Blurring distinctions between the empirical and the theoretical? The roles of examples in the proving process*

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Resumo

Este artigo examina as diferentes formas pelas quais alunos utilizam evidências empíricas em suas tentativas na redação de provas matemáticas. Exemplos de construções de alunos relacionados com atividades de Álgebra e Geometria são apresentadas para ilustrar como tais evidências podem ter uma variedade de funções no processo de prova – por exemplo, como testes de uma conjectura, como exemplos genéricos em argumentos dedutivos e como casos especiais para enfatizar propriedades particulares quando argumentos de natureza mais indutiva são desenvolvidos. As análises sugerem que o envolvimento na construção de objetos matemáticos durante interações com o computador pode encorajar alunos a identificar estruturas gerais quando da manipulação de casos particulares.

Palavras-chaves: prova; computador na educação matemática.

Abstract

This paper considers the different ways in which students make use of empirical evidence as they attempt to write valid mathematical proofs. Examples of students' proof constructions related to both algebra and geometry activities are presented to illustrate how that type of evidence can play a variety of different roles in the proving process. For example, it can act as tests of a conjectured conditionality, as generic examples in deductive arguments and as special cases to highlight particular properties when more inductive arguments are developed. It is suggested that involvement in the construction of mathematical objects during computer interaction can encourage students to identify general structures when they manipulate particular cases.

Key-words*: proof; computers in mathematics education.*

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A consistent theme in research into proof in school mathematics is the relationship between empirical evidence and analytic argument. In general, this relationship has been seen as a problematic one, with the vast majority of students far from clear about the distinction between inductive and deductive reasoning. Some researchers have suggested that the cognitive gap between different modes of reasoning parallels a profound epistemological gap between ordinary argumentation (in which appeals to empirical evidence are accepted and commonplace) and mathematical proof (Balacheff, 1988; Duval, 1991). Rather than focussing on discontinuities, a number of recent studies, on the other hand, have stressed connections between different aspects of the proving process. Examples include Simon's idea of transformational reasoning (Simon, 1996), the cognitive unity of statement, proof and theory proposed by Mariotti, Bartolini Bussi, Boero, Ferri and Garuti (1997), and the consideration of role of abductive reasoning in the construction of proofs by Arzarello, Micheletti, Olivero and Robutti (1998a).

This paper too concerns the building of connections between different aspects of the proving process. A number of examples of the kinds of the proof constructions produced by English and Welsh students[1] will be presented in order to consider the ways in which they co-ordinate (or not) empirical and theoretical modalities and how this co-ordination is shaped by different approaches to teaching proof.

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First examples, then explanations

3 + 5 = 8   5 + 7 = 12
11 + 7 = 18   31 + 19 = 50

odd number = even no + 1

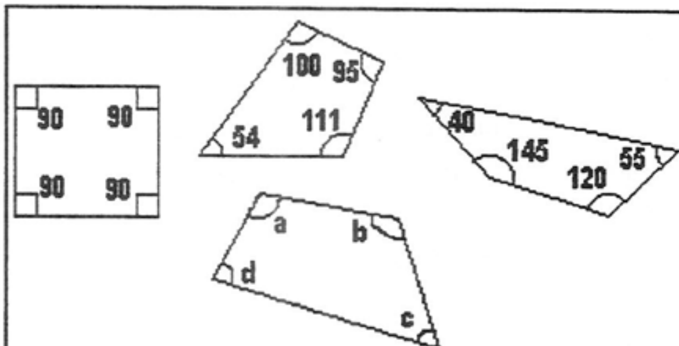
So when you add two odd numbers it is the same as adding two evens and 2 1's

odd + odd = even + even + 1 + 1

This makes an even number becuase it has already been proved that two evens make an even and two odds is even + even giving even + 2 which is even too.

