D models of light dispersion from a point source to a 3D-spherical model, and based on the their difficulties in making this connection in the second MXA, the teacher had designed a 3D representation (see Figure 7) that she used to engage students in acting out and visualizing the light intensity phenomena. The students moved from the dots representation, to a table representation (see Figure 4), to a symbolic representation, to images of rays and spheres, and to the formula for the surface area of a sphere.

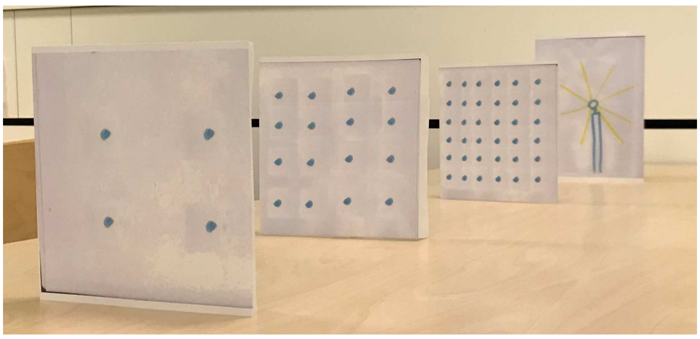


Figure 7. A 3D-model of light dispersion from a point source

At this juncture, the teacher was again faced with deciding what to do next. Rather than guide the students through bringing these ideas together, the teacher turned these elements of representing their model of changing light intensity back to students, asking them to think "about all these ideas and put some of this together … One of the questions is why do you think light behaves this way [as an inverse square]?" She encouraged them to use the representations that had been discussed as "ways to reason about that" and thus develop and refine their representations of changing light intensity. The final task for the students was to complete a written report that summarized their findings about how light intensity changes with respect to distance from a light source. In the report the students described and explained their procedures of collecting and analyzing data, fitting a function to their data, generating and describing various graphs, and explaining why the function and graphs made sense in the context of the phenomena.

Towards the end of the MDS, students were beginning to orally express explanations for why the inverse square behavior of light intensity relates to spherical geometry. However, the lab reports showed that most students in their written accounts did not move beyond describing their data of the phenomena of light dispersion from a point source. Only six of the 19 reports attempted to explain why the phenomena qualitatively is explained by an inverse square function, and five did so successfully. In five of the six reports the students drew either 2D or 3D representation mirroring those in Figure 6 and 7. All the students had been taught the inverse square behaviors of light intensity in prior physics courses, and some of the students could recall it. However, our data provide no evidence that any of the students had understood why this model explained the behavior of light intensity, but many students started to express their understanding and representations of the inverse square area model towards the end of the MDS.

6. Discussion and future research

The MDS analyzed in this paper was the first experience for the students in developing an explanatory model and presented them with new expectations. Our results show that the students, having found an exponential function fitting the data, were satisfied having developed a descriptive model answering the *how* question and did not see any need for an explanatory model. To answer the *why* question requires an explanatory model, but the students struggled with relating the context of light intensity to the structure and representational aspects of the underlying spherical model. Students had difficulties in shifting to a 3D model based on the 2D dots representation the teacher introduced, but by supporting the students to develop language around the phenomena, visualizing their ideas, and introducing a new representation, the teacher facilitated the students in expressing how their emerging models connected with the spherical model. The MDS provided the researcher and teacher with a tool to structure the students' learning based on what was produced in class in terms of the design of the tasks. From a models and modeling perspective and analogously to research and study paths (Barquero, Bosch & Gascón, 2013), the sequences of questions motivating and guiding the students' work was designed to create a need for an explanatory model. However, given that students need multiple opportunities to develop a generalized model that can be re-used in a range of contexts (Doerr & English, 2003; Lesh et al., 2003), we suspect that engaging students in one or more MAAs from other contexts having the same underlying inverse square structure (such as two body gravitational forces, two point charges or sound intensity) would have further manifested the explanatory power of the spherical model for the students. This, however, needs to be investigated further.

Our results overall highlight the importance for (1) the activities students work with to be dynamic, motivated by both *how* and *why* questions, and engaging students in self

-evaluating their emerging models; and (2) the teacher to support students in developing language and representations when learning mathematical content to facilitate the development of explanatory models. The MDS shows that engaging students in self-evaluation can be built into the MDSs by the design of the MEA and be facilitated by suitable teaching strategies. These strategies, accompanied by MDS task design, provide a feasible and productive way for the teacher to respond to the openness of modeling activities discussed by Maass (2011) and Lingefjärd and Meier (2010). The first MXA illustrated how the teacher can reveal and use students' thinking and ideas by focusing on students' language use and representations so that students could self-evaluate and further develop their models by sharing them with other students. The design of the MAA engaged students in collecting data that enabled them to self-evaluate the goodness of their emerging models about how light intensity changes with respect to distance from a light source. It resolved the teaching dilemma that would otherwise have confronted the teacher as to how to resolve students' conflicting ideas. However, to move beyond simply descriptive models of best fit, the design of two additional model exploration activities enabled the teacher to support students in exploring and interpreting area based representations of light intensity and dispersion that were connected to their own images of light and to the spherical geometry, and ultimately provided an explanatory model for changes in light intensity. These aspects of designing and teaching using model development sequences continue to need attention from research, and a key goal for such research should be to focus on formulating, testing and evaluating design principles and implementation strategies for model exploration and model application activities that move students' models from descriptive to explanatory.

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Moving beyond descriptive models: Research issues for design and implementation

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We draw on a models and modeling perspective on the teaching and learning of mathematics and report on the design, analysis, and results of a model development sequence focusing on how light intensity changes with respect to distance from a light source. Working collaboratively with the teacher and using a model development sequence as a framework for task design, we highlight how the tasks in the sequence supported the students' development of a model that is not only descriptive but also has explanatory power. This sequence was implemented by an experienced teacher with 35 students as part of a six week summer course, preparing them for their first year of university studies in engineering. The model development sequence starts with a model eliciting activity on how light intensity changes with respect to distance and on how light disperses from a point source. The students' ideas elicited by this task are revised and developed in subsequent model exploration and model application activities. Many students had an initial model of linear decrease; other students represented light intensity as decreasing non-linearly drawing on familiar models of exponential decay. In addition to supporting students in developing language about their representation and the context of light intensity, the teacher encouraged them to self-evaluate their developing models of changes in light intensity. Thus, the teacher engaged students in a model application activity to evaluate their models of linear or exponential decrease by collecting and analyzing light intensity. Many students found an exponential function that provided a reasonably good descriptive fit of their data and a very few students found an inverse square function to fit their data. Only the latter function can provide the basis for an explanatory model of *why* light intensity changes as it does. To move to an explanatory model, the teacher implemented two additional model exploration activities designed to connect to students' earlier ideas and representations about light dispersion as rays from a point source and to connect to an existing model, namely the sphere geometry. By exploring and interpreting area based representations of light dispersion connected to students' images of light and to the sphere geometry, students ultimately developed an explanatory model for changes in light intensity based on an inverse square model. In order for students to move beyond descriptive models and to develop explanatory models, our results highlight the importance of (1) the activities to work with to be dynamic, motivated by how and why questions, and engaging students in self-evaluating their emerging models; and (2) the teacher to support students in developing language and representations when learning content in order to promote explanatory models. The model development sequence presented in the paper shows that engaging students in self-evaluation across the model development sequence is supported by task design and teaching strategies. These strategies, accompanied by model development sequence task design, provide a feasible and productive way for the teacher to respond to the openness of modeling activities and to support students in developing explanatory models.