

B. Russell's "introduction to mathematical philosophy"

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Abstract

Bertrand Russell is an important and interesting figure, undoubtedly the most read, honored, and reviled English-speaking philosopher of the twentieth century. And his "Introduction to Mathematical Philosophy"¹ is no less fascinating. Indeed, Russell's above-mentioned work, published in 1918, has sometimes rightly been called "an admirable exposition of the monumental work Principia Mathematica".

The main object of Russell's book is number, and everything belonging to number, to arithmetic, and to the logic of arithmetic. The foundations of arithmetic always remained the focus of Russell's interest in logic and mathematics, and his views had a profound influence on the reform movement of mathematics education that began around 1960. Since the beginning of the 19th century, mathematics showed a strong tendency towards arithmetization, because space and the continuum seemed beset with seemingly intractable otherness. By the end of the 19th century, even number appeared not to be so transparent and so immediately given anymore and the question "what numbers are" arose to explain and to complete the foundations of arithmetic itself by means of logical analysis and set theoretical construction.

Key-words: mathematical fundaments; new math; number concept construction.

Resumo

Bertrand Russell foi um intelectual importante do século XX, sem dúvida o autor mais divulgado e mais lido da atualidade. Ele foi o filósofo mais elogiado, assim como o mais injuriado de sua época. Seu livro Introduction to Mathematical Philosophy, publicado em 1918, não só foi uma introdução fascinante em sua obra prima Principia Matemática, que revolucionou os fundamentos de toda a matemática, mas foi também o texto mais influente na reforma educacional chamada Matemática Moderna dos anos 1960-1970. O tema principal desse livro é o "número". Russell tentou construir o conceito do número em termos da lógica e da teoria de conjuntos. É proposta deste artigo tanto analisar como apontar falhas e ganhos dessa tentativa.

Palavras-chave: fundamentos de matemática; matemática moderna; construção do conceito de número.

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1 "Introdução à Filosofia Matemática" – tradução para o português da 10ª impressão, de 1960, por Giasone Rebuá e publicado por Zahar Editores, no Rio de Janeiro, em 1963.

I. Presentation

Bertrand Russell is an important and fascinating figure, no doubt the most read, most honored, and most reviled English-speaking philosopher of the twentieth century. He was born in Wales in 1872 and died there in 1970. His lifetime thus extends from the times of Bismarck and Queen Victoria to the atomic bomb and the cold war period after 1950.

Russell's "Introduction to Mathematical Philosophy" is no less fascinating reading and it is a book of the kind only someone like him was able to write, while in prison with no resources at his disposal, compelled to jettison all technical ballast and to take the subject in a bold fashion trying to provide a readable, almost popular presentation of the subject. Indeed, Russell's "Introduction to Mathematical Philosophy" of 1918 has sometimes and rightly been called "an admirable exposition of the monumental work *Principia Mathematica*". The book is even more, because it builds upon the results of all of Russell's fundamental work since about 1900. And it is at the same time something else, as it provides a most original, lively and self-contained introduction to the foundations of mathematics and of epistemology. It contains, in fact, as it were in a nutshell all the fundamental questions of mathematical epistemology.

In contrast to standard present day texts on the philosophy of mathematics, Russell always allows the reader a glimpse into his thinking, as it develops, without trying to cover up his presumptions and errors. The reader can literally experience Russell's enthusiasm in dealing with his subject. Russell has always been an independent spirit to whom narrow specialization remained alien, and who strove to simultaneously keep up with all the developments in mathematics, philosophy, in the experimental natural sciences and in politics as well. And during his lifetime he wrote on an impressive variety of fields, in most cases very lively, a fact which made many of his books popular science in the very best sense of the term. Not only his comprehensive studies enabled him to do so, but also his conviction that his true vocation was to write.

As Russell is one of the eminent protagonists of modern scientific empiricism, and one of the founders of the analytical philosophy of mathematics dominant until today, his writings give the rare chance of a unique and deep insight into the philosophical, scientific, technological, and political turmoil characteristic for the first half of the 20th century.

While the program of the logical world view, as inaugurated, among others, by Frege and Russell, and later to be continued by their "disciples" – Carnap and Quine – may seem to be very dry, technical and perhaps even objectivistically inhuman, the underlying intentions were firmly anchored in a humane belief in the merits of scientific progress and enlightenment.

The orientation towards a strictly logico-scientific mode of thought intended to counteract dumb irrationalism and traditionalism, as it became prevalent after World War I. As Rudolf Carnap expressed it in 1928, logical philosophy is supported by the belief that the future belongs to a mentality which "demands clarity everywhere, but which realizes that the fabric of life can never quite be comprehended", and which consequently puts "clarity of concepts", "precision of methods", and "responsible theses" into the service of a progress in cognition by cooperation (Carnap 1968, preface to the first edition). In view of the fact that the philosophy of mathematics and the present analytic theory of science, while adopting Russell's "technical achievements", do not utter a single word about the philosophical and historical concerns inseparably linked to it, it would seem appropriate to point out the historical origins of modern set-theoretical mathematics and epistemology, and the cultural and political context of their emergence.

II

The main object of Russell's book is number, and everything belonging to number, to arithmetic, and to the logic of arithmetic. The foundations of arithmetic always remained the focus of Russell's interest in logic and mathematics. Since the beginning of the 19th century, mathematics showed a strong tendency towards arithmetization, because space and the continuum seemed beset with seemingly intractable otherness. By the end of the 19th century, even number appeared not any more so transparent and so immediately given and the question arose to explain "what numbers are" and to complete the foundations of arithmetic itself by means of logical analysis and construction. Russell's concern is at first not so much with clarification of the ontological foundations of arithmetic, resp. of mathematics as a whole, but rather with differentiation of the conceptual apparatus, and specification of proof methods. As his argument unfolds it became obvious, however, that the existence claims on which to build, themselves were very insecure.

To answer the question “what is a number” a “new logic” was necessary. Russell was first confronted with this new logic at the International Congress of Mathematics in Paris in 1900, in the person of Peano. And however impressed he was by Peano’s logical precision, he did not perceive the form of logic grown from the algebraic and axiomatic approach to be in agreement with his own natural disposition and with his own interests focussed on the arithmetization of mathematics. Russell himself describes the development as follows:

In itself, mathematical logic had for a long time not been a novel discipline any more at this period (i.e. about 1900). [...] Boole had published his *Laws of Thought* in 1854; C.S. Peirce had elaborated a logic of relations, and Schröder had published a comprehensive work in Germany in three volumes in which he summarized everything achieved up to then. Whitehead had treated the Boolean calculus in the first part of his own *Universal Algebra* (which had appeared in 1898, treating besides the above authors Grassmann, De Morgan and others, my insertion, M.O.). Most of these works were known to me, but I did not have the impression from them that they made the logical grammar of arithmetic appear in a new light.

Nevertheless, Russell remembers,

the Congress was a turning point in my intellectual life, because I there met Peano. I already knew him by name and had seen some of his work, but had not taken the trouble to master his notation. In discussion at the Congress I observed that he was always more precise than anyone else, and that he invariably got the better of any argument upon which he embarked. As the days went by, I decided that this must be owing to his mathematical logic. (Russell 1967, 144)

The logicians of the algebraic school like Peirce, Peano or Schröder, it is true, proceeded mathematically, transposing the mathematical laws to the realm of logic, resp. understanding logic as a universal algebra, without intending to revise the methods of mathematics. Modern axiomatic method represents nothing but the ultimate step towards a mathematization of all phenomena and areas of reality with mathematics

itself now being finally mathematized. The concept of number, in particular, has always marked the heart of mathematical thought. Still it seems legitimate to ask why arithmetic was to be axiomatized only in the second half of the 19th century, that is more than 2000 years after Euclid's axiomatization of geometry.

Now mathematical philosophy and foundational research grew out from two rather different ways of thinking in mathematics itself. The first one, to which the name of set-theoretical reductionism seems appropriate, began with Bolzano (1781-1848) and Cauchy (1789-1857) and culminated in the achievement of Russell. The other, usually called axiomatic or postulational method, originated in the works of Poncelet (1788-1867) and Grassmann (1809-1878), who continued and radicalized the approach of his father Justus Grassmann (1779-1852) and reached full development in the works of Peano and Hilbert (1861-1943). While set-theoretic reductionism was mainly the work of analysts and philosophers, the new structural or axiomatic thinking was primarily established by geometers, algebraists and engineers.

It were mainly problems of teaching and communication which had always led to algebraic exposition of analysis and which after this seemed no longer appropriate enforced a search for new foundations of the number concept itself. Until then these two traditions do not seem to have become aware of their real difference, or of the possibility to propose themselves as two alternative foundational perspectives, not least because modern axiomatics itself came in two rather different forms, that of Pasch (1843-1930) on the one hand and that of Hilbert on the other.

And only after people began to inquire into the foundation of arithmetic towards 1870, looking for specific logical justifications of the latter, they were compelled to become concerned with the traditional epistemological systems. The self-referential character of the process of mathematization as it addresses mathematics itself, and this means arithmetic now, inevitably leads to raising fundamental logical and epistemological problems. In the foundations of geometry one had, since Descartes always, explicitly or implicitly, made use of number and arithmetic. Any mathematization of the continuum requires the number concept in some or other ways (see: Hölder 1899; and for an alternative Grassmann 1844). And whereas Xenon's Paradoxes for a long time had been considered mere sophisms or anachronisms, which may admittedly have some charm, they have since the 19th century become the focus of

an intense and profound debate, because now the consistency of a discrete picture of the world itself grew into a serious problem (see W. C. Salmon (ed) 1970).

When dealing with the foundations of pure mathematics and of arithmetic in particular people often change all their epistemological attitudes as well as their methodological rules and become more or less obsessed with finding ultimate truths about the nature of reality. Russell provides a shining example of this attitude. On the one hand, Russell makes it clear to us, for example, that we cannot get farther in our own grasp of natural phenomena than to a mathematical representation of the relations and the relational structures – the “Book of Nature” has been written in a mathematical language, as Galilee already stated – and that we have to admit that we know much more about “the form of nature than about its content”. On the other hand, where mathematics and in particular the concept of number is concerned, Russell makes an effort to answer “what is” questions in an absolute and definite way.

