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NOTE FOR THE THIRD HILBERT PROBLEM: A FRACTAL CONSTRUCTION

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Abstract. Hilbert's Third problem questioned whether, given two polyhedrons with the same volume, it is possible to decompose the first one into a finite number of polyhedral parts that can be put together to yield the second one. This finite equidecomposition process had already been shown to be possible between polygons of the same area. Dehn solved the problem by showing that a regular tetrahedron and a cube with equal volume were not equidecomposable. In this paper, we present an infinite fractal process that allows the cube to be visually reconstructed from a tetrahedron with equal volume. We have proved that, given two tetrahedrons with the same volume, the first one can be decomposed into an infinite number of polyhedral parts that can be put together to yield the second one. This process makes it possible to obtain the volume of a tetrahedron from the volume of the parallelepiped, without the use of formulas or the Cavallieri Principle.

1. Introduction

In 1900, Hilbert proposed 23 problems that opened research lines in different branches of Mathematics [8]. Some of these yet unsolved problems remain as a challenge for current mathematicians. On the other hand, the Third problem was solved before Hilbert's lecture was delivered [3]. However, the interest on this matter remains.

In particular, the Third problem, in which prestigious mathematicians such as Gauss had already shown interest [6], stated that given two polyhedrons with the same volume, it is possible to decompose the first one into a finite number of polyhedral parts that can be put together to yield the second one. Dehn, [2], one of Hilbert's students, provided a negative answer

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to this problem and proved that a regular tetrahedron and a cube with the same volume are not equidecomposable. In other words, the former cannot be decomposed into finitely many polyhedral pieces that can be put together to obtain the latter. The main idea behind Dehn's proof was to define an invariant that remains unchanged in the process of decomposition into finite polyhedral pieces, and to show that the cube and the regular tetrahedron had different values of the invariant. In 1965, Sydler, [6], proved that two polyhedra are equidecomposable if and only if they have the same volume and the same Dehn invariant.

The above highlights the difference between the Euclidean plane and the Euclidean space, as the Wallace-Bolyai-Gerwein theorem [1] states: *Polygons are equidecomposable if and only if they have the same area.* Theile [7] has suggested that on Hilbert's view, the need of infinite processes could contradict the metaphysical principle that the Universe is governed in such a way that a maximum of simplicity and perfection is done. However, infinite processes can also result of great simplicity and perfection, as happens with fractals, which exhibit approximations to infinity of great beauty and mathematical regularity [5].

In this paper we describe an infinite fractal process to decompose a regular tetrahedron into infinite pieces that can be put together to form a cube of the same volume. The process is repeated successively in different scales showing a fractal nature.

This process is generalizable to any tetrahedron, obtaining an original visual demonstration of an "infinite" version of the Third problem for tetrahedrons: *given two tetrahedrons with the same volume, it is possible to decompose the first one into a finite number of polyhedral parts that can be put together to yield the second one.*

To ease the visual comprehension of this process without resorting to the use of formulas of volumes or the Cavalieri principle, the decomposition of a particular tetrahedron, the trirectangular tetrahedron, is initially shown. Subsequently the generalization to any tetrahedron is presented.

Before the main results we introducing some concepts and notations in order to clarify the paper.

2. Concepts and notations

- A trirectangular tetrahedron is a tetrahedron with all three faces angles at one vertex are right angles. The three edges that meet at the right angle of a trirectangular tetrahedron are called legs.
- C_n denotes a cube of edge length n.
- T_n denotes a trirectangular tetrahedron of equal legs of length n.
- P_n denotes the regular tetrahedron obtained by joining four of the 8 vertices of the cube C_n. More specifically, picking every other vertex of a cube so that no two are joined by an edges but any pair is joined by a diagonal of the cube's face we can form a regular tetrahedron.