So odd + odd = even

Figure 1: Explaining with reference to structure



a	b	c	d	a+b+c+d	
90	90	90	90	360	✓
100	95	111	54	360	✓
40	55	120	145	360	✓

By drawing some quadrilaterals and adding the interior angles, you can see that it is always 360°

$$a + b + c + d = 360°$$

Figure 2: Explaining with reference to action

Figure 1 presents an argument constructed to prove that the sum of two odd numbers is always even, while the argument in Figure 2 is an attempt to prove that the sum of the interior angles of a quadrilateral is always 360°.[2] There are some similarities between the structure of these two arguments, but also a rather important difference. Both arguments contain a set of examples which confirm the conjectured conditionality and which is followed by a written observation of why the given statement is true, but the nature of the respective observations indicates substantially different interpretations of what it means to explain. In the first, the explanation focuses on the mathematical properties underlying the examples, whereas the second explanation involves a description of the actions through which confirming evidence was produced.

Amongst the 2459 high-attaining mathematics students (14-15 years old) who attempted to construct proofs for these two statements, arguments with this structure – examples followed by observation – were the most common constructions produced (see Healy and Hoyles, 1998). This is not all that surprising since, in our mathematics curriculum, students are encouraged to approach proving in this way. Proof and justification activities are located largely in activities collectively known as "investigations" where data are to be generated, synthesised into the articulation of a general conjecture to be explained and, if possible, proved. As suggested within the hierarchy of levels by which our mathematics curriculum is organised (Department of Education, 1995), the different aspects of the proving process are interpreted as representative of ascending levels of reasoning, with inductive processes associated with lower levels than deductive ones. The result is that the former are introduced before, and usually independently from, the latter. A few students seem to be able traverse the implied developmental passage from the empirical to the theoretical for themselves and, when this happens (as can be seen in Figure 1), the arguments produced are meaningful and creative. In general, however, the generation of an appropriate set of examples does

2 These two questions appeared in a proof survey administered to students in England and Wales. For a complete description of the survey and its results see Healy and Hoyles (1998).

not necessarily motivate in students a need for deductive proof. We can say that there seems to be no natural progression from empirical and theoretical reasoning.

According to Duval (1998), any model of mathematics learning in which different ways of reasoning are organised according to a strict hierarchy is inappropriate. Rather than being representative of higher (or lower) levels of thinking, he argues that different kinds of cognitive activity have their own specific and independent development. This might suggest structuring activities to separately address specific types of thinking processes. Instead, we chose to develop computer-based situations so that students might face the empirical, the visual and the theoretical simultaneously.

We devised two teaching experiments (one using a Logo microworld and the other Cabri-Géomètre) during which students worked on activities with the following structure: first, mathematical objects are constructed on the computer; second, by attending to the construction procedure, the properties and relations underpinning these objects are to be identified and described; third, the computer resources are used to generate and test conjectures about further properties and to inform explanations as to why they must hold; fourth, the arguments generated during the computer activity are organised into logical deductive chains in the appropriate formal language.[3]

Before presenting examples of the students proof constructions formulated during the experiments, it is important to stress that, unlike the survey where students were given the conjectures to be proved, these activities involved students in both the processes of generating and proving conjectures. Boero, Garutti and Lemut's (1999) suggest that students' exploration during these two processes are similar in nature but differ in function. In their analysis of the different ways through which students' generate conditionalities, the central role played by empirical evidence is clear. The following sections present some of the ways in which evidence is used in the second process: the process of proving. These examples are

3 Details of the two teaching experiments can be found in Hoyles and Healy (1999).

by no means exhaustive, but have been selected to focus on different ways in which particular cases were incorporated in the construction of analytic arguments.

