Steering between skills and creativity: a role for the computer?

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Resumo
O conhecimento matemático de crianças e adultos parece entrar em crise com freqüência – crise de habilidade ou de criatividade. Embora cauteloso quanto a soluções universais, este artigo sugere, como uma saída para esse impasse, a inclusão, no currículo, de atividades computacionais planejadas para auxiliar os alunos a associar seu conhecimento informal às práticas convencionais da matemática.
Palavras-chave: prova, currículo de matemática, computador na educação matemática.

Abstract
Children's and adults' mathematical knowledge frequently appears to be in a state of crisis - a crisis of skills or a crisis of creativity. While cautioning against universal solutions, this paper suggests that the incorporation into the curriculum of computer-based activities designed to help students connect their informal knowledge with the conventional practices of mathematics is one way out of this impasse.
Key words: *mathematics curriculum, computer in mathematic education.*

BACKGROUND

Children's and adults' mathematical knowledge frequently appears to be in a state of crisis – a crisis of skills or a crisis of creativity. In UK and USA, there are now waves of enthusiasm for basic skills, mental arithmetic, and target setting. A huge multi-million pound National Numeracy project is now underway in UK and we await the final publication of our Government's Numeracy Task Force. In its preliminary report, (Numeracy Matters, 1998), the TIMSS studies (see for example, Harris, Keys & Fernandes, 1997) were cited as one reason for this new focus.

Studies comparing England's performance in mathematics with other countries show this country to be performing relatively poorly in comparison with others. For example, evidence from the Third International Mathematics and Science Survey (TIMSS) indicates that our Year 5 pupils (aged 9 and 10) are amongst the lowest performers in key areas of number out of nine countries with similar social and cultural backgrounds. (p. 8)

At the same time, the news from the Pacific Rim reports rather different pressures for change. For example, Lew, (in press) describes Korea, a country which scores very highly on most international comparisons of mathematics attainment, as being in "total crisis" in mathematics. He illustrates graphically how most students seem quite unable to relate their well-developed manipulative skills to the real world. Lew argues that 'the direction of the mathematics curriculum in Korea should change from emphasis on computational skills and the 'snapshot' application of fragmentary knowledge to emphasis on problem-solving and thinking abilities'. Similarly, Lin and Tsao present a picture of test obsession in Taiwan where college entrance examinations dominate students' (and parents') lives (Lin and Tsao, in press). Both of these countries are planning to encourage more 'open' curricula to include opportunities for mathematical creativity: that is, adapt their curricula to be more like those now being vilified in UK and USA!

Other data from TIMSS suggest that English children are comparatively successful at applying mathematical procedures to solve practical problems and are generally positive about mathematics. Is it possible to retain these strengths while at the same time consolidating arithmetic skills and developing the ability to construct rigorous and systematic arguments? (The latter area is one in which we have shown out students to be surprisingly weak - see Healy and Hoyles, 1998). The challenge for the international mathematics education community perhaps appears at first sight to be the design of a globally-effective balanced curriculum. From a UK perspective, this would build on the wealth of informal mathematical knowledge students bring to school, while at the same time drawing their attention to mathematical structures and properties and introducing them more systematically to mathematical vocabulary. The mathematical curriculum of the next millennium should harness children's motivation without losing their mathematics - and we envisage that the computer might offer just the context to help us to do this.

A ROLE FOR THE COMPUTER

I was inspired in the early 80's by Seymour Papert's radical vision of a mathematics that was playful and accessible, but at the same time rigorous and serious. We[1] dreamed (and still do!) of children actively expressing mathematics in different ways. We wanted children to learn by conjecture, reflection on feedback and debugging, as part of their own meaningful projects that required planning, sustained engagement with mathematical ideas and the bringing together of a range of skills and competencies. Logo was the vehicle or the catalyst for many of us to try to achieve those dreams. In doing this work, our eyes were opened to students' strategies and potentials - computer interaction was a window on to possibilities, an environment to illuminate pupil meanings and interpretations (Hoyles 1985, Noss and Hoyles, 1996).

Since that time, we have designed a range of microworlds around different 'open' software and have further developed the notion of technology as a means by which knowledge can be concretised and connected. We have also undertaken more systematic investigation of the nature of the child's activity and how it can be better understood and guided (Hoyles and Noss, 1992). Inevitably the boundary of what is and is not mathematics has been explored (see Papert, 1992): some say that working experimentally with the computer is mathematics, some that it is not, and many are not sure. The software may have changed but the issues have not and the location of this boundary is still a matter of hot dispute, brought even more into focus in an international forum such as this.