While he responds to the “non-mathematical mind”, to whom the abstract character of our physical knowledge appears unsatisfactory, by saying that these abstractions may personally not please her or him, but are useful from a practical as well as theoretical viewpoint, he considers the opposite to be true with regard to mathematics. And against the complaints of commonsense empirism or artistic imagination, he justifies the mathematical representations of phenomena by their fertility and their uses:

Abstraction, difficult as it is, is the source of practical power. A financier, whose dealings with the world are more abstract than those of any other ‘practical’ man, is also more powerful than any other practical man. He can deal in wheat or cotton without needing ever to have seen either: all he needs to know is whether they will go up or down. This is abstract mathematical knowledge, at least as compared to the knowledge of the agriculturist. Similarly the physicist, who knows nothing of matter except certain laws of its movements, nevertheless knows enough to enable him to manipulate it. After working through whole strings of equations, in which the symbols stand for things whose intrinsic nature can never be known to us, he arrives at last at a result which can be interpreted in terms of our own perceptions, and utilized to bring about desired effects in our own lives. (Russell 1971a, 138f)

In contrast to that, he would like to constitute these mathematical abstractions themselves by pure reasoning and by ensuring their applicability in a completely *a priori* way in a somewhat Kantian spirit. We should reduce, he seems to recommend, everything in science and daily life as far as possible to number and arithmetic. What number itself is, and what the concept of number means, however, shall be determined independently of any application by pure thought and logical analysis. Sense perceptions and numbers have been forming the foundations of our cognitions since Descartes, Russell says. By clarifying the meaning of the number concept, logic is now replacing arithmetic, bridging the gap between empirism and mentalism.

While Russell believes that science can only determine its domain of investigation up to an isomorphic mapping, structural isomorphism hence marking the limits of our possibilities of symbolic representation of the world and of our mathematical descriptions of the same, and while he thinks that this might even be an advantage in the realm of the natural sciences, because "in the mathematical treatment of nature, we can be far more certain that our formulae are approximately correct than we can be to the correctness of this or that interpretation of them" (Russell, 1971a, 131), he would nevertheless not like to see mathematics be described simply by its own structure.

Where the understanding of nature is concerned, Russell states the insight "that, as reasoning improved, its claims to the power of proving facts grow less and less" (Russell 1971a, 135). This is precisely what he refuses to accept with regard to mathematics.

This difference in attitude and cognitive behavior seems due to the fact that in mathematics we have, differently from the natural sciences, simultaneously to secure the existence of our universe of discourse as well as to construct the conceptual means of dealing with the existent. Bateson has affirmed that from an epistemological point of view we seem to have the option of treating as real either objects resp. events or signs resp. messages. And he has given the following description of these two options:

The difference between the Newtonian world (the world of objects, my insertion M.O.) and the world of communication is simply this: that the Newtonian world ascribes reality to objects and achieves its simplicity by excluding the context of the context – excluding indeed all meta-relationships – a fortiori excluding an

infinite regress of such relations. In contrast, the theorist of communication insists upon examining the meta-relationships while achieving its simplicity by excluding all objects. This world, of communication, is a Berkeleyan world. (Bateson 1973, 221)

Now mathematics seems to belong to both worlds and the explanation of mathematics resp. foundational research on mathematics thus hence shares the exigencies resulting from both.

III

Russell begins his work with a chapter on “the series of natural numbers”, in which the latter are introduced on the basis of Peano’s axioms, the concept of ordinal number being the only one to have a role here. There is no mention of cardinality in the beginning.

The five primitive propositions which Peano assumes are:

- (1) 0 is a number.
 - (2) The successor of any number is a number.
 - (3) No two numbers have the same successor.
 - (4) 0 is not the successor of any number.
 - (5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers”.
- (pp. 5-6)

It might be suggested, Russell says at the end of this chapter,

that, instead of setting up ‘0’ and ‘number’ and ‘successor’ as terms of which we know the meaning although we cannot define them, we might let them stand for *any* three terms that verify Peano’s five axioms. They will no longer be terms which have a meaning that is definite though undefined: they will be variables.

(pp. 9-10)

This is, indeed, the common understanding of the axiomatic approach. It can also be expressed by saying that arithmetic is not about concretely existing things, but rather about general relations or ideal objects (cf. for this also section VII). But this point of view does not satisfy Russell.

On two accounts, Russell says, Peano's approach

fails to give an adequate basis for arithmetic. In the first place, it does not enable us to know whether there are any sets of terms verifying Peano's axioms; [...] In the second place [...] we want our numbers to be such as can be used for counting common objects, and this requires that our numbers should have a *definite* meaning, not merely that they should have certain formal properties. (p. 10)

[...] if we start from Peano's undefined ideas and initial propositions, arithmetic and analysis are not concerned with definite logical objects called numbers, but with the terms of any progression. We may call the terms of *any* progression 0, 1, 2, 3, [...], in which case 0, 1, 2, [...] become "variables". To make them constants, we must choose some one definite progression; the natural one to choose is the progression of finite cardinal numbers as defined by Frege. (Russell 1954, 4)

According to the Frege's definition of number which Russell, without knowing it beforehand, had rediscovered on his own after the Paris conference

[...] primitive terms are replaced by logical structures, concerning which it is necessary to prove that they satisfy Peano's five primitive propositions. This process is essential in connecting arithmetic with pure logic. We shall find that a process similar in some respects, though very different in others, is required for connecting physics with perception. (Russell 1954, 4)

Logic on this account is interpreted in an utterly realistic way.

The minimal requirements to justify the concept of number, or the propositional function "x is a number" are showing that it has instantiations, that it is not "empty" in the Kantian sense. We have to understand "number as the number of a quantity" and to provide an application for the concept thus defined by demonstrating the existence of sets of arbitrary cardinality. This can obviously only be done axiomatically. In doing so, however, the notion of axiom must not be understood in the Peano-Hilbertian sense; the term must rather be conceived of according to the classical Euclidean tradition, that is as an

intuitively evident truth and as a precondition of mathematics. This is why Russell introduces an “axiom of infinity”.

We have to ascertain or render plausible that there are in fact infinite collections or sets in the world to be able to found number (p. 77). Infinite sets of arbitrary cardinality, Russell assumes, do exist and thus can be referred to in our reasoning. Set-theory must not be understood here as a merely formal *façon de parler*, it should rather be seen as an intuitive model of reality. Arithmetical intuition is hence replaced by set theoretical intuition. This might appear strange as the axiomatization of arithmetic has been caused by the feeling that we might be unable to intuit or fully grasp number and must therefore settle for the formal laws numbers satisfy. Russell now apparently replaces number by the intuitive concept of set as foundation of these formal laws. Nearly half a century later, mathematical education worldwide tried to repeat this move, with little success.

It is instructive to compare Russell’s concern with that of Rudolf Carnap, on whom Russell exerted a crucial influence in the 1920s. Carnap calls it one of the objectives of his project “Der logische Aufbau der Welt” to establish a “cognitive-logical system” of concepts and objects which permits to derive or “constitute” all concepts “step by step from certain basic concepts”, the result being “a genealogy of concepts in which every concept finds a secure place” (Carnap 1968). But the main characteristic of Carnap’s book is its structuralism:

Each scientific statement can in principle be so transformed that it is nothing but a structure statement. But this transformation is not only possible, it is imperative. For science wants to speak about what is objective, and whatever does not belong to the structure but to the material (i.e., anything that can be pointed out in a concrete ostensive definition) is, in the final analysis, subjective. (Carnap 1968, 29)

Theories make, according to this view, statements only about structure. Theoretical objects are determinable only up to the structure of their relationships. Structure is exactly what can be represented by mathematics and logic. There seems something wrong with this view, because we cannot grasp structure directly and without the mediation by some incorporation or application of it. And we would always have to observe the influence of these mediations. In contrast to Carnap, and to

logical positivism as such, Russell, in his analysis of mathematics, is not concerned simply with structural descriptions, but his own logical analysis is always mixed with the notion of an absolute notion of truth and meaning. It is this concern which conveys a certain weight to the interpretation of the deductive systems, and in this connection an eminent role is due to the example of the interpretation of axiomatized arithmetic. The concept of set now serves to "link arithmetic to pure logic", and hence to interpret the axiomatic characterization in logical terms (Russell 1954, 4). In this connection, Russell also speaks of an overestimation of number. By giving up number, one attains "a gain in logical purity".

Russell is aware that his own understanding of theory, application, and truth, is not "provable", and that "it seems to be a matter of individual taste whether one accepts or rejects what is called the realistic hypothesis" (a.a.O.). For his own part, he is interested in realism and in the meaning of the word "truth", tying this interest to the question whether there are "elements, or constructions built up from such", which meet the conditions established in a given system of axioms. In this sense, he deals with numbers and with their logical constitution in a set theoretical context.

If one believes that the origin of axiomatization lies in mathematization, accepting Hilbert's conviction that "everything which can be an object of scientific thought (falls to) the axiomatic method and hence immediately to mathematics, as soon as it is ripe for the formation of a theory" (Hilbert 1964, 11), Russell's concern seems perfectly consistent. For arithmetic, to be axiomatizable, must be conceived of as a specific object field. The axiomatic theory alone, just like the natural laws, does not supply complete descriptions of its object field. It is only the application or interpretation of the axioms resp. of the natural laws which yields proper knowledge. Laws as well as axioms as such represent only possible knowledge. The undefined terms, which appear in the axiomatic characterizations, do not refer to specific singular objects, but rather serve to represent possible connections between indeterminate or general objects, free variables. Axioms in the modern sense therefore are considered mere axiomatic schemes.