Generic Examples: Using a specific case to convey a general property

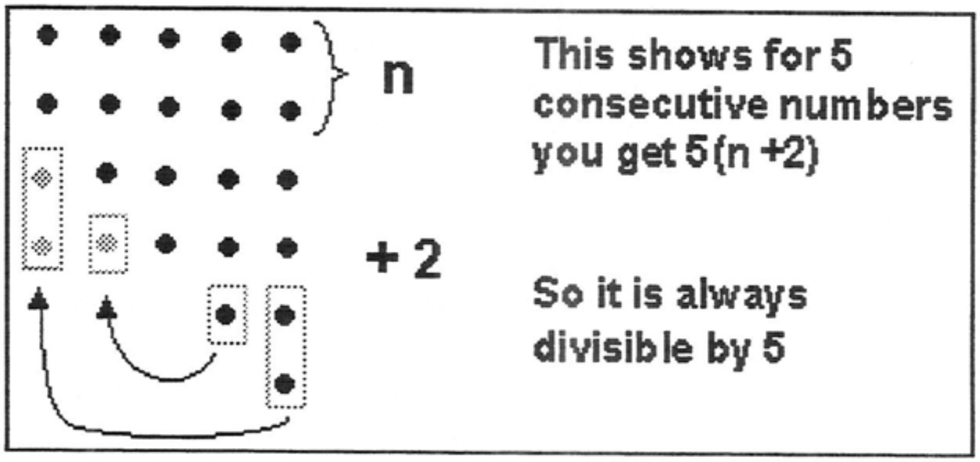


Figure 3: Manipulating to prove

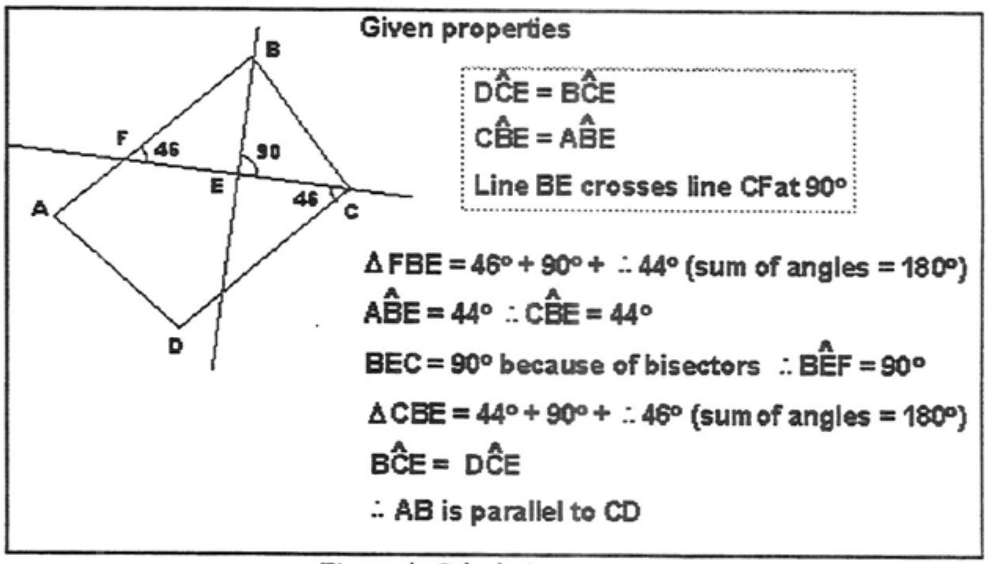


Figure 4: Calculating to prove

Figure 3 presents an attempt to prove that the sum of five consecutive numbers is always a multiple of five. It was written by a student who first constructed a variable Logo procedure to generate a column of n dots, and then used this to produce a visual representation of the five consecutive numbers 2, 3, 4, 5 and 6. The student manipulated the figure in such a way that the conjecture and its proof emerged simultaneously – in one moment the student identified both *that* and *why* the property holds. No more examples were deemed to be necessary as there was nothing special about the choice of 2 for the first number – or rather what was special about it was that it represented both the variable n and the first 2 dots in every column.