3. Main result

Lemma 3.1. *A Fractal Decomposition of a trirectangular tetrahedron in a cube is possible.*

Proof. The three diagonals of the faces of a cube that meet in a vertex comprise a trihedron angle which is the tip of a regular tetrahedron. By joining the other vertices of these diagonals with those from the other sides of the cube, we obtain a tetrahedron with equal angles and legs of the same length, the diagonal of the face of the cube C_1, therefore it is a regular tetrahedron P_1 (Figure 1). □

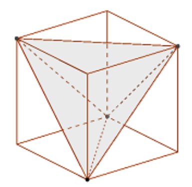
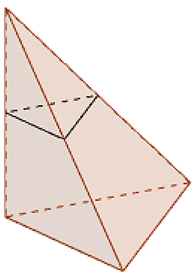


Figure 1. Regular tetrahedron P_1 within the cube C_1.

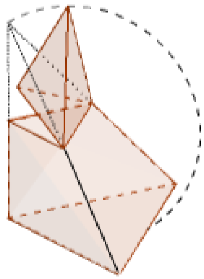
Outside the tetrahedron there remain four identical figures, namely triangular pyramids, with 3 faces comprising rectangular and isosceles triangles and one face comprising an equilateral triangle (the face of the tetrahedron). So, C_1 is decomposed in four trirectangular tetrahedrons T_1, and a tetrahedron P_1 (Figure 2).

FIGURE 2. Decomposition of C_1 in P_1 and four T_1.

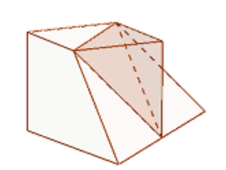
We start the fractal decomposition of the trirectangular tetrahedron T_1. In order to determine the portion of the cube occupied by T_1, we will truncate it by means of half-planes parallel to one of its bases. A new trirectangular tetrahedron then appears, $T_{\frac{1}{2}}$, in the upper part of the truncated one (Figure 3a). Letting such $T_{\frac{1}{2}}$ rotate (Figure 3b), with axis equal to the cutting line between the plane and the equilateral face we obtain three pieces: a cube $C_{\frac{1}{2}}$ and two $T_{\frac{1}{2}}$ adhered to it (Figures 3c, 3d).



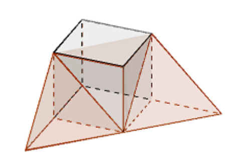
(A) Step 1



(B) Step 2



(C) Step 3



(D) Step 4

FIGURE 3. From left to right, top to bottom: Truncation of T_1 to obtain one cube $C_{\frac{1}{2}}$ and two $T_{\frac{1}{2}}$.

Repeating the truncation process for $T_{\frac{1}{2}}$, two cubes $C_{\frac{1}{4}}$ and four new $T_{\frac{1}{4}}$ are obtained (Figure 4).

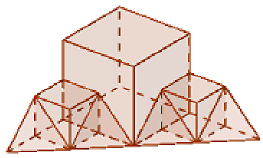


FIGURE 4. Truncation of the two $T_{\frac{1}{2}}$.

This process can be successively iterated as it is shown in Figure 5 obtaining a series of cubes and trirectangular tetrahedrons that can be summarized in Figure 6.

(A) Truncation of the 8 $T_{\frac{1}{4}}$. (B) Recurrent process

FIGURE 5. Recurrent process of the truncation.

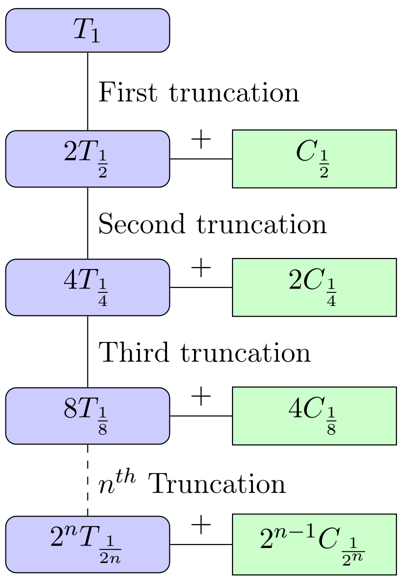
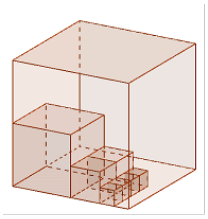


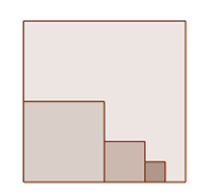
FIGURE 6. Number of trirectangular tetrahedrons and cubes obtained after each truncation.