To complete an axiomatic description one thus has to also indicate the intended applications. As the intended applications of arithmetic are not finite, however, the majority of mathematicians has tried to ascertain the existence of infinite sets by epistemological arguments, and not by ontological ones. If we intend, for instance, to constructively generate

such sets, we might have to postulate the power of the mind for endless repetition of certain of its operations, e.g. of the operation of counting. Dedekind was not ready to imagine a straightforward axiomatic definition of number, because after recognizing the essential characteristics of such a system “the question arises: does such a system exist at all in the realm of our ideas” (Dedekind in his letter to Keferstein in 1890). Dedekind tried to provide an infinite totality of things by ascribing to us human subjects the ability to infinitely repeat certain ideas or mental actions, like adding another stroke. Dedekind considered his thought experiment to be a logical existence proof and he was not concerned, as Russell was, with the meaning of individual number symbols.

Doubts with regard to the plausibility of such a construction respecting with its assumptions arise if one is not willing to rigorously separate between our inner mental world and the outer empirical world. Otherwise, one could “prove” Euclid’s parallel axiom precisely in this manner, by iteratively drawing infinitely further two parallel line segments of say 1-inch length. Why does the possibility to count an infinite set appear to us actually more plausible than such an illustration of geometry’s parallel axiom? Why is there so much more confidence in conceptual constructions than in intuitive representations?

The answer could be the following: Either these two ideas represent only possibilities without any existence claims involved, then there is no difference. Or one of these, counting, is something mental and not subjected to any limitations, resp. to purely logical ones, while the other, i.e. geometric construction, has to respect objective constraints. Then the conditions for the two forms of control would nevertheless have to be analyzed first and one would have to develop a notion of (objective or logical) possibility. In contrast to this, many have been convinced, together with Frege that

in arithmetic we are not concerned with objects which we come to know as something alien from without through the medium of the senses, but with objects given directly to our reason and which being their own kind are utterly transparent to it. (Frege 1884, §105)

The logical paradoxes proved this to be a mere illusion and showed that our mental world is no less complex and intransparent than outer

reality and that there are constraints to our mental activities as there are to our concrete actions. In short, it is not possible to conduct a proof of existence without further assumptions. We do not know how Russell might have considered the difference between number and space, as representations of our inner resp. outer world. In any case Russell accordingly thinks that one can never attain infinite totalities by mere enumeration, and he considers it an empirical fact "that the mind is not capable of endlessly repeating the same act". Russell says: "So the reader, if he has a robust sense of reality, will feel convinced that it is impossible to manufacture an infinite collection out of a finite collection of individuals" (p. 135). We cannot prove that infinite numbers or sets exist. This is probably consensus today while the explanations given are very diverse and controversial.

Russell's critique of Dedekind's construction concerns also the relationship between object and concepts, which is relevant both for logic and for epistemology. While one must postulate, on the one hand, that concept and object are to be distinguished, even of different logical type, that is belong to different categorical strata, concepts, on the other hand, appear in the next step of iteration as extensions of second order concepts. As the assumptions necessary to justify such notions are by no means logical axioms, the situation becomes intransparent. Russell therefore emphasizes "that concepts do not have a factual existence in the common sense" and hence cannot be treated like things, which is, however, contrary to common mathematical practice.

Indeed the topologist Salomon Bochner rightly conceives of the iteration of abstraction as of the distinctive feature of the mathematics of the Scientific Revolution of the 17th century.

In Greek mathematics, whatever its originality and reputation, symbolization [...] did not advance beyond a first stage, namely, beyond the process of idealization, which is a process of abstraction from direct actuality, [...] However [...] full-scale symbolization is much more than mere idealization. It involves, in particular, untrammled escalation of abstraction, that is, abstraction from abstraction, abstraction from abstraction from abstraction, and so forth; and, all importantly, the general abstract objects thus arising, if viewed as instances of symbols, must be eligible for the exercise of certain productive manipulations and operations, if they are mathematically meaningful. (Bochner 1966, 18)

Russell also rejects all attempts at proving the existence of an infinite set for the reason that they violate the requirements of his own theory of types. On the other hand, as was said, mathematical development demands just this.

IV

What then is Russell's theory of types about? In order to repair certain paradoxes of logic and set theory, Russell had introduced the rule "Whatever involves *all* of a collection must not be one of the collection" (Russell 1971, 63) An all-concept referring to a totality cannot belong to the totality. Now Dedekind, in his "Was sind und was sollen die Zahlen?" (cf. Dedekind 1969, 14), had founded the proof for the existence of infinite sets on the antinomial set (of all things) "which can be object of my thought" (Russell's set of sets that are not members of themselves certainly is a possible object of thought). Cantor had called Dedekind's attention to the inconsistency in the latter's foundation of number already in 1899.

Despite the fact that it is not the principle of type classification in itself, but rather its too rigid, reifying interpretation, which is problematic, Russell himself characterizes it as only a negative move, as a principle of interdiction. In particular certain limitations are to be established with regard to the range of the quantified variables in propositional functions. An expression like "all propositions are either true or false", for instance, makes no sense anymore. And with respect to arithmetic it becomes clear that formal axioms could characterize number only in connection with a class of intended applications. Hence existence claims become the main concern of the philosophy of mathematics as they had already been since the beginning of the 19th century in mathematics proper. Existence can, however, be assured only by means, like ostension, which Carnap had outruled as being subjective. In this sense ideas, concepts, laws or other universals, strictly speaking, do not exist. Do Russell's sets exist?

Another consequence is, that mathematical axioms must be understood in analogy with natural laws, as hypothetically conditional propositions, something which Russell, in contrast to other logicians, seems to accept, even if he repeatedly vacillates in doing so (cf. Gödel, 1944, 127). Russell adopts a realistic position as expressed, for instance, when stating that

logic is concerned with the real world just as truly as zoology, though with its more abstract and general features. To say that unicorns have an existence in heraldry, or in literature, or in imagination, is a most pitiful and paltry evasion. What exists in heraldry is not an animal, made of flesh and blood, moving and breathing of its own initiative. What exists is a picture, or a description in words [...] There is only one world, the "real" world [...] The sense of reality is vital for logic. (pp. 169-170)

Hence, Russell would like to ensure that the mathematical (resp. the logical) concepts be obtained by abstractions, like the empirical ones, rather than constructively. This is why he postulates the existence of infinite sets.

Russell accepts – and this might seem astonishing for an empiricist position – that thoughts and feelings are real, while considering the "objects" with which our thoughts and feelings are concerned, to be widely unreal. Shakespeare's thoughts for example, "in writing Hamlet are real. But it is of the very essence of fiction [...] that there is not [...] an objective Hamlet" (p. 169). "A unicorn' is an indefinite description which describes nothing. It is not an indefinite description which describes something unreal" (p. 170). Such a kind of realism may blunder easily with respect to the existence claims of objects described. In the interest of the intended applications of mathematical theories the question arises which ontological status, then, mathematical, i.e. ideal objects possess.

On the one hand, type theoretical distinctions are quite self-evident and common. The totality of chairs is no chair, and the class of red things is not a red thing, but a hypostatic abstraction like redness, or a propositional function like "x is red", or whatever. While concept and object, menu and meal, map and territory are easily distinguished, this distinction, on the other hand, becomes a relative one from the perspective of mathematical activity and its dynamical development. The assumption of absolute boundaries between essence and existence, or between sign and object, is problematical from an epistemological point of view, as only that may appear as motive and driving force of cognizing activity which somehow appears to really exist. The map, or the mathematical model being instruments of exploration first of all, may also become objects of cognition in just the same way they were means of study before. Or to take the example of number theory: numbers are established in

measurement as relations between quantities, but are in number theory considered as definite objects of investigation.

It is indeed mathematics which distinguishes itself for the fact that the process of abstraction is continued indefinitely and recursively, and the number of semantic levels appears to have increased quite considerably again in computer science as compared to traditional mathematics. This means it is the complexity of mathematics, of the “map” itself, which leads to problems. Hence the limitations of type theory. The theory of types touches the very same problem of contextualization and interpretation which had stimulated Russell’s objections against modern axiomatic.

If one conceived the theory of types in an absolute or “ontological” sense, the natural numbers themselves already would all be of different type, not only on Dedekind’s account, but also according to the set theoretical interpretation which Russell himself gave of them. Arithmetic activity, by contrast, requires them all to be of the same type. Russell attempts to meet this requirement later, by introducing additional logical axioms.

But numbers and sets of numbers or number concepts are of different logical type. We cannot proceed inductively in arithmetic, no more than in any empirical field, to gain general number concept. Is then a variable only a placeholder for the individual numbers, and must propositions containing it be interpreted in the sense of an infinite logical conjunction? Or do ideal mathematical objects exist and if so, in what sense? The two views coexist in mathematics. The difference between an affirmation about a general number and an affirmation about all numbers must be introduced into mathematics, as even Russell himself admits, because deduction depends on free variables, or ideal objects. The natural laws cannot be understood as infinite conjunction either, say according to the pattern “stone a drops”, and “stone b drops”, and “stone c drops”, etc. We cannot give individual names to all the stones or all the atoms in the universe, no more than we could name all numbers or all triangles. Peano’s axioms must indeed be understood meaning that if a number is given it obeys the axiomatic description and the axioms are also not to be interpreted as an infinite conjunction of definite statements.

For example, the statement that if a is a numerical symbol, then $a+1=1+a$ is universally true, is from our finitary perspective

incapable of negation. We will see this better if we consider that this statement cannot be interpreted as a conjunction of infinitely many numerical equations by means of 'and' but only as a hypothetical judgement which asserts something for the case when a numerical symbol is given. (Hilbert in: Benacerraf/Putnam (eds.) 1983, 194)

A natural laws' field of intended application must remain open and indeterminate in a certain sense, and it develops together with the evolution of theory. And even in arithmetic we have to accept non-standard models as soon as we confine ourselves to logical descriptions. On the other hand, however, the object field of a theory cannot remain completely indeterminate, for the natural laws alone do not yield any definite knowledge. Russell's efforts at awarding an ultimate meaning to numbers thus show that a theory's development and application are intertwined in a very complicated way. It is true that geometry and arithmetic are treated differently in general. Frege, for instance, who did influence Russell deeply held the view that in arithmetic there need be no appeal to intuition whereas geometry cannot be divorced from intuition. The analogy between mathematical axioms and natural laws would then apply only to geometrical axiomatic and consequently the notion of variable would also be different in arithmetic and geometry. In arithmetic variables were placeholders for definite entities and hence Russell's axiom of infinity.