If we want to design investigative environments with computers that will challenge and motivate children *mathematically,* we need software where children have some freedom to express their own ideas, but in ways constrained so as to focus their attention on mathematics. Are there lessons to be learned from all the work that has been done with these sorts of environments over several decades? What do we actually know about how children can better learn mathematics with technology?

Mathematics comprises a web of interconnected concepts and representations which must be mastered to achieve proficiency in

1 We, in this text, refers to my close colleague in Mathematical Sciences at the Institute of Education, Richard Noss.

calculation and comprehension of structures (for elaboration of this theoretical framework, see Noss and Hoyles, 1996). Mathematical meanings derive from connections - intramathematical connections which link new mathematical knowledge with old, shaping it into a part of the mathematical system; and extra-mathematical meaning derived from contexts and settings which may include the experiential world. Yet how are these meanings to be constructed? How is the learner to make these connections? To what extent can the software tools encourage this process of meaning-making and connection-making?

A critical weakness of many mathematical learning situations has been the gap between action and expression and the lack of connection between different modes of expression. Does technology magnify these problems of fragmentation and lack of connection or help to solve them? Clearly it depends how the technology is used; a lesson certainly worth reiterating! Technology does nothing in and of itself! Over many years, our central research priority has been to find ways to help students build links between seeing, doing and expressing (see for example, Noss, Healy & Hoyles, 1997). We have shown that technology can change pupils' experience of mathematics but with several provisos: the users of the technology, (teachers and students), must appreciate what they wish to accomplish and how the technology might help them; the technology itself must be carefully integrated into the curriculum and not simply added on to it (see Healy and Hoyles, in press), and most crucial of all, the focus of all the activity is kept unswervingly on mathematical knowledge and *not* on the hardware or software.

COMPUTERS AND THE CURRICULUM

To date, work with computers in mathematics education has largely been concerned with construction and the potential of software to aid the transition from particular to general cases - specific instances can be easily varied by direct manipulation or text-based commands and the results 'seen' on the computer screen (see, for example, Laborde and Laborde, 1995). Yet, even if students develop a sense of how certain 'inputs' lead to certain results, there remains the question of how to develop a need to explain, a need to prove, as part of, rather than added on to, this constructive process. In countries like UK, where proof has all but disappeared from the curriculum, this issue must be addressed urgently if we are to avoid limiting the mathematical work for most children by the introduction of computers. If we fail, the majority of our students will simply be subjected to even more convincing empirical argument - for example, using powerful dynamic geometry tools simply to measure, spot patterns, and generate data.

There is an alternative which we are in the process of investigating. We have designed activities where, through computer construction, students have to attend to mathematical relationships and in so doing are provided with a rationale for their necessity. Thus, the scenario we envisage is one where students construct mathematical objects for themselves on the computer, conjecture about the relationships between them, and check the truth of their conjectures with the tools available. This forms part of a teaching sequence which also includes reflection away from the computer guided by the teacher, and the introduction of mathematical proof as a particular way of expressing one's convictions and communicating them to others. It is in this way, we suggest, that constructing and proving can be brought together in ways simply not possible without an appropriate technology: formal proof is simply be one facet of a proving culture, revitalised by the 'experimental realism' of the computer work (Balacheff and Kaput, 1996).

Over the last few years. Lulu Healy[2] and I have devised algebra and geometry teaching sequences which follow these criteria. Our activities were developed after analysing students' responses to a nationwide paper-and-pencil survey to assess students' conceptions of proving and proof (Healy and Hoyles, 1998). This questionnaire was completed by 2,459 fifteen year-old students of above average mathematical attainment from across England and Wales. Each teaching sequence was designed 'to fit into the curriculum' and to plug at least some of the gaps our survey had revealed in the understandings of our students. Overall 18 students from three schools have worked through the sequences, each of which took nearly 5 hours of classroom contact supplemented by homework.

I will now present snapshots from the case studies of two students who engaged in these sequences. The first case study illustrates the gains

2 ESRC project, *Justifying and Proving in School Mathematics*, Ref R000236178. I wish to acknowledge the central work of Lulu Healy in all aspects of this project.

that can be made by connecting skills to creative exploration through computer interaction; the second points to potential pitfalls in planning 'the best' mathematics curriculum incorporating technology.

TIM: MAKING THE STEP TO *EXPLAINING* IN ALGEBRA

Tim was a quiet and diligent student who knew about proof as something that involved verification and explanation, only recognised it in the context of algebra – a natural consequence of our curriculum with its emphasis on generalising and explaining number patterns (see Figure 1).