By conveniently placing these cubes above half of the original cube, we obtain a sequence from which two views are shown (perspective and lateral) in Figures 7a-7b.

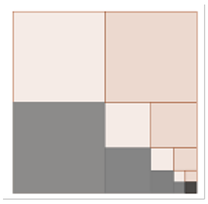
According to Nelsen's in [4], and by comparing quantities, the sum of the surface quantities of the square faces is equal to one-third of the surface quantity of the square (Figure 7c).



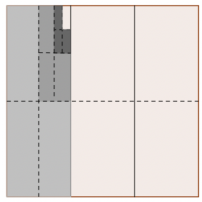
(A) Figure 7a



(B) Figure 7b



(C) Figure 7c



(D) Figure 7d

FIGURE 7. From left to right, top to bottom: location of the cubes in the truncation in order to obtain the volume.

We also note that this follows from the geometric series of ratio $\frac{1}{4}$ and first term $\frac{1}{4}$ and

$$\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n = \frac{1}{3}.$$

This comparison allows us to say that the sum of the volume magnitudes of all the cubes in which we decomposed T_1 is one-third of half of the cube.

A new placement of these cubes (Figure 7d) divided into three pieces allows the construction of a prism of equal height of T_1 and rectangular base of dimensions of the base $\frac{1}{3}$ and $\frac{1}{2}$. To do so, each square of Figure

7c is divided into three equal rectangles. Two of them are left together and the third is placed on the first of those mentioned previously, as it can be perceived in the recursive process of Figure 7c.

Visualizing this construction in perspective as it is shown in Figure 7a, it is perceived that T_1 has decomposed into a right prism of $\frac{1}{6}$ of the volume of C_1. As any two given prisms of equal volume are equidecomposable [1], we have proved the following result.

Proposition 3.2. *T_1 has been decomposed into an infinite fractal process into a prism of which the volume is $\frac{1}{6}$ the volume of the cube with its same edge (Figure 8).*

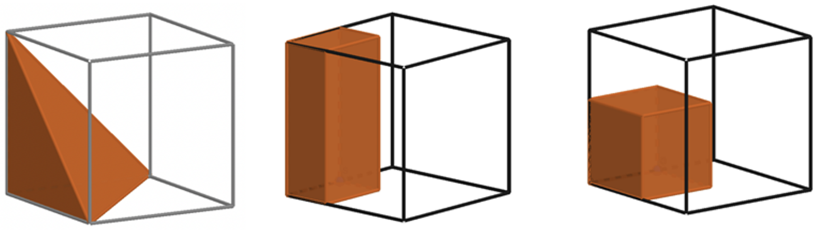
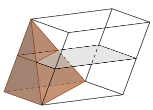


FIGURE 8. Equidecomposition from Ttri to a cube.

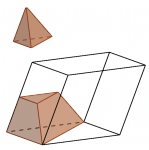
Now we can generalize the result for any tetrahedron.

Theorem 3.3. *A fractal decomposition of a tetrahedron in a cube is possible.*

Proof. The above fractal process is generalizable to any tetrahedron. We show with images the sequence of steps. The only difference is that in the truncation process it is necessary to perform a symmetry (Figures 9–10). □