Russell indeed seems to assume that the axiom of infinity is ontological in nature, and that it is more plausible to assume the existence of infinite sets than to adopt the opposite finitist hypothesis, even though neither can be proved. It is precisely a case of an axiom in the classical sense. And Russell himself appeals to intuitive plausibility. According to our "empirical evidence and the divisibility it would seem favorable to assume that there is an infinite number of objects in the universe", but we cannot know anything about it *a priori*, and "whether the axiom is true or false, there seems no known method of discovering" (p. 143). Strangely enough Russell fails to see that infinite divisibility actually does not mean anything but Dedekind's postulate of the infinite iterability of one and the same operation, would one not want to assume an unknown world of things in themselves, like Kant.

V

A comparison to Kant is in fact suggestive here. Kant's understanding of mathematics is based on the exemplary character of the Euclidean method as discussed at length in Aristotle's *Analytics*.

According to Aristotle mathematical definitions give the essence of a thing but do not say anything about its existence. Hence the existence of a thing must either be given directly in intuition or it must be proved showing *why* a thing is, that is, providing reasons for its existence. Aristotle says that this is particularly the case in geometry, where only points and lines must be assumed to exist, while the other notions must be given through some construction. Thus what cannot be intuited must be constructed in intuition.

In his Critique of Judgment Kant raises the question whether it is possible to provide a general characterization of the human intellect in difference to all other modes of knowing and he discovers it in the fact that we cannot know intuitively but rather are discursive minds with depend on concepts. God's intelligence is intuitive he cannot, comments Cassirer,

think of a thing without, by this very act of thinking, creating and producing the thing. ... Human knowledge is by its very nature symbolic knowledge. It is this feature, which characterizes both its strength and its limitations. And for symbolic thought it is indispensable to make a sharp distinction between real and possible, between actual and ideal things. A symbol has no actual existence as a part of the physical world; it has a "meaning". (Cassirer 1962, 28th printing 1977, 56; 57)

Never shall we humans possess a direct, immediate access to the things in themselves. We can only think by means of symbols or concepts, but "concepts without intuitions are empty". Kant writes:

I can think what I please, provided only I do not contradict myself; that is, provided my conception is a possible thought, though I may be unable to answer for the existence of a corresponding object in the sum of possibilities. But something more is required before I can attribute to such a conception objective validity, that is real possibility the other possibility being merely logical. (Kant, B XXVII)

To exhibit the real possibility of a mathematical conception one has to construct it. "The construction of a conception is the presentation a priori of the intuition which corresponds to the conception" (Kant, B 741). This corresponds to Aristotelian method.

The objectivity of mathematical knowledge is based on the legislative character of pure intuition in the process of knowledge application and concrete experience, according to Kant. It is well known that Kant inquires how it can be possible that the *a priori* conclusions of mathematics lead to real knowledge. And he answers this question by attempting, in a way quite similar to Russell, to determine the conditions of the applicability of mathematical judgments *a priori* and in principle. Kant localizes these conditions in pure intuition, which, differently from mere logical analysis and conceptualization, furnishes objective possibility. Following Kant, we have to distinguish, in a way which seems to anticipate Russell's theory of types, between the general conditions of applicability, and the application itself, between the conditions of the possibility of experience on the one hand, and individual experiences on the other, for Hume has taught us that inductive inference from the particular to the general is not possible "without regulative (not constitutive) principles of experience *a priori*".

The general conditions of the possibility of experience have to be determined *a priori*, quite in the sense of Russell's undertaking, and these *a priori* explanations refer to the conditions which have to be met by objects of possible experience. An epistemology from the perspective of the finite human subject compels us to define ontological questions epistemologically, and Kant thus determines the reality and objectiveness of experience from its productive conditions rather than on the basis of metaphysical assumptions. We obtain our cognitions, however, by intuition, even if we may not, like empiricism does, identify intuition with the perception of mere sense data.

Ernst Cassirer, too, compares Kant's views to those of Russell, writing in this connection:

that our concepts have to refer to intuitions means that they have to refer to mathematical physics, and to prove fruitful in how they are formed. The logical and mathematical concepts shall no longer form the tools we use to erect a metaphysical world of thought. They have their function and their legitimate application

merely within empirical science itself. It is this, their very limitation, which assures reality to them. In this sense, Kant has formulated the supreme principle of all synthetic judgments: "*The conditions of the possibility of experience in general are likewise conditions of the possibility of the objects of experience, and have for that reason objective validity in an a priori synthetic judgment* (B 197). (Cassirer, 1907, 43)

Frege or Russell would perhaps respond to this by saying that there is objectivity in our reasoning and judging without pre-established objects. I hence understand, says Frege for example,

objectivity to mean independence from our feeling, perceiving and intuition [...] but not independence from reason. For to answer the question what things are independent of reason would be as much as to judge without judging, or to wash the fur without wetting it. (Frege 1884, § 26)

Hence there is a strong similarity between Frege, Russell and Kant, apart from the question of intuition. Intuition and space were meant to justify existence claims and were indispensable, because for Kant the (possible) existence of objects was a prerequisite of knowledge and judgement. Frege, as we have seen, wanted to get rid of this prerequisite. But he could not define the identity of concepts or functions without it. Frege had no solution at all to the identity of concepts, which, although buried, remains a virulent issue to this day. And if our grasp of identity and hence "of the extensional equivalence depended on intuition, Frege's account would in its essentials collapse into Kant's" (Potter 2000, 80). This is in fact also the case with Russell's approach as based on the axiom of infinity.

It must be recalled that Russell, in his "Problems of Philosophy" of 1912, formulated a fundamental epistemological principle which strongly reminds of Kant's emphasis on the indispensability of intuition. He states there that "every proposition which we can understand must be composed wholly of constituents with which we are acquainted" (Russell 1997, 58), rather than knowing them merely by description. In the preceding chapter above, one might remember, we have seen that the existence claims involved in our descriptions of the world mark the cornerstone of Russell's realism.

In contrast to Kant, Russell proposes a logical construction of the theoretical concept, a construction, in which the axiom of infinity, replaces Kant's forms of pure intuition. Like Kant, Russell wants to constitute the applicability of mathematical concepts *a priori*. And both consider this possible by investigating into the structure of a separate world – the world of intuition vs. the world of logic.

And however strongly Russell repeatedly criticizes Kant for making too much use of intuition while entering too little upon logic and logical analysis, Russell's philosophy can nevertheless be justly characterized, as has often been done, by the formulation "away from Kant and back to Kant". Is the idea of set in the Russell-Cantor conception of mathematics not indeed just the substitute for space? And must not the set theoretical models, which supplement axiomatic mathematics, be understood in a way similar to how geometrical intuition was to be understood in its relationship to Euclid's axiomatic?

In particular Peano's fifth axiom, the axiom of mathematical induction, does not belong to logic in the proper sense (first order predicate logic) and cannot be replaced by logical axioms either (not even by infinitely many, a fact proved only in 1934 by Thoralf Skolem). In 1900 already, mathematicians like Poincaré had interpreted this axiom as an indication to the synthetic character of arithmetic in Kant's sense. Russell considers Poincaré's views to be erroneous, for

mathematical induction is a definition, not a principle. There are some numbers to which it can be applied, and there are others [...] to which it cannot be applied. We *define* the 'natural numbers' as those to which proofs by mathematical induction can be applied, i.e. as those that possess all inductive properties. (p. 27)

This definition, however, depends on the foundation of the concept of number on the notion of set, and thus on the axiom of infinity, and on other "disputed" assumptions like the axiom of choice, which Russell himself considers to be "unprovable" (p. 117, see for this also Heinzmann, 1993).

VI

While we now know, having followed Russell's argument so far, that the concept of "number" is not "empty", to express it in Kantian terms, because there are finite and infinite sets to which it may be referred, we do not yet know its concrete meaning, its content, except for the fact that we have to assume that it is a concept of set theory in the general sense. It is one thing to postulate that infinite sets exist, and another to establish a correspondence between numbers and sets. We have to concretize this correspondence in particular in such a way that the entities thus given will respect Peano's five axioms. For this purpose, Russell constructs a set-theoretical model the details of which need not interest us here.

"Awareness of universals is called conceiving, and a universal of which we are aware, is called a concept" (Russell 1997, 52). We do not only perceive individual gradations of yellow, we also are aware of universals, general ideas like yellowness. This universal is the subject in judgments like ,yellow is different from blue". In what, then, does awareness of a universal like "yellow" differ from one like "three"? Number words do not function as predicates one might answer. This is not a very profound answer, however.

Russell notes that "number" is the characteristic in numbers, just like "man" is the characteristic in men, or "triangle" the characteristic in triangles. "A number is anything which is the number of some class" (p. 19). Russell defines the individual numbers whose essence is summarized in the general concept of number as designations of respectively determined equivalence classes of sets, and "the number of a class is the class of all those classes that are similar to it" (18). In this determination, the concept of class or set is taken as a yet undefined basic concept, resp. we act as if sets were real things, and were, in fact, the only existing things.

Everybody who has been to school during the last 30 years will feel this to be somehow familiar, the catchword being "New Math". It is even more interesting to note that this reform movement which represents an attempt to create a place for modern mathematics in common sense, an attempt that was certainly as fascinating and heroic as it was ill advised, had been divided in itself from the very outset. One main current, which intended to reduce the meaning of all concepts to logic and set theory,

was from the beginning opposed by another more oriented toward structure and axiomatic.

One of the most eminent reformers, for instance, Georges Papy from Belgium, held a lecture at the Düsseldorf Academy in 1967 in which he pointed out in particular the importance of the axiomatic method. Axiomatic, he said,

has sometimes been presented as the greatest mathematical discovery of the 20th century. It is axiomatic indeed which offers a key to modern pedagogy. It is important that we understand each other well at this point. There are different axiomatic methods, and different kinds of axiomatic expositions. The most perfect and highest among them is the formal axiomatic representation. The objects are not defined and have a role in theory only by means of the abstract relations which have been introduced by the axioms.