The argument presented in Figure 4 shows an attempt to prove that a quadrilateral in which two consecutive angle bisectors cross at right angles will have one set of parallel sides. In common with the previous argument, one specific case only is included and it was through manipulations performed on this example that the student managed to construct his proof. Like the consecutive numbers example, the student-generated conjecture (that segment AB is parallel to CD) emerged from the consideration of just one case. In contrast, the process of determining why was far from immediate and it was only after considerable computer exploration that the proof was attempted. During these investigations, a variety of configurations of the quadrilateral were created – in some the given properties were preserved and in others they were purposefully violated. The first critical moment in the construction of a proof occurred, ironically, when the general quadrilateral was turned into a specific case – that is when the measures for two carefully chosen (alternate) angles were obtained. From this point on, no further manipulations of the figure were made. The *calculation* of the value 3rd angle in the triangle FBE, a value which strictly speaking is unnecessary, provided the second vital step and the obtained value was used as the basis to deduce the parallel property.

In both these student proofs, the particular case is presented as a carrier of its underlying relationships, it serves as a representative for the class of possible examples. As such, they are both what have been termed *generic examples* (Pimm and Mason 1984; Balacheff 1988). Typically, generic examples have been presented as inferior to arguments formulated in more general terms (Balacheff, 1988; Harel and Sowder 1998), although Rowland (1998) has questioned recently whether this pejorative view is justified. He argues that generic examples provide a powerful and accessible means of for conviction and explanation and, at the very least, they might serve as a "half-way house" between empirical generalisation and generalised formal proof.

But – apart from the danger that this brings us back to the hierarchical model of learning that we wanted to leave behind – what does this mean in contexts where the distinction between the empirical and the theoretical is blurred? This is the case in both the microworlds we used: A Cabri figure is simultaneous a figure and a drawing; and working with general Logo procedures also enables students to experience simultaneously the general relationships and their specific manifestations. In some senses, this implies that every Cabri figure and every instantiation of a general Logo procedure is generic. The mathematical properties of any particular screen object are well-known to the student – they received explicit attention in the construction process. A specific example is hence one of the possible representations through which an object can be expressed. The construction process is another way of expressing of the same object – more tangible perhaps in the case of Logo, where its symbolic encoding is easily accessible, than in Cabri. From this perspective it makes little sense to consider a generic proof as inferior to a similar argument that happens to be presented using more general terms.

Constructing a justification from a special case

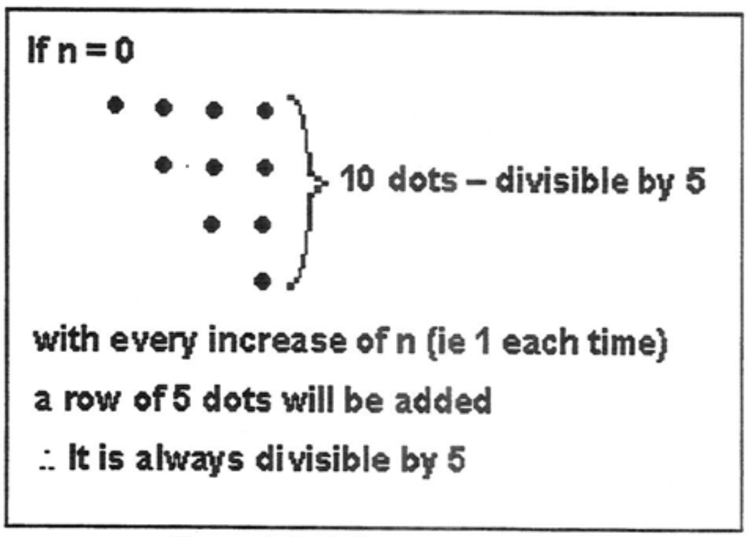
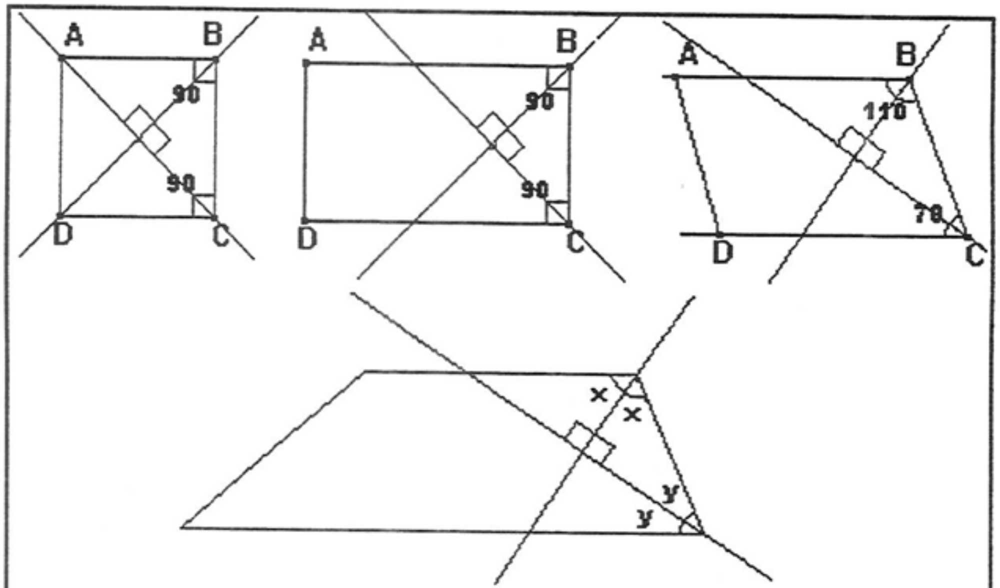