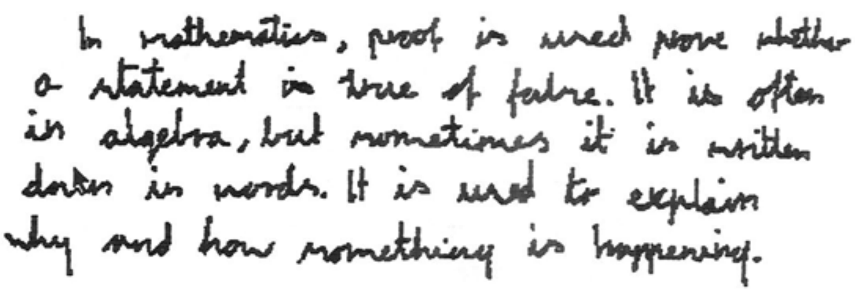
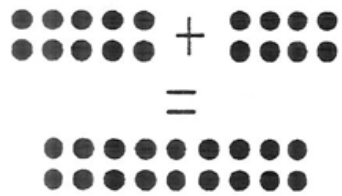

In mathematics, proof is used prove whether a statement is true of false. It is often in algebra, but sometimes it is written down in words. It is used to explain why and how something is happening.

Figure 1: Tim's initial view of proof

It was also clear from Tim's choices in the questionnaire, that he had a preference for visual argumentation: he evaluated the visual 'proof' by Yvonne in exactly the same way as a 'correct' formal algebra proof, and when asked about this, it was clear he 'saw' the general structure *through* this particular visual example (see Figure 2).

Yvonne's answer



So Yvonne says it's true.

Yvonne's answer	agree	don't know	disagree
Has a mistake in it	1	2	③
Shows that the statement is always true	①	2	3
Only show that the statement is true for some even numbers	1	2	③
Shows you why the statement is true	①	2	3
Is an easy way to explain to someone in your class who is unsure	①	2	3

Figure 2: Tim's evaluation of a visual proof

In the first algebra session of our teaching sequence, students are introduced to our microworld, *Expressor,* in which they build 'matchstick' patterns of number sequences by constructing simple programs. They are encouraged to connect their computer constructions with corresponding mathematical properties, and find a general formula for the number sequence explaining why any conjecture is true or false by reference to computer feedback *and* to the mathematical structures they have constructed. Similar work with more complex number sequences is undertaken in the third session.

Tim found this work of generalising through programming both engaging and challenging – in fact, he described it as the most enjoyable parts of our teaching. He also saw a strong connection between proving and his computer work.

> T – I liked the programming stuff – that helped [to write proofs] because it sort of showed how it was constructed so... It helped prove because it showed you how they were made... How that construction was made step by step.

In the second session, students are introduced to writing formal algebraic proofs and helped to 'translate' their Logo descriptions of the mathematics structures into algebra. They are also taught how to construct deductive chains of argument; systematically to start from the properties they had used in their constructions and to deduce further properties. Both of these activities are unfamiliar to UK students.

Let me give an example. Students are asked to investigate the properties of the sums of different sets of consecutive numbers. They construct by programming a visual representation of numbers as columns of dots (shown in Figure 3 below). Students can for example move the bottom right dot to the bottom left, see that it would 'even up' the three columns, and convince themselves that the conjecture that the sum of 3 consecutive numbers is divisible by 3 is always true.

Although these moves can be achieved by, for example, using counters, in *Expressor,* the visual arrangement has a simultaneous 'algebraic' description which is constructed by the children. In Fig. 3 a program `col`, has been written to generate 6 (`n`), 7 and 8 columns. The dots can be dragged into columns as with real counters; but as this is done, a recorded 'history' of the actions is stored (see the `history` box in Figure 3*)* in the form of fragments of computer program. This code is executable: that is, it can be 'run' to produce the output (or part of the output) which produced it. There is, therefore, a duality between the code and the graphical output of the dots; the action (on the dots) to produce a new visual arrangement and the expression (in the form of pieces of program) are essentially interchangeable and the code is a rigorous description of the student's action to construct a particular image, and her actions are executable as computer programs. A box `n` is used to store the smallest of the three numbers and our student might see that what is in the box `n` hardly matters, and therefore that the theorem is independent of the first number.