One should not say, Papy continues, that this mode of thought was inappropriate for beginners. "One will quite conversely take the physicist's point of view who in his best moments often formulates axioms without realizing it, just like Molière's Monsieur Jourdain formulated prose" (our translation).

Here again, we have an echo of the connection between mathematization and axiomatization indicated by Hilbert. Russell's conception is thus confronted, in all the fields where we have to do with mathematics, with the other view that numbers and all universals of the like have to be conceived of functionally, in their role as means of thought. Rudolf Carnap says that logical empirism assumes "that the object and its concept are one and the same". This identification being a "functionalization" of the object rather than intending a reification of the concept (Carnap 1968, 10). And Moritz Schlick, founder of the "Vienna Circle", thinks that

to make use of concepts in the business of reasoning requires no other of their properties than that certain judgments are valid (e.g. the axioms about the basic concepts of geometry). For exact science, which joins conclusion upon conclusion, the concept, consequently, is indeed nothing but that about which certain judgments hold. Thereby it is hence to be defined. (Schlick 1979, 51)

In the axiomatic approach, the objective seems reduced to the function of making the notion of truth definable.

Russell was not a logical positivist, and he called the predominant trend of logical positivism a new kind of scholasticism which, by concentrating too strongly on formal structure, “may forget the relation to fact that makes a statement true” (Russell 1966, 380). Russell, while deeply interested in logical analysis, always warned against the formalism of thought.

A logically perfect language, if it could be constructed, would not only be intolerably prolix, but, as regards its vocabulary, would be very largely private to one speaker. That is to say, all the names that it would use would be private to that speaker and could not enter into the language of another speaker. [...] For in a logically perfect language, there would be only a single word for every simple object. [...] That is one reason why logic is so very backward as a science, because the needs of logic are so extraordinarily different from the needs of daily life. One wants a language in both, and unfortunately it is logic that has to give way, not daily life. (Russell 1966/1998, 198)

The intention to reduce arithmetic to logic, Russell says, only follows “a rule which is recognized in all of mathematics” and which has the objective of “making the results of a given process of deduction applicable”. The mathematician, however, does not share the intentions of the logician in trying to dissect the concepts into their elementary components. The mathematician does not seek a language at all, but rather a model or a structure. The mathematician generalizes, as a rule, by weaken the hypotheses of his theorems, thus making them true about more structures and by making his structures less conditional in order to be able to enlarge the range of their possible applications. This is effected by the axiomatic foundation of mathematical theories.

Quite in contrast to Russell’s view, one would claim it is precisely under aspects of application that the manifold interpretability of axiomatic systems makes the strength and significance of the axiomatic method. It is in particular difficult to understand why the meaning of all concepts should be totally fixed prior to all application. In the end concepts would be nor more than complete description of individual entities. Mathematical axiomatic, in contrast, is often characterized by Hilbert’s pronouncement

according to which, in the axioms of plane geometry, the terms 'point' and 'line' could just as well be substituted by the terms 'beer jug' and 'table'; "in other words: each and every theory can always be applied to an infinite number of systems of basic elements" (Hilbert in a letter to Frege dated Dec 29th, 1899). Still it is true, we must find at least one application, to solve the problem of existence.

Even intuitionistically and constructivistically minded logicians and philosophers, who do not assign any functions at all to the axiomatic method, emphasize its significance for a praxis of application. In particular, the method seems well-suited to clarify the structure of theories, and "to fuse theories which were originally quite different by uncovering relationships of isomorphy" (Heyting 1934, 30). Systems of axioms thus are properly speaking meta-theoretical entities; this aspect has a rather important role in present-day mathematics because a large part of applications and processes of mathematization occur within mathematics itself (like in the search for inter-theoretical relations).

Only when we have a special intended application in mind, the problem of how to construct the necessary correspondences arises. During the entire 17th and 18th centuries, for instance, mathematics had been understood as the science of quantity. As the new phenomena of electricity and magnetism had to be mathematized in the 19th century, it became necessary to extend the concept of quantity to vector quantity (directed quantity) and to generalize the quantity operations correspondingly (e.g. by introducing a non-commutative vector product). One can as a rule not derive the axioms from the "essence" of the relata (from the meaning of the latter's concepts), simply because one may only know the things with regard to some few relations between them. All objective knowledge is, in fact, relational knowledge. Kant therefore insisted that we cannot know the things in themselves. Axiomatic theories then are synthetic in this sense, and their value lies in their fertility, not in objective or logical justification.

Russell was indeed not unaware of Hilbert's axiomatic method. For him, however, the interpretation of a deductive system always is of fundamental philosophical significance, as it shall serve to clarify how an empiric theory can be linked to perception, and how a mathematical theory, like the theory of arithmetic, can be linked to logic. Despite all his open-mindedness with regard to future developments of mathematics and of the exact natural sciences, Russell repeatedly stresses his interest in the absolutely invariant elements of cognition.

Russell deems the axiomatic method to be incomplete, as unspecified terms occur within the axioms. These uninterpreted terms must indeed be specified in a way, which permits to establish a connection to the intended application. An absolute or ultimate interpretation of mathematical concepts, however, is generally neither possible nor desirable. The axiomatic determination of mathematical concepts will always be incomplete, inasmuch one always has to take into account the possibility that a concept has an empty extension (inasmuch the axioms can be inconsistent), or that it is ambiguous (a property desirable under the aspect of application). If one intends, against that, to introduce all concepts by complete definitions, one must necessarily make metaphysical and psychological assumptions about the world, as it is in itself, which is a futile undertaking. The concept of set which Russell tries to use to specify the meaning of number now is certainly cognitively not less sophisticated than the number concept itself. Russell, however, deems the concept of set both logically more fundamental and empirically simpler.

Technically, there are two problems which have already been mentioned several times. Firstly, the contrast between constructive versus inductive foundation of the concepts or theories. With respect to arithmetics, Russell, like brands of constructivists (Hölder, for instance) prefers to base number on cardinality, defining the cardinal number of a given collection as the set of all equally numerous collections, rather than using the axiomatics of ordinal number.

Secondly, the problem of indeterminacy: the more we try to control our own concepts, and the more we codify and formalize our language, the more we restrict our notion of existence, the less we can maintain the claim to say *a priori* something about the applicability of our mathematical or scientific theories, or about their ontological foundations, for a logical language would not only be, as Russell himself says, a speaker's (resp. theory's) private language, but it would also be useless with regard to application and generalization. I think that this is the real meaning of Einstein's famous quip: "as far as the theorems of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality" (Einstein, quoted after the reprint in: Strubecker (ed), 1972, 414). This is not to mean that mathematics is dispensable, or not useful for application, but only that we cannot "explain" its applicability itself in a mathematical or logical theory. Theories are not at the same time theories of their own application.

They are not mere relational structures either. The meaning of our theoretical concepts depends rather more on the methods of their application, like methods of experimentation and measurement for instance, than on notions about the world. The axiomatic perspective, and in so far Russell is right, must always be supplemented by the perspective of the active subject, placed within a concrete context of life and activity. We must entertain other relationships to the objects of our cognitions than merely theoretico-conceptual ones.

VII

There is no possibility of ultimately determining the meaning of "number", not even within the framework of set theory. The widely known and repeatedly reprinted collection on the philosophy of mathematics (cf. Benacerraf/Putnam 1983), in which parts of Russell's "Introduction to Mathematical Philosophy" were reprinted as well, also contains an essay by Paul Benacerraf titled "What Number could not be" in which he shows that the number concept can be reduced to the concept of set in many, very different ways, without there being an opportunity to separate out the correct set theoretical interpretation from all the possible ones. Benacerraf hence concludes that numbers cannot be sets, or sets of sets, at all, as there exist very different accounts of the meaning and reference of number words in terms of set theory. And even Quine, who shared Russell's dislike for the "disinterpretation of mathematics" and who pointed out that words like "two" and "four" do not occur uninterpreted anywhere in our language, emphasizes that every set theoretical interpretation of number words – Frege's, von Neumann's, or Zermelo's – is used "opportunistically to suit the job in hand, if the job is one that calls for providing a version of number of all" (Quine 1960, 263).

Now either one understands set theory as a means of construction not unlike ruler and compass in Euclidian geometry, being thus prepared to accept, like in geometry, that things can be constructed in various ways and that every construction quite naturally highlights different characteristics of the thing constructed – Benacerraf's various set-theoretical reconstructions of the number system agree in overall structure while disagreeing when it comes to fixing the referents for the particular number terms –, or one would have to say that the concept of set does not represent the last, ultimate, adequate basis for establishing the

existence of number. Existence cannot be provided by symbolic means and has to be assumed from the beginning.

One could thus come to see Russell's set theoretical reductionism, or should we now rather say: constructivism, on the one hand and the axiomatic method in Peano's and Hilbert's sense, on the other as complementary views on mathematics exactly like in classical geometry constructive and deductive method have been complementary to each other (cf. Casari 1974, 49-61; Putnam <1967>, 1975). This complementarity is also expressed in the notions of concept and object in mathematics.

Numbers, for instance, are universals, or general concepts. While from one point of view the general is conceived of as a predicative general, as a concept which (rightly or wrongly) assigns a certain predicate to (concrete or ideal) objects, the general also might appear on the other hand as an ideal object whose generality stems from its (relative) indeterminateness. Any proposition can be written as relation between concrete and universal objects. Instead of saying, for instance: "honey is sweet, one could say: honey possesses sweetness" and could then add sentences in which "sweetness" enters into the subject position like "the sweetness of honey is different from the sweetness of sugar". This implies that we have two types of universals (relations and relata). We claim, for instance, that numbers having certain properties exist, just like we speak of the existence of empirical concepts; like "energy" or "society", for instance. This is in line with the axiomatic view. A system of axiom never defines a determined, individual concept or object, but rather an undetermined universal object. Axiomatic theory refrains deliberately from establishing its field of application in detail – this being Russell's very complaint – but it possesses its "objects" in a general sense.