Figure 5: An inductive argument



$$x + y + 90° = 180° \text{ (angles in a triangle)}$$

$$\therefore x + y = 90°$$

$$\therefore 2x + 2y = 180°$$

Figure 6: Special quadrilaterals

In contrast to generic examples chosen to be representative of their class, the examples presented in Figures 5 and 6 were chosen precisely because of their specific properties.

The argument in Figure 5 shows a another form of explaining why the sum of five consecutive number is always a multiple of five. This time, the conjecture was produced as a result of intensive empirical investigation in which various sets of five numbers were generated and re-arranged. In the visual view that emerged from these activities, the sum of five consecutive number was seen to consist of a rectangular block – of width 5 – and a triangular tail of dots. The proof was constructed using the reasoning that if the number of dots in the tail is a multiple of five, than the sum will be too. Testing this hypothesis involved producing a very particular case, when n = 0, and then explaining the relationship between this case and subsequent examples. The proof is hence a visually inspired example of inductive reasoning, but an inductive reasoning considerable more developed than that behind the argument in Figure 2[4].

The argument presented in Figure 6 was also developed from special rather than generic cases. The proof was constructed in response to the geometry problem described above, the investigation of properties of a quadrilateral in which two consecutive bisectors cross at right angles. It started from the (correct) hypothesis that a square would satisfy the given properties. Since the student already knew about various properties of a square, his next task was to identify which of these properties were shared by the other quadrilaterals which also satisfied the givens. He chose to focus on the sum of the two consecutive angles that had been bisected and, to help in his explorations, he decided to measure them. Then, he transformed his square into other well-known cases, a rectangle, a parallelogram and, finally, a trapezium. He conjectured that the properties shared by all these quadrilaterals was that the sum of the two angles is 180° and his subsequent proof was based on another very familiar construction, the right-angled triangle. The strategy employed in the production of this proof is very similar to that described by Arzarello, Micheletti, Olivero and Robutti (1998b) and, actually, both the Cabri-inspired proofs constructed presented in this paper involved what they describe as abductive as well as deductive reasoning.

The two examples in this section were intended to illustrate how special cases can form the basis of a logical argument. Both involve transformations of specific cases, but the nature of the final proofs was not the same. The first argument was driven by inductive concerns, a search for the difference between adjacent cases, while in the second, finally expressed in a deductive form, the focus was on identifying the

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4 The arguments in both Figure 3 and Figure 5 could be criticised as restricted only to positive numbers. Undoubtedly, the majority of students who used the Logo microworld were thinking primarily of the positive cases (although we have some evidence of students mentally constructing visualisations of the negative cases that were impossible to construct using the microworld tools). But we can only be sure that this is not the case for students who use more general modes of representation if they make this explicit – it is quite possible, even likely, that most of them too consider mainly positive cases.

properties shared by the generated cases. Of course, it could be argued that in the geometry example, the student did not actually use any specific examples, but that the square, for example, was general – the (unknown) measures of the sides of the square were clearly irrelevant to the activity. This only goes to show that the distinction between the specific and the general in the geometry context is far from fixed.