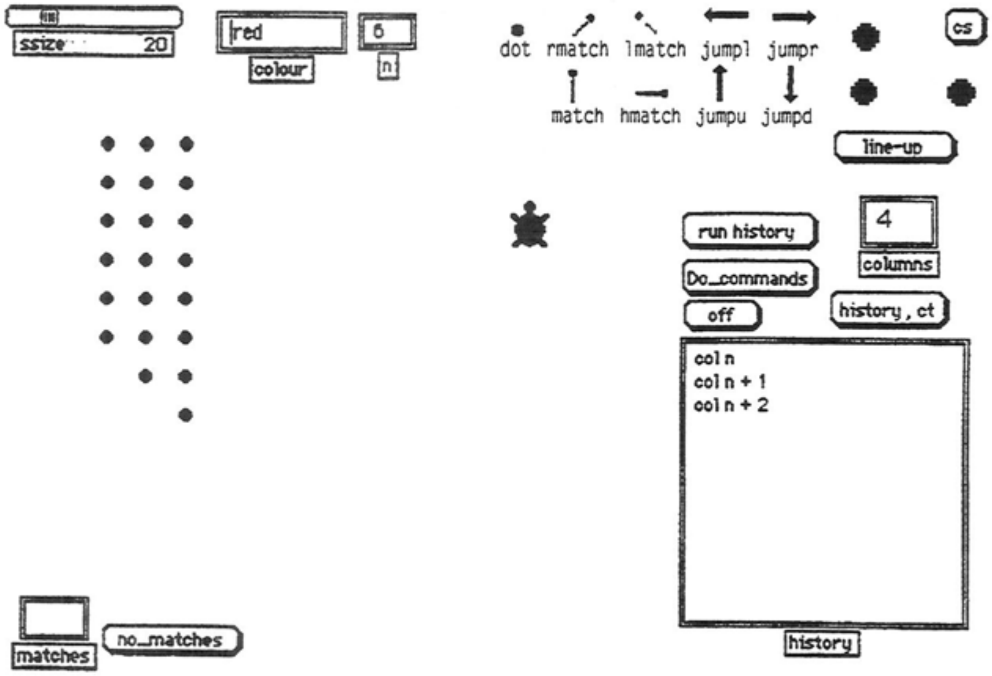


Figure 3: A typical *Expressor* screen to explore the sum of 3 consecutive numbers

How did Tim cope with this activity? In his first session, he had been seeking explanations for a general rule in the general symbolic expressions he had constructed (in the form of programs). He constructed his three columns of dots in Expressor and was faced with a screen rather like Figure 3. Then he wrote: $n+(n+1)+(n+2) = 3n+3$.

But, he obtained this equivalence *not* a result of a manipulating algebra but by reference to our microworld to 3 columns of length and a 'tail' of 3. He then argued correctly as his proof that the sum of 3 consecutive numbers always had a factor of 3: "if you add 3 to any factor of 3, then it is still a factor of 3" (he used factor instead of multiple throughout!). Tim generalised this method to find factors of sums of different numbers of consecutive numbers - always considering columns of dots and a tail, but flexibly using his visual argumentation. For example, to show that it was impossible for the sum of 4 consecutive numbers to have a factor of 4 and so could never add up to 44, he visually moved dots, as he described in Figure 4:

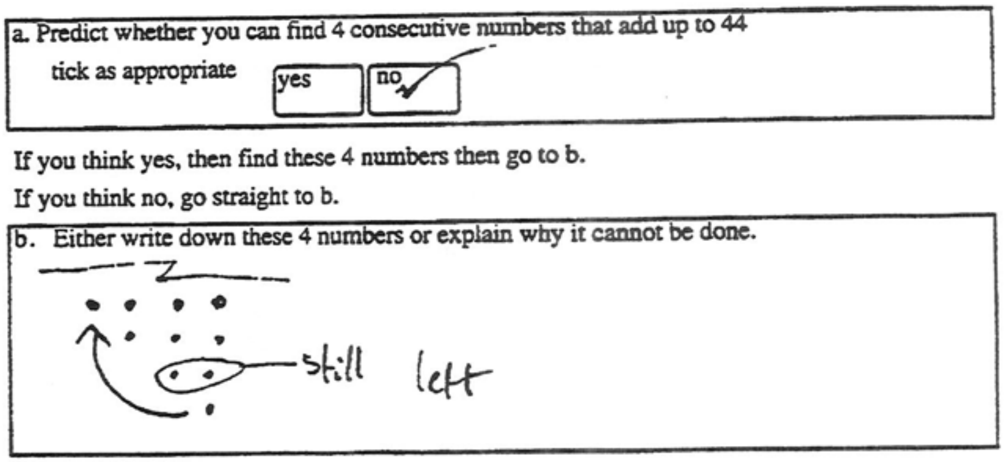


Figure 4: Tim's proof that the sum of 4 consecutive numbers is not divisible by 4

Finally together with his partner, Tim also came up with a brilliant inductive. visual 'proof' that the sum of 5 consecutive numbers had a factor of 5, again using visual reasoning but in yet another way (see Figure 5).