Let us take another empirical example, that of colors. Redness is to be understood predicatively, and at the same time as an object, as a color, which occurs in an infinite variety of hues. One can now ascribe the generality to its predicative character (that is to class), or to its relative indeterminateness. In the first case, the general is exclusively of predicative nature or, as Russell would say, it resides in propositional functions, and accordingly redness is to be understood as the propositional function "x is red". In the second case, one assumes the existence of general or ideal objects in a more or less Platonist sense, a view expressed in the fact that hypostatic abstractions like "redness" or "sweetness" assume in certain expressions the position of the subject.

In the first case, I have an attribution of things and judgments, in the second case, other judgments are inferred from given ones. In the first case, the generality is tied to the impossibility of finding a counter-example. In the second case, I can apply the law of contradiction only if I confine myself to the questions admissible within the theoretical context. It makes no sense to inquire into the color of the number 3 or into the spiritual life of the falling stone. As mathematics claims to establish objective truths as well, as it is essentially conceived very often also in terms of deductive reasoning and formal proof, both types of universals, functions and objects, are indispensable to its enterprise.

Where deductive inference is concerned, it is indeed not necessary for us to assume the existence of concrete objects determined in every respect. As mathematical deduction reduces to material implication which interprets the proposition "X implies Y", for two sentences X and Y to say that either X is false, or Y must be true (or both being the case), I need not assume the existence of an object x which would make Russell's propositional function "x is a general triangle" true. An index ensuring the reference is sufficient. General objects are nothing but relations or signs.

Let us consider the elementary geometrical concept of "general triangle". It has been the object of many discussions since Locke and Berkeley, which properties are due to a "general triangle". The essential point lies in that the "general triangle" does not possess any determinate properties at all, but that it represents merely a possibility to determine a triangle appropriate for a particular context. Just like the continuum contains the possibility of determining individual points, while not being constituted by points itself. A general triangle is a free variable, and not a collection of determinate triangles.

Which properties are due then to a "general triangle" in a certain situation depends on context. If the task, for instance, is to prove the theorem that the medians of a triangle intersect in one point, the triangle on which the proof is to be based can be assumed to be an equilateral one without loss of generality – because the theorem in case is a theorem of affine geometry and triangle is equivalent to an equilateral triangle under affine transformations. This fact considerably facilitates conducting the proof because of such a triangle's high symmetry. The truth of a proposition about the "general triangle" then means nothing but that this proposition is provable in a certain way, i.o. that a certain proof scheme applies. If

one operates solely on the basis of the object's axiomatic characterizations one does not use the concepts involved referentially at all. Both mathematical and empirical theories as a rule use further resources to solve their problems, and this is how the problem of interpretation of deductive systems becomes significant.

VIII

Let us recapitulate what has been presented up to now. Empirical thought is initially objective thought without worries. It is about the properties of familiar objects, and about dealing with things. Mathematical thought, in contrast, as Aristotle already says, begins with the Pythagoreans, with "theoremata" like: "The product of two odd number is odd". Or: "If an odd number divides an even number without rest, it also divides half that number without rest". These are theorems which, as one says, go beyond what can be experienced concretely, because they state something about infinitely many objects. Actually, they do not state anything at all about objects (e.g. about numbers), but are about universals or ideal objects. They are analytic sentences, which unfold the meaning of certain concepts. This kind of conceptual inference finds its most exalted expression in the modern axiomatic method, a method that is not at all confined to mathematics and to logic.

Now in a period in which the problem of mathematics' applicability had become ever more significant, Immanuel Kant had put forward the claim that real mathematical cognition did not correspond to this structuralist conception, and that mathematics, while being *a priori*, is nevertheless, like any other cognition, both constructive and dependent on experience. Mathematical thought is objective, too, Kant says. Its objectivity shows precisely in intuition. Only, mathematics must construct its objects on the basis of certain conceptual determinations, being unable to take these abstractively from empirical reality because mathematical forms are the basis of all our scientific experience. And what is a precondition of experience cannot be its result. The definition of a mathematical concept, Kant say, thus has to be followed by making sure that the concept created is not "empty", but possesses possible applications. As the mathematical concepts proper designate connected actions, one must ascertain the possibility of their application in a mental intuition or anticipation.

How do we prove, for instance, those seemingly analytic propositions like the already quoted "the product of two odd numbers is odd"? We intuitively represent certain activities. We will say, for instance, if an odd number is divided by 2, there will per definition remain a rest of one. From this we now infer that there is for each odd number X another number N such as that $X = 2N + 1$ holds. If we now have two odd numbers represented in this way before us, and if we multiply these, the said theorem will result quasi automatically by applying the distributive and commutative laws. Mathematics typically proceeds by constructing (algebraic or geometric) diagrams and by observing them, rather than by analyzing the meanings of mathematical concepts. And this results from the fact that mathematics deals with relations and that relations are extrinsic to the relata, as Hume had taught Kant. Mathematics conceived of as diagrammatic reasoning is synthetic a priori, because it cannot itself further justify the rules according to which it proceeds (i. e. the commutative and distributive laws in the example given).

Now according to the structuralist or axiomatic approach one would define the concept exactly by the relations as specified by the axioms (distributive and associative laws) and thus would call the proof analytical. If in contrast one either shares Kant's epistemological approach, that means his fundamental affirmation about the objectivity of the subject, or his view that the objects in question are the individual numbers, rather than the structure itself, then one might be compelled to acknowledge that an equation involving large numbers is mediated by calculation and diagrammatic reasoning to check it and that hence the theorems of arithmetic do not flow directly from the nature of the numbers involved (an argument which Kant uses himself (B 16) such that it seems strange to see Frege apparently using it against Kant (Potter 2000, 60)), and one would have to understand the arithmetical laws (i.e. the arithmetical axioms) apodictically and objectively given and hence consider arithmetic as synthetic.

An empiricist like Russell finally might want to gain these laws inductively from examples and then has to proceed to arguing that for this purpose we need some principle of continuity or principle of the permanence of forms, and would then be forced to find out how this universal principle itself can be justified. According to Kant's prediction, he would have to accept it apodictically, which once more would show that arithmetic is in some sense synthetic.

Russell, on account of his realistic conception of the laws of logic, seems indeed near to some kind of Kantianism, but thinks that Kant restricts the validity of *a priori* statements too much inasmuch he seems to make them dependent on the subject. It could be objected against this that the logical principle of contradiction ultimately remains subject-referred, just like Kant's principle of pure intuition.

An essential objection against the analytical view results from the observation that mathematical structures as well as proofs do not exist or have no effect without being concretely represented or applied. Mathematical objects are intensional objects in the first place. Two mathematical objects can be extensionally identical, but intensionally different by being presented differently. This is why mathematical affirmations as a rule have the form of equations $A = B$. In our example we had: $X = 2N + 1$.

It is important to observe at this point that it does not make sense to ask of *any* two symbols whether they represent the same object or different ones. Identities of the form $A = B$ are context dependent referring to things from a certain perspective. Two commodities, like shoes and chairs, may be considered equivalent from the point of view of economical value. They are equivalent as representatives of some concept or universal. The law of the conservation of energy marks the most important discovery of natural science during the era of Industrial Revolution. Where from do we know, however, that motion and heat are different phenomenal forms of one and the same entity, we call energy? Is it really a case of a discovery, or is what we designate as energy merely a fiction? In our example we compared number symbols and would have to establish the number concept beforehand to establish the essential context. The approach of Russell or Frege in contrast demands a universal and uniform ontological furniture of the world. Logic would really have to deal directly with the universe itself. As Frege had put it:

If we are to use the symbol A to signify an object, we must have a criterion for deciding in *all* cases whether B is the same as A , even if it is not always in our power to apply this criterion. (Frege 1884, § 62) (we shall come back to this in section XII)

IX

Russell felt the time when he made his discoveries, in 1900, after his visit to the Second International Congress of Mathematicians in Paris, to be a time of "intellectual excitation". He writes: "Intellectually the month of September 1900 was the highest point of my life" (Russell 1967, 145). It is well known that Russell's own discovery of the set theoretical antinomies one year later put an end to the belief in ultimate logical foundations. During the attempt to solve the logical problems arising from that effort and to bring the logicist program to a close – an attempt from which the monumental work "Principia Mathematica" evolved, which was published between 1910 and 1913 in three volumes – logicism had attained a culminating point, and simultaneously its definite crisis, as Russell found himself compelled to hypothetically accept non-logical axioms like that which ensures the existence of infinite sets. What proves to be problematic, however, is the reifying conception of set itself, say as a collection of any kind of things. By the paradoxes he had discovered, Russell saw himself prompted to found the concept of set logically.

Where set theory is concerned, Russell's familiar paradox can be avoided if we distinguish, in the sense of type theory, between the set and the totality of its elements, for in this case it is true for every set that it is not an element of itself, and Russell's "set of all sets which do not contain themselves as an element" would be the most comprehensive imaginable set indeed and would represent something like a largest cardinal number. Such a concept, however, is contradictory. Cantor had inferred this on the basis of his own power set axiom, and Leibniz already had connected the question of the infinite with the problem of the contradictory character of such universal concepts. To put it quite generally: our highest level concepts or principles cannot be unequivocally specified or defined.

In his letter of July 1899, in which Cantor pointed out to Dedekind the inconsistency in the latter's foundation of arithmetic, Cantor had already established the principle that only that can be called a set or a "consistent multiplicity" which lends itself to being combined to "one thing", whereas he calls multiplicities for which "the assumption that all of its elements are together leads to a contradiction", it being impossible to conceive of the multiplicity as of "one completed thing", absolutely infinite or inconsistent multiplicities. Independent of Cantor, Russell

arrives at a similar conclusion, defining sets, as we shall see, finally as extensions of propositional functions.

From objects to relations or propositional functions, from construction to deduction, and from intuition to language and to logic, this is the path of mathematical philosophy, which Russell embodies in such a most exemplary way. This is how language and speech then become the “conditions of the possibility of cognition”, the latter being nothing but linguistically modified Kantianism.