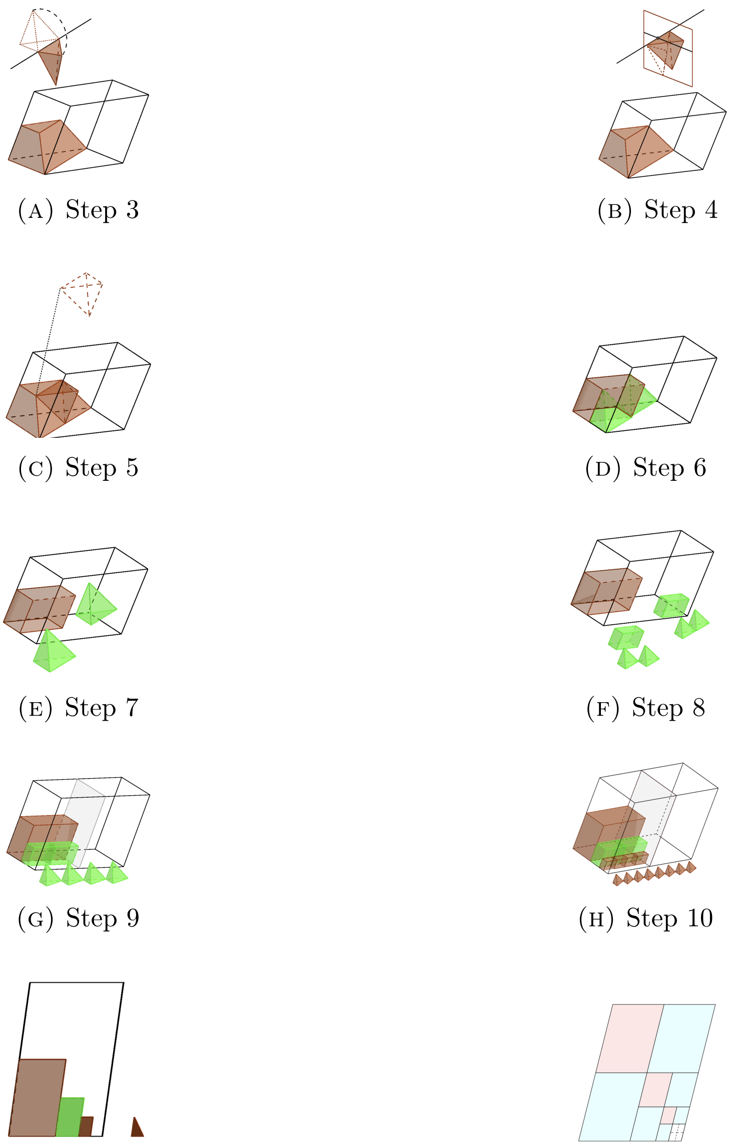


(A) Step 1



(B) Step 2

FIGURE 9. Fractal process: Steps 1 and 2.



(A) Step 3 (B) Step 4

(C) Step 5 (D) Step 6

(E) Step 7 (F) Step 8

(G) Step 9 (H) Step 10

(I) Step 11 (J) Step 12

FIGURE 10. Fractal process: Steps 3 to 12.

As in the T_1 case, these parallelepipeds divided into three pieces can be repositioned to obtain a new parallelepiped of volume 1/6 of the original parallelepiped.

Finally, by equidecomposition of prisms of equal volume, we obtain a cube of equal volume to the original tetrahedron, and therefore of volume equal to $1/6$ of the parallelepiped that contains the original tetrahedron. Therefore, we can formulate the following result:

Theorem 3.4. *Given two tetrahedra of equal volume, it is possible to decompose the first one into an infinite number of polyhedral pieces that can be put together so as to yield the second one.*

Proof. The proof is based on decomposing the first tetrahedron into a prism with volume $1/6$ of the parallelepiped and then repeating the reverse process to reconstruct the other tetrahedron. Furthermore, as a consequence of this process, what is obtained is that the volume of a tetrahedron is $1/6$ of the parallelepiped that contains it. An infinite process has been necessary, but neither the formulas nor the Cavalieri Principle have been used. This construction can also be used for the case of the regular square pyramids, because they can be decomposed in 4 T_1 (Figure 11). □

FIGURE 11. Regular square pyramid (left) and comprising four trirectangular tetrahedrons (right).

As pointed out by the Dehn Theorem answering to Hilbert's problem, the comparison of the volume of a pyramid requires an infinite process. The explained procedure makes an iterative comparison based on self-similarity, which allows to obtain the ratio between the volumes of the tetrahedron and the regular square pyramid, and between both and the cube.

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