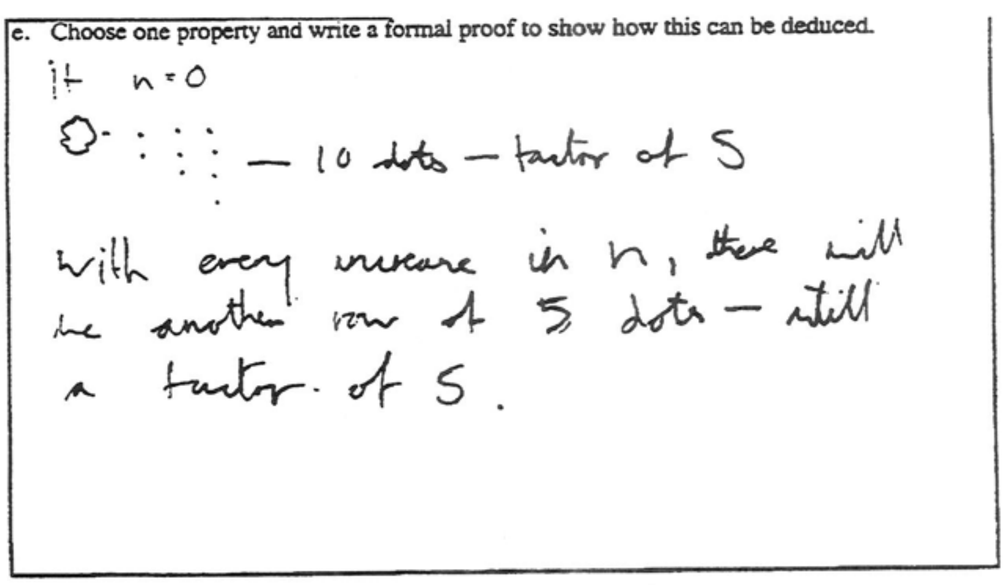


Figure 5: Tim's inductive Proof

By this time it was clear that Tim had found two ways to explain which seemed to be well connected: constructing symbolic code and manipulating visual expressions. His explanations came from linking logical and general arguments with visual representations (columns of dots) - and not from algebra, even though he clearly recognised its importance. This gap in his repertoire of skills is well illustrated in his final homework (Figure 6). Tim creatively generalised 'the dots microworld' into thinking of multiplication as a rectangular array of dots, whose rows could be paired off leaving 'one left over'. But, he was still unable to multiply out brackets correctly!