“Analytic” philosophy is one more variant of Kantian philosophy, a variant marked principally by thinking of representation as linguistic rather than mental, and of philosophy of language rather than “transcendental critique”, or psychology, as the discipline which exhibits the “foundations of knowledge”. This emphasis on language [...] does not essentially change the Cartesian-Kantian problematic, and thus does not really give philosophy a new self-image. (Rorty 1979, p. 8)

From Russell’s type theory it follows, as has been said, that sets cannot simply be what common sense imagines them to be and what everybody has learned who has been to school in the times of the “New Math” reform movement. Although Russell makes large efforts over extensive parts of the book to reduce the concept of number to the concept of set, treating the latter as a fundamental concept in doing so, the way he develops his own argument shows that such a conception must be revised. The last but one chapter hence is about sets, and there Russell’s concern is to “realize why classes cannot be regarded as part of the ultimate furniture of the world [...] We cannot take classes in the *pure* extensional way as simply heaps or conglomerations” (pp. 182-183). If we conceived of sets extensionally as objects, Russell thinks, it would be impossible for us to understand

how there can be such a class as the null-class, which has no members at all and cannot be regarded as a “heap”; we should also find it very hard to understand how it comes about that a class which has only one member is not identical with that one member. (p. 183)

Kurt Gödel has considered this an overreaction to the paradoxes, saying that Russell's argumentation at best shows "that the null class and the unit classes (as distinct from their only element) are fictions, not that all the classes are fictions" (Gödel 1944, p. 141). The two particular classes mentioned above could be treated, Gödel continues, like the points at infinity in geometry, and like similar mathematical generalizations. This, however, would require historically determined borderline concepts or general principles, like the principle of continuity, or the principle of permanence of relations, and would then lead to understanding the theory's context as the primary thing and foundation. With this, however, we should be back to the axiomatic view and then would again have to bother about Russell's' problem of how to determine the applicability of an axiomatized theory.

In the course of the argument of his book Russell gradually gives up his realistic attitude and the basic entities of his reconstruction become more and more turned into "logical fictions". In his penultimate chapter, Russell indeed defines sets as equivalence classes of propositional functions. Such a definition could be conceived of to be circular if one were to assume that the concept of function requires identity itself. A propositional function, Russell says, is "a function whose values are sentences". Russell, for instance, defines the zero class as the extension of a contradictory propositional function, i.e. a propositional function, which ascribes contradictory things to its subject. Here, the attempt at logical precision causes the tendency to push assumptions of existence and ontological notions as far back as possible, or to eliminate them altogether, and to advocate an intensional view of set theory. The ontological status of these intensions, concepts or functions, however remains somewhat unclear. In his chapter II, Russell had already expounded the advantages of an intensional conception of sets. The same tendency, however, characterized predominantly the axiomatic method, and thus set theory also came to be axiomatized by Zermelo, in order to exclude the paradoxical set formations. The problem of application arises anew.

From relations and functions (of axiomatic) to objects (sets) and from there back again to functions (propositional functions): this to and for path demonstrates the logician's desperate quest for a thing absolutely given and existent. On this path to make things ever more precise, Russell at times wanders into a position of extreme empirism, one that insists on regarding nothing but the immediate sensual data for real.

Here we find a remarkable development in Russell's thought lasting two or three decades. A seemingly straightforward epistemology is dragged by logic and a theory of meaning into one of the most exotic metaphysics yet presented. We call it "logical atomism". In the beginning Russell thought that the things with which I am acquainted are the immediate objects of experience. (Hacking 1975, 72)

But finally he ends up with pure sensual data. To make his point, Russell holds up a piece of chalk and says: "This is white [...] I do not wish you to be thinking of this piece of chalk in my hand, but of what you see if you direct your gaze to this piece of chalk." Hacking comments on the fact that the "this" in Russell's statement "this is white" is not intended to designate an object, but a sensual datum, by pointing out that Russell "almost adopted Berkeley's position" in this way. Russell himself, after having arrived at a position of extreme empirism, closes by assuming that empirical objects like chair, table, etc. only exist in one's head as well. This is in some way a Berkeleyism "without God", and hence without anybody who will still perceive my writing table, thus conveying existence to it, after I have turned my back on it. As Russell lacks Berkeley's God whose perception conveys duration to the world, his position is even more complicated or exquisite than the latter's.

X

We should like to briefly discuss the book's last chapter on the relationship between mathematics and logic, before we come back to Russell's notion of set. The chapter begins by stating that

mathematics and logic, historically speaking, have been entirely distinct studies. Mathematics has been connected with science, logic with Greek. But both have developed in modern times: logic has become more mathematical and mathematics has become more logical [...] This view is resented by logicians who, having spent their time in the study of classical texts, are incapable of following a piece of symbolic reasoning, and by mathematicians who have learnt a technique without troubling to inquire into its meaning or justification. (p. 194)

Nevertheless, it is even truer today that the philosophy of mathematics means something different to mathematicians than to logicians.

Frege brought about the extension of logic, which forms the basis of all of Russell's work by drawing on the mathematical notion of function. This notion hence offers the opportunity of establishing a relationship between mathematics and logic, even if it is true that, in line with the separation of the fields of work proper to mathematics and to logic, different exemplary representations of the term of "function" exist in mathematics on the one hand, and logic on the other. For mathematics and the exact sciences, the concept of natural law yields the prototypical case of a function. For logic, against that, a function is either a propositional function, or an algebraic formula.

Notwithstanding these different intuitions, it makes sense to illustrate the difference between mathematics and logic by considering the problematic of the function concept. It is a fact that mathematics has always since the 17th century been busy with great emphasis to construct ever more general functions. Mathematics is essentially relational thinking, and logic did not begin to modernize itself before it learnt from mathematics during the 19th century to construct a logic of relations. Bolzano, de Morgan, and Jevons presumably were the first to treat propositions as relations, or functions.

Let us thus speak of functions quite generally. Functions initially were simply correspondences between a domain of inputs, that is the area of the function's arguments, and an area of values of the functional transformation, the so-called co-domain or outputs. Propositional functions assign sentences, resp. judgments about things to things, In this sense, one can also conceive of concepts as of abbreviated judgments, or functions.

But just as the user of a computer program is not interested in the details of programming, but rather in the function thus provided, mathematics, logic, and science as such are always concerned with dissociating themselves from the concrete modalities by which a function is given. Mathematics and logic worked at elaborating such a general concept of function throughout the entire 18th and 19th centuries, until they came to the conclusion to understand a function as an equivalence class of concrete representations of the same, the relation of equivalence being given by the "axiom of extensionality". This axiom says that functions are equivalent if they yield the same values when applied to the

same arguments. Only to such equivalence classes of concrete representations can be attributed important properties, for instance the property of continuity, of differentiability etc. (cf. Otte, 1994, chapter X). And we are then even able to define algebraic or logical operations for such classes. In this way, the equivalence classes become the objects proper of mathematics and logic. This respective method of establishing functions or concepts in general is called "definition by abstraction".

Now it is possible to hold the view that mathematics has essentially to do with intensional objects, i.e. with objects, which are determined conceptually, resp. by their properties. One or another representation of the function can be of particular importance, according to the respective goal. If I have, for instance, two views of a house, and desire to know how the front looks like a picture of the front side will surely be more useful than one of the back side. The same is true with the ideal objects of mathematics and with logical fictions. Which of the properties are significant for the representation of a "general triangle" in elementary geometry is dependent on the task at hand. We have already commented on this. And this is where the entire problem lies which consists in finding a specific representation of such an equivalence class or of an ideal object, an extensionally equivalent, but intensionally quite different representation.

XI

In Frege's presentation of the distinction between meaning and reference, the same asymmetry occurs which we have established for the identity criteria of functions, and which is based on existence claims with respect to the arguments or objects, while concepts (intensions) or functions merely possess an identity. Russell has treated this problem in his famous theory of descriptions which he summarizes in its essential parts in the 16th chapter of his "Introduction". He corrected and extended Frege's interpretation of $A = B$, resp. of $A = A$ by introducing a distinction between designation and reference. Frege had treated the difference between these two forms of an equation by his own distinction between sense and meaning, concluding that singular descriptions function like designations as one usually understands them referentially. Russell considers this to be an error, for

a proposition containing a description is not identical with what that proposition becomes when a name is substituted, even if the name names the same object as the description describes. "Scott is the author of *Waverley*" is obviously a different proposition from "Scott is Scott" (p. 174), and further on: "If 'x' is a name, $x = x$ is not the same proposition as 'the author of *Waverley* is the author of *Waverley*'" (p. 176). $x = x$ is always true because x being an index (a name) is taken as an existence claim, rather than as a description. In fact, propositions of the form "the so-and-so is the so-and-so" are not always true: it is necessary that the so-and-so should *exist*. It is false that the present king of France is the present king of France, or that a round square is a round square. (p. 176)

To speak of contradictions makes no sense without any existence claims: Green unicorns are yellow. Or aren't they? Even the paradox of the perpetually lying inhabitant of Crete or that of Russell's "barber who shaves exactly those men that do not shave themselves" can be reduced in this way to problems caused by implicit illegitimate existence claims (cf. L. Wen 2001). We cannot generate the existence of an objective reality by concepts and language alone, and without this reality, there are no contradictions either, Kant would say (cf. *Critique of Pure Reason*, B 622).

Russell himself obviously passes here, where the concept of function is concerned, into the complementary extensional position as well (he had spoken of "two kinds of propositional functions with which we have to do"). In other terms: the "being" of functions now does indeed depend on the domain of the quantified argument variables, thus supposing an object classification in the function's field of application. In classical mathematics a theory of functions requires a theory of quantities first. Russell's theory of types alone imposes strong restrictions on the range of the quantified variables. In logic proper (first level predicate logic), the terms "all" and "there are" may only be applied to objects (subject variables), but in no case to predicate variables (concepts). This requirement is largely not met by Peano's axioms, for in his fifth axiom, there occurs predication over all properties.

