Can x=3 be the solution of an inequality? A study of Italian and Israeli students

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Abstract

This paper describes a study regarding Israeli and Italian students' solutions to algebraic inequalities. The findings presented here show similarities in students' correct and incorrect solutions, in both countries. Fischbein's notions of algorithmic, intuitive and formal knowledge are used to analyze the data. The findings indicate that students generally worked in an algorithmic manner, intuitively drawing analogies to the solutions of related equations. We conclude by suggesting some educational implications.

Key-words: inequalities resolutions; algorithmic knowledge; intuitive knowledge.

Resumo

Este artigo descreve um estudo que focaliza soluções dadas, por estudantes de Israel e da Itália, para inequações algébricas. Os resultados apresentados aqui mostram similaridades nas respostas corretas e, também, nas incorretas, nos dois países. As noções de Fischbein, sobre conhecimento algorítmico, intuitivo e formal, são usadas na análise de dados. Indicam que os estudantes usualmente trabalharam de um modo algorítmico, formando, intuitivamente, analogias com soluções de equações. Em conclusão sugerimos algumas implicações pedagógicas.

Palavras-chave: *resoluções de inequações; conhecimento algorítmico; conhecimento intuitivo.*

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Introduction

There is a wide call for using students' ways of thinking and their mistakes in teaching (e.g., NCTM, 2000). Even though, this recommendation seems to be "speaking for itself", any attempt to take it from theory to practice, shows how complex and demanding this is. Among the various prerequisites for such teaching, are familiarities with students' various correct and incorrect reactions to different related types of tasks, with possible reasons for students' errors, with different mathematical solutions to the given tasks and with available teaching approaches to be considered under specific circumstances. All of the above are needed, but do not guarantee that the student whom we taught would gain mathematical understanding. Thus, a critical reflection of the teaching process, i.e., planning, designing and carrying out instruction may contribute to a constructive use of students' ways of thinking when teaching.

May be it is needless to say that the teacher should be well acquainted with the relevant mathematical issues. As it is the students' conceptions and errors that are to be considered, there is naturally a need to have the

Inequalities play an important role in mathematics. They are part of various mathematical topics including algebra, trigonometry, linear planning and the investigation of functions (e.g, Chakrabarti & Hamsapriye, 1997; Mahmood & Edwards, 1999). They also provide a complementary perspective to equations. Accordingly, the American Standards documents specify that all students in Grades 9-12 should learn to represent situations that involve equations, inequalities and matrices (NCTM, 1989). They further recommend that students would "understand the meaning of equivalent forms of expressions, equations, inequalities and systems of equations and solve them with fluency" (NCTM, 2000, p. 296). To implement these NCTM recommendations it is crucial to consider students' ways of thinking about inequalities.

However, so far, research in mathematics education has paid only little attention to students' conceptions of inequalities (e.g., Dreyfus & Eisenberg, 1985; Linchevski & Sfard, 1991; Tsamir & Almog, in press). Most of the related articles dealt with teachers' and researchers' suggestions for instructional approaches, usually with no research support. They recommended, for instance, the sign-chart method (e.g., Dobbs &

Peterson, 1991), the number-line method (e.g., McLaurin, 1985; Parish, 1992), and various versions of the graphic method (e.g., Dreyfus & Eisenberg, 1985; Parish, 1992; Vandyk, 1990).

Those few studies, which have been published, tended to describe students' reactions to a few inequalities of the types commonly presented in class, and usually reported only one or two difficulties. For instance, studies pointed to students' tendency to make invalid connections between the solution of a quadratic equation and its related inequality (e.g., Linchevski & Sfard, 1991; Tsamir, Tirosh, & Almog, 1998). Other studies related to students' tendency to regard transformable inequalities as being equivalent. They further identified the need to use logical connectives (Parish, 1992), and found the solutions of inequalities with "R" or "f" results extremely difficult (Tsamir & Almog, 1999).

The present study was designed in order to extend the existing body of knowledge regarding students' ways of thinking and their difficulties when solving various types of algebraic inequalities. During discussions at conferences for the psychology of mathematics education (PME22, 1998; PME23, 1999), it was found that in both Italy and Israel, algebraic inequalities usually receive relatively little attention and are commonly discussed only with mathematics majors in the upper grades of secondary school. Discussions are usually limited, emphasizing the "practical" algorithmic perspective of algebraic manipulations. Attention is usually paid mainly to providing students with rules for solving, with no relation to "Why solve it this way?" "Are there additional ways to solve it?" or "How can I be sure that the solution I have reached is the correct solution?" Moreover, in both countries, the two researchers witnessed students' and teachers' frustration with the difficulties encountered when dealing with inequalities. Consequently, an Italian-Israeli collaborative study was designed to investigate students' ways of solving standard and non-standard tasks with similar, underlying mathematical ideas. The students were given six tasks, presented in the manner to which they were accustomed in their classes, i.e., "solve" tasks, designated as "standard tasks". They were also given nine tasks, related to the same mathematical issues, which were presented in a non-customary manner, and designated as "non-standard tasks".

In this paper we focus on 2 of the 15 tasks that were give to the students. Both tasks dealt with the same underlying mathematical situation, i.e., single-value solutions to inequality tasks. The main related research question was: Do Israeli and Italian secondary school students accept the expression $x = a$ as the solution of an inequality – Once, presented in a standard multiple-choice "solve" task, and once as a "reversed order" task, asking whether a given set can be the truth sets (the solution) of any equation or of any inequality, and are the students' reactions to the two tasks consistent?

Methodology

Participants

One-hundred-and-seventy Italian high school students and 148 Israeli high school students participated in this study. Both the Italian and the Israeli participants were 16-17 year old mathematics majors. That is, in both countries we examined students who were aiming to take final mathematics examinations in high school. Success in these examinations is a condition for acceptance to academic institutions, such as universities.

In their previous algebra studies, the participating students had studied the topic of algebraic inequalities, including linear, quadratic, rational and absolute value inequalities. In both countries, the participating students were taught this topic in a conventional way, being presented with different methods for solving the different types of inequalities. For example, parabolas or the number line to solve quadratic inequalities, and "multiplying by the square of the denominator" for the solutions of rational inequalities.

Tools

A 15-task questionnaire was administered in both countries. Italian and Hebrew versions were given to the Italian and Israeli students respectively. The two tasks analyzed here are Task 1 (a non-standard task) and Task 9 (a standard task).

Task 1

Consider the set S= {xÎR: x=3} and check the following statement:

S can be the solution of both an equation and an inequality.
Explain your answer.

Task 9

Indicate which of the following is the truth set of $5x^4 \leq 0$

A={x: x>0}   B=R   C={x: x<-5}   D={x: 0<x<1/5}
E= ∅   F x=0   G={x: x≤0}

Task 1 demanded proving the existence of a case where x=3 is the solution of an equation, and also proving the existence of a case where x=3 is the solution of an inequality. The easiest way to go about this was by providing suitable examples. This kind of assignment, asking the students to examine the existence of a