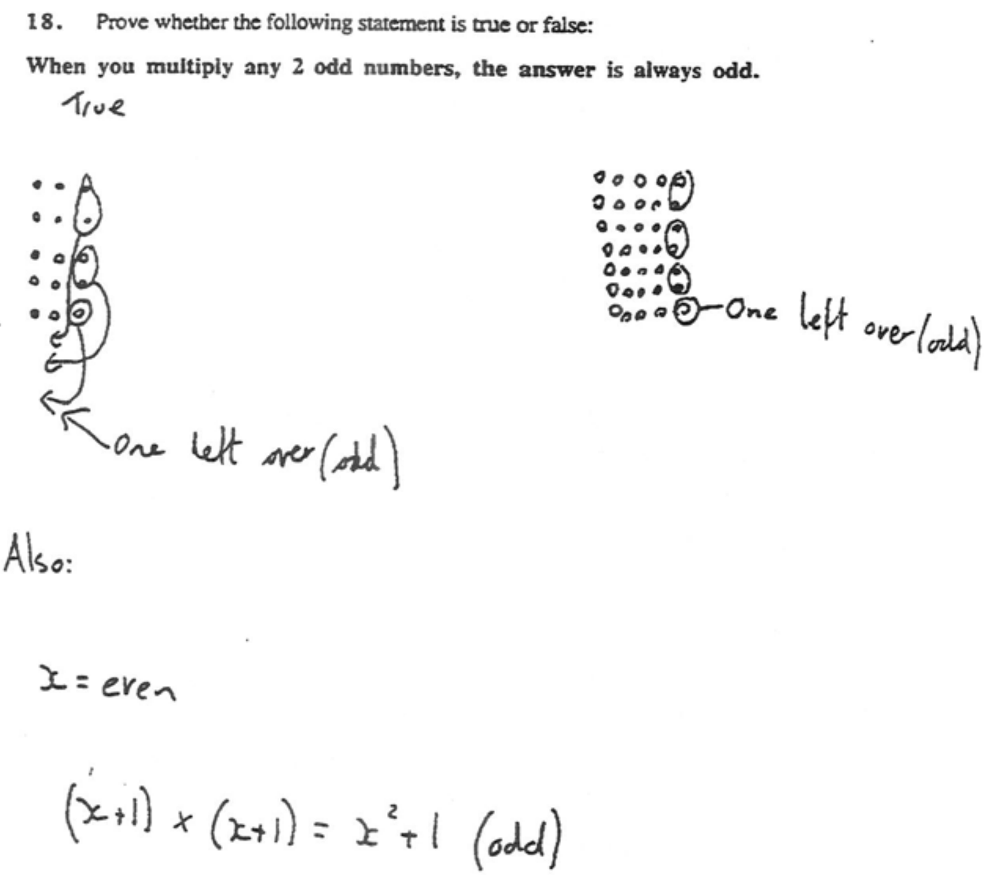


Figure 6: Tim's two explanations

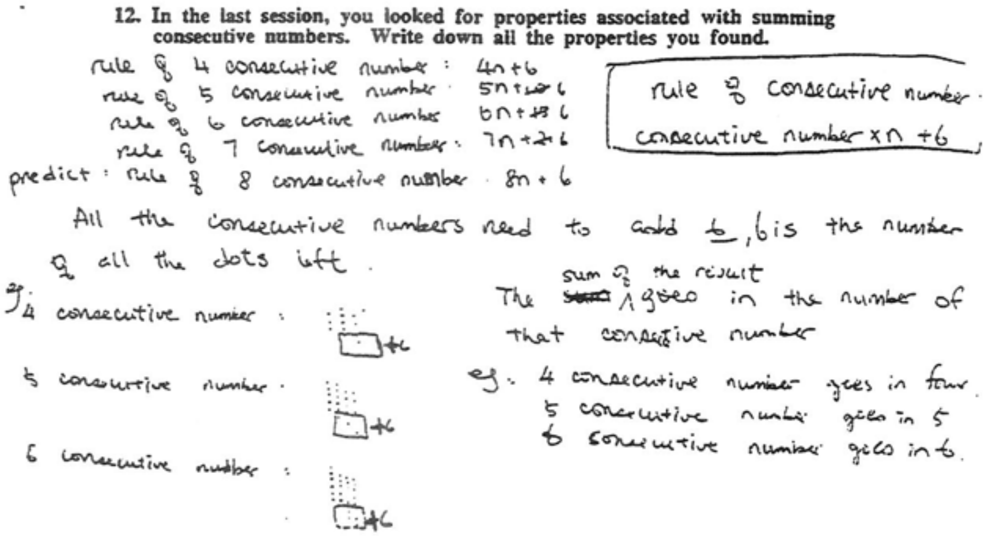
SUSIE: AN INAPPROPRIATE INTERVENTION IN ALGEBRA?

In contrast to Tim, Susie could say nothing about what proof was about and was clearly confused about the generality of a mathematical argument. She selected empirical arguments as her own approach in all the multiple-choice 'proofs' in our survey, in geometry and in algebra, and described these as both general and explanatory. She thought mathematics was quite complicated and, in fact, admitted to hating it. Although Susie offered no description of proof or its purposes, it emerged in interview and watching her computer work that she did have a view about proof - it was about examples (*many* examples)! It was enough to have shown a statement was true many times! Additionally, for Susie, there was another important aspect of proof which was a rule or formula. *But,* its role was to obtain more marks from the teacher rather than to confer generality - the examples were enough for this.

Although Susie could write formal proofs, she did not see them as general and found them no more convincing than empirical evidence - her two 'modes of proving'- examples and formal proofs - apparently completely disconnected. She believed for example that even after producing a valid proof that the sum of two even numbers is even, more examples would be needed to check that the statement holds for particular instances.

In our teaching experiments, both in algebra and geometry, we noticed that Susie followed all the instructions carefully, but rarely if ever experimented with the computer. She found it hard to see the computer as means to try things out when unsure, to learn from feedback.

I will illustrate Susie's work in algebra by reference to the same activities described earlier. Susie was considering the sum of four consecutive numbers. She constructed the columns of dots and came up with the formula, $4n + 6$, ostensibly by making the connection of the '$+6$' with the 'tail of dots'. For 5 consecutive numbers, she apparently used the same method to come up with the correct sum of $5n + 10$. Then she changed her mind. She crossed out the $+10$ explaining this by writing that she had checked and 'it was 6'. From this point on, her written work and explanations were disconnected from any generality suggested by the visual display she had constructed in the microworld, *except* she persisted in showing pictures of columns of dots with a 6 dot tall, as illustrated in her homework following this session.