Frameworks for proof?

Up to this point, proof constructions associated with two different teaching approaches have been considered. In the first approach, the approach prescribed in the statutory Mathematics Curriculum for England and Wales, students are expected to start by experimenting with data and identifying regularities and only later focus explicitly on mathematical properties (and later still on the relationships amongst properties). The second approach involved the use of computer microworlds in which students construct mathematical objects in order to provide the data from which they can abstract further regularities. It has been argued that the first approach can have the effect of confining students to empirically based reasoning, while in the second students engage simultaneously with specific configurations and general relationships. Even the limited examples included in this paper illustrate how interacting with the tools of the Cabri and Logo microworlds can provoke students to develop a variety of reasoning approaches and facilitate reflections upon the steps made in constructing and manipulating new objects. The student proofs presented above show too how the reflections could be successfully reorganised into coherent mathematical arguments. Not surprisingly this did not always happen. One situation in which all students experienced considerable difficulties was when the construction of a geometrical object did makes visible adequate information for a proof – that is when it was necessary for students to add further constructions to their figures. This leaves us with a question to consider: What activities might help the student who produced the argument presented in Figure 7 develop the necessary steps to complete the proof?

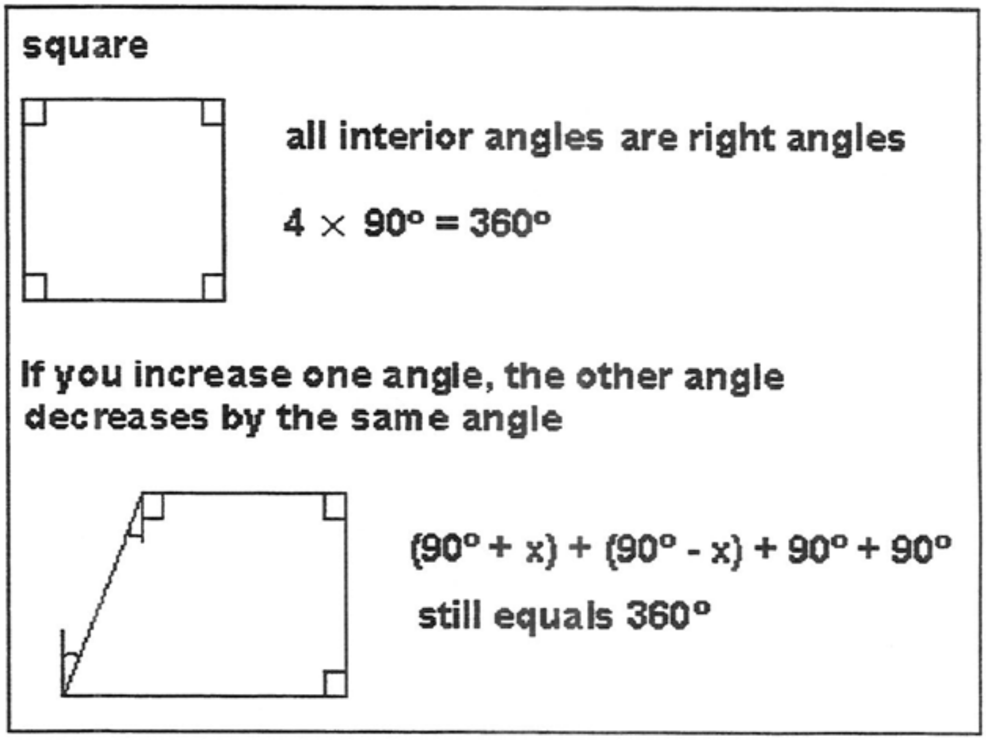

Figure 7: Where next?

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