Above all, however, Russell wishes to see established, before all science and mathematics, an absolute and universal ontology, to see "what there is". This kind of fundamentalism is of very traditional nature, and is being treated by Russell in an increasingly restrictive way. In modern

mathematics, however, activity is primary, not ontology and Russell himself acts in this way. Russell himself remains silent with respect to his permanent assumptions concerning the existence or identity of concepts he employs. The questions of ontology are determined relatively, with reference to an activity, resp. to a theory. Sets, functions, concepts and the like cannot be conceived of in a purely extensional way. Mathematics was, during long periods of its history understood constructively.

Equations of the $A=B$ kind can only be regarded within their context; the problem of individuation cannot be solved either in a purely logical way, nor empirically. Russell's famous theory of description shows that answers of any kind could be given to the question "what there is". It depends on the perspective taken. The theory or the system of description actually determines what there is, and what is admitted as possible. But even to-days structuralist philosophy of science which assumes that any decomposition of our world of experience into ontological categories is theory-dependent, conversely finds itself impelled, in the question as to the criteria of identity for theories, to accept pragmatic determinations. Let us take Newton's particle mechanics as an example.

Of course, the initially purely formally characterized term, like that of "particle", must additionally be given an interpretation which is correct with regard to content; the "particles" mentioned here are to represent genuine particles, and no loving souls. However: what are genuine particles? [...] The answer given by the structuralist approach is very differentiated. The first thing established is that theories contain intended applications as constituents – in a comprehensive sense of application. [...] From this follows that it is the pragmatic relationship between the epistemic subject who tries to apply the theory, and the theory itself, which determines what is to be regarded as a real, or genuine, object. (Moulines 1994, 187f)

We have already pointed out the example of electromagnetism, and the connected reinterpretation of the concept of quantity. Logicians and mathematicians never operate only in one homogenous object field, but are intent on obtaining and opening ever new applications for their structures. If they are asked by what they let themselves be guided, the most frequent answers are: intuition, analogy, experience, etc. And surely

we will have to understand these concepts in a pragmatic sense. What is "intuitive" in each case is quite relative, and human experience can never be fully expressed in words.

Logicians like Frege and Russell do not want to bother with intuition. If we intend to form equivalence classes, however, we must introduce a perspective from which we consider things as similar or equivalent. In a certain sense everything seems similar to everything. Ideal objects like number, energy, society, etc. are indeed not given to us in themselves, but only mediated by an activity and some feeling of "similarity" or "equality of species". The concept of similarity, however, resists "the reduction to less dubious concepts like that of logic or set theory" (Quine).

The subject matter of mathematics being defined in terms of similarity and difference as well, we have also to introduce strict divisions and classifications. Already Aristotle's thinking was riddled with two orientations diametrically opposed to each other. Aristotle is most often regarded as the great representative of a logic and mathematics, which rests on the assumption of the possibility of clear divisions and rigorous classification.

But this is only half the story about Aristotle; and it is questionable whether it is the more important half. For it is equally true that he first suggested the limitations and dangers of classification, and the non-conformity of nature to those sharp divisions which are so indispensable for language [...] (Lovejoy 1964, 58),

and for mathematics, as one might add.

XII

Now logic and mathematics do not only differ as to their views about mathematics, but views differ within logic as well. We have already hinted at this in the examples given by Russell on the one hand and Peano on the other and have called these two orientations set-theoretical reductionism on the one hand and axiomatics on the other. J. van Heijenoort has characterized the two rather different conceptions of logic connected with these different foundational programs. The first considering logic as universal and as a language, the other viewing logic

more as a part of an algebraic calculus in the sense of Boole, Grassmann, or Peano. Russell, who like many other authors was strongly influenced by Frege, rather more advocates the conception of logic as a language, which remains inevitably imbedded in our thinking.

The universality of logic expresses itself in an important feature of Frege's system. As is well known, according to Frege, the ontological furniture of the universe, divides into objects and functions, (Heijenoort 1967, 325)

The other conception of logic does not know of the notion of a fixedly given universal ontology. Rather, the ontology can be changed.

The universe of discourse comprehends only what we agree to consider at a certain time, in a certain context. For Frege it cannot be a question of changing universes. One could not even say that he restricts himself to *one* universe. Not necessarily the physical universe, of course, because for Frege some objects are not physical. Frege's universe consists of all that there is, and it is fixed. (Heijenoort 1967, 325)

Just as there were two views of logic in the 19th century, there were two different conceptions about what could be called the "foundation of mathematics". Besides the axiomatic movement, which climaxed with Hilbert's "Grundlagen der Geometrie" of 1899, there was the movement of the so-called "arithmetizing rigor" which strove to reduce all mathematics to numbers. Russell's focus, too, went increasingly towards "understanding number" after 1897, as he considered arithmetic to be the foundation and the starting point of all mathematics. He interpreted this in the sense of the goal of reducing arithmetic to logic, resp. to logic and set theory. With this, we have returned to our starting point.

This gives us occasion to sum up the presentation of Russell's philosophy of mathematics in a thesis which places the latter within a more general epistemological context. Russell's epistemology, and in particular his philosophy of mathematics, is based in all its particularities and difficulties on the attempt to specify the problem of application and the reference to reality of our (mathematical) cognition by a-priori means, and thereby to remove its independent dynamic and insecurities from

the problem of application. This resides in the nature of a logical conception of the world. Russell's philosophy appears in some way as Kantianism turned inside out logically. In contrast to pure mathematics and to logical positivism, Russell is concerned with application. But he attributes absolute priority to logic, thus eliminating the connected problem of the interchange between cognition and reality.

The effect of this is that reading his texts is instructive in a twofold sense, both in his achievements and logical innovations and in his deficiencies and errors. Any introduction to mathematics and to mathematical philosophy should start as a commentary on Russell's project.

Bibliography

- BATESON, G. (1973). *Steps to an Ecology of Mind*. Paladin, Frogmore UK.
- BENACERRAF, P. and PUTNAM, H. (eds.) (1983). *Philosophy of Mathematics*. Cambridge UP.
- BOCHNER, S. (1966). *The Role of Mathematics in the Rise of Science*. Princeton, Princeton University Press.
- CARNAP, R. (1968). *The Logical Structure of the World*. London, Routledge & Kegan Paul.
- CASSIRER, E. (1907). *Kant und die moderne Mathematik*. Kantstudien Bd. 12.
- _____ (1962). *Leibniz' System in seinen wissenschaftlichen Grundlagen*. Darmstadt, Wissenschaftliche Buchgesellschaft.
- CASARI, E. (1974). Axiomatical and Set-theoretical Thinking. *Synthese* 27.
- DEDEKIND, R. (1969). *Was sind und was sollen die Zahlen?* Vieweg, Braunschweig.
- FREGE, G. (1884). *Grundlagen der Arithmetik*. Breslau.
- GÖDEL, K. (1944). "Russell's Mathematical Logic". In: SCHILPP, P. A. (hgb.). *The Philosophy of Living Philosophers*.
- HACKING, I. (1975). *Why does Language Matter to Philosophy*. Cambridge University Press.
- HEIJENOORT, J. (1967). Logic as Calculus and Logic as Language. *Synthese*, v. 17, pp. 324-330.
- HEINZMANN, G. (1993). *Zwischen Objektkonstruktion und Strukturanalyse*. Vandenoek Ruprecht, Göttingen.

- HEYTING, A. (1934). *Mathematische Grundlagenforschung, Intuitionismus. Beweistheorie*. Heidelberg, Springer Verlag.
- HILBERT, D. (1964). *Hilbertana*. Wissenschaftliche Buchgesellschaft, Darmstadt
- HÖLDER, O. (1899). *Anschauung und Denken in der Geometrie. Inauguration Lecture*, Leipzig.
- KORIAKO, D. (1999). *Kants Philosophie der Mathematik*. Hamburg, Meiner Verlag.
- KRAJEWSKI, W. (1977). *Correspondence Principle and Growth of Science*. Dordrecht, D. Reidel Publishing Company.
- LOVEJOY, A. O. (1964). *The Great Chain of Being: A Study of the History of an Idea, (The William James Lectures 1933)*. Harvard Univ. Pr, Cambridge, Mass.
- MOULINES, C. U. (1994). "Wer bestimmt was es gibt?" In: *Zeitschrift für philosophische Forschung*, Bd. 2 (175-196).
- OTTE, M. (1994). *Das Formale, das Soziale und das Subjektive*. Frankfurt/M, Suhrkamp Verlag.
- POTTER, M. (2000). *Reasons Nearest Kin*. Oxford University Press.
- PUTNAM, H. (1967-1975). *Mathematics without Foundations in ders. Mathematics, Matter and Method*, CUP Cambridge
- QUINE, W. V. (1960). *Word and Object*. Cambridge, The Mit Press.
- RORTY, R. (1979). *Philosophy and the Mirror of Nature*. Princeton University Press.
- RUSSELL, B. (1954). *The Analysis of Matter*. London, Allen & Unwin.
- _____ (1967). *The Autobiography of Bertrand Russell*. London, Allen & Unwin.
- _____ (1970). *Introduction to Mathematical Philosophy*. London, Allen & Unwin.
- _____ (1971). *Logic and Knowledge: Essays 1901-1950*. MARSH, R. C. (ed.). London, Allen & Unwin.
- _____ (1971a). *The ABC of Relativity*. London, Allen & Unwin.
- _____ (1973). *Essays in Analysis*. LACKEY, D. (ed.), London, Allen & Unwin.
- _____ (1985). *The Philosophy of Logical Atomism*. La Salle Open Court Publ. Co.
- _____ (1997). *The Problems of Philosophy*. Oxford University Press.

- SALMON, W. C. (ed.) (1970). *Zeno's paradoxes*. Indianapolis, Ind. Bobbs-Merrill.
- SCHLICK, M. (1979). *Allgemeine Erkenntnislehre*. Frankfurt/M., Suhrkamp Verlag, 1. Aufl. 1925
- STRUBECKER, K. (hgb.) (1972). *Geometrie*. Wissenschaftliche Buchgesellschaft, Darmstadt
- WEN, L. (2001). Semantic Paradoxes as Equations. *Math. Intelligencer*, v. 23, pp. 43-49.

Recebido em abr./2000; aprovado em jun./2000