12. In the last session, you looked for properties associated with summing consecutive numbers. Write down all the properties you found.

rule of 4 consecutive number : 4n + 6
rule of 5 consecutive number : 5n + 6
rule of 6 consecutive number : 6n + 6
rule of 7 consecutive number : 7n + 6
predict : rule of 8 consecutive number : 8n + 6

rule of consecutive number :
consecutive number × n + 6

All the consecutive numbers need to add 6, 6 is the number of all the dots left.

eg. 4 consecutive number : +6
5 consecutive number : +6
6 consecutive number : +6

The sum of the result goes in the number of that conseqtive number
eg. 4 consecutive number goes in four.
5 consecutive number goes in 5
6 consecutive number goes in 6.

Figure 7: Susie's rule for consecutive numbers

We can explain the rupture of connection between particular examples and generalisations by reflecting on what we had discovered as Susie's goal in mathematics - to find examples and then a *rule*. She had achieved this: found a rule in which numbers could be substituted and even had pictures to illustrate it!

I must mention Susie's story is not completely negative. She did make progress after engaging in our sequences, By constructing matchstick patterns, Susie was beginning to appreciate how an algebraic expression could express generality (and serve merely as something to be manipulated), and, although proving for Susie remained solidly 'a rule plus examples', she did seem to be beginning *to want to explain* as well.

SOME SNAPSHOTS FROM THE GEOMETRY SEQUENCE

I will mention briefly some insights we gained from our teaching sequence in geometry, simply to illustrate further some points raised in the previous sections. This sequence followed a similar pattern to that in algebra. In the first session, students are encouraged to construct simple geometrical objects on the computer with dynamic geometry software, to describe their constructions, connect each with a corresponding mathematical property, and use the computer to explore or reject conjectures. In the second session, students are encouraged to construct familiar geometrical objects (parallelograms, rectangles) on the computer, identify the properties and relations of a geometrical figure that had been used in their constructions and distinguish some properties that might be deduced from those given by exploring with the computer. In much the same way as in algebra, students are also taught at this point to construct logical deductive chains of argument and write formal proofs based on their computer constructions. In the third session, students are faced with more unfamiliar constructions and proofs, which again they can tackle experimentally on the computer.

So how did Tim fare in geometry? Geometry for Tim, as for most of our students, was far more problematic than algebra. He did make some progress in that he learnt to write clear descriptions of his constructions, translate them into given properties and 'see' deduced properties. The computer work helped Tim 'see' relationships and convinced him of their necessity, but the links he could make between constructions and proofs or even explanations were much more tenuous than in algebra.

> T – Well you could actually see like if they were congruent - you could take however much you were allowed to take and actually make a triangle. If it was congruent then you could... tell it was.
>
> C – Tell it how?
>
> T – Just by seeing.
>
> C – And did that help you write your formal proofs?
>
> T – Not really - not the formal stuff. — But — well it made it more enjoyable.

Tim found it hard to appreciate and reproduce 'the game' of proving - that is, systematically to separate givens from deduced properties and produce reasons for all his steps. He found the language of formal geometry proofs inhibiting - it stopped him 'seeing it all'. The construction task in the third session was important in his progress. He had to construct a quadrilateral where adjacent angle bisectors were perpendicular and to describe and justify its properties. Tim found this hard, but, after much experimentation and measuring lots of angles, he eventually 'saw' the key relationship - two parallel lines - not be 'just seeing them' but by noticing two equal angles and dragging. The important point is that the measurements for Tim were not simply collecting empirical evidence, they were not only part of the conjecture but also and crucially part of his proof. When he talked about two angles of 44°, it was clear to us that he was seeing *through* the numbers to the general case -just as he had done in *Expressor.* As in algebra, Tim was using the computer interaction to help him to find explanations.

Susie again presents us with a different picture. When it came to constructing proofs, Susie's responses were quite unlike the majority of students in our survey. Her proofs in geometry were far better than in algebra and the proof she constructed for the more complex geometry question, (a standard Euclidean geometry proof), was much better than almost all the survey students[3]. Yet we found very little evidence that Susie made any progress in geometry as a result of undertaking our teaching sequence. At the start, formal proof was a ritual, disconnected from any appreciation of the generality of the mathematical properties and relationships she used. As in algebra, she believed that even after proving a statement, its validity had to verified in any specific set of cases (see also, Chazan, 1993). In our case, Susie was certain that she needed examples to check that the statement, the sum for the angles of a triangle is always 180° held for right-angled triangles. The computer interaction did not seem to help Susie to come to appreciate the generality of a proof and the proving process. Also, before she started our teaching sequences, Susie could already construct formal geometry proofs, but only in the context of familiar and fairly routine problems. Faced with more unfamiliar situations as that described above, she was lost and, unlike Tim, was unable to use the computer to help her.

We can throw light on this lack of progress, by reference to two factors: her interactions with the computer and her interpretation of feedback. First, as I have mentioned, Susie did not exploit the computer to test hypotheses or try things out. But the success of our tasks *relied* on

3 She produced na almost perfect formal proof – something only achieve by 4.8% of the students in the survey and which 62% of students did not even start.

experimental interaction - we did not expect our students immediately to know what to do. Second, Susie interpreted the feedback from the dynamic geometry software in a way which certainly was unexpected. For her, dragging a Cabri construction was *not* testing a relationship, exploring a property - but merely a way of generating many examples. Once we had noticed this, we could see it was completely consistent with Susie's view of proof! Susie's reflections on the use of the computer in mathematics are relevant. When interviewed, it was clear that she thought the computer had given her ideas about 'what it was all about' and had done it quickly. But rather crucially it makes examples and checks them[4].

I have to mention that Susie did change for the better in her response to mathematics. All through the teaching experiments, Susie picked the most enjoyable aspect of her mathematical work as 'finishing it', 'getting it right', 'writing down the results'. Yet, in algebra, we were beginning to catch glimmers of enjoyment and engagement: Susie began to mention *the activity* rather than simply its end point. In her interview too, she spontaneously said how much she had enjoyed the work the computer, although it must be admitted this was only as a contrast with 'normal maths'. Even so, this more positive attitude might be the key to Susie's further development.

DISCUSSION AND CONCLUSION

To begin an explanation of the two very different student responses to our teaching and the work with computers, we have to consider cultures and curricula - huge issues beyond the scope of the paper but which simply cannot be ignored. Susie's profile is somewhat less 'odd' if it is known that she had only studied mathematics in an English school for one year - she was in fact from Hong Kong and had been educated there, although the language of instruction was English. Unlike most other students in our survey, Susie had been taught formal geometry proofs as well as algebraic formulae and manipulation and had little experience of 'doing investigations'.

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4 She also thought that the computer helped her to remember, but there was a disadvantage 'you can't use the computer in exams'!

As I have tried to show, Susie's lack of progress might at least partly be explained by the disjuncture between the assumed starting points of our activities, particularly those with computers - in terms of sense of proving and student-computer interaction - and Susie's world. We had students like Tim in mind when we designed our sequences; students reared in an investigative culture - who wanted to explain but who lacked the tools to do it. Susie was at odds with this culture in terms of her beliefs about mathematics and about proof. Our activities did not build on *her* existing framework for proof, did not help her to connect her informal mathematics to our agenda. Our story of Susie provides compelling evidence that we must take seriously prevailing beliefs about mathematics and about computers in our curriculum planning, and resist the temptation to import 'exemplary activities' from other cultures.

The comparison of Tim and Susie's work cautions against any assumption that the computer will lead to a set of learning outcomes or bring about particular changes. We can only design optimal activities within very limited parameters, given that how children interact with and learn from software depends on their expectations and beliefs. Curricula must seek to build on student strengths - in the case of UK on a confidence in conjecturing and arguing - and connect these strengths to new dimensions. Students like Tim respond positively to the challenge of attempting more rigorous proof alongside their informal argumentation. Susie was less successful as the culture which shaped our teaching and task design was not shared by her.

Clearly, not all of UK students are like Tim or students from Hong Kong like Susie. But the purpose of elaborating their stories is to guard against the stupidity of 'transferring' curricula simplistically across cultures, the replacement of a curriculum which over-emphasises an empirical approach with one in which students are simply 'trained' to write formal proofs. It is all too easy for countries simply to flip between two states of skill and creativity crisis while attempting to model curriculum innovations which look so alluring to the distant observer.

So, returning to my initial question about the desirability of a globally-effective mathematics curriculum. I can only conclude that this goal is fundamentally misguided. We should not set our sights on the same curricula sequences and targets, because these are not the same in any reality. Incorporating what look like comparable activities into our curricula, will not mean that the meaning derived from them will be comparable[5]. Cultural effects might even be magnified when activities involve technology, which carries its own sets of beliefs and agendas. I have tried to illustrate how the power of microworlds to engage our students with mathematics rests first and foremost on what our students believe about curriculum goals and intentions.

Our aim in mathematics education maybe to reach a common goal - mathematical literacy comprising a better balance between skills and competencies and engagement with mathematical thinking. We might even agree that the computer might have a useful role to play. Although it is deeply illuminating and exciting to move beyond the surface features and slogans of international comparisons and focus on what *mathematics* and what *education* we strive to achieve in our countries, to learn from each in international meetings like this, ultimately we have to tease out different routes to this same goal.

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5 Similar points have been made in relation to the meanings of test items in TIMMS which are not the same because it is the same test (Keitel and Kilpatrick, in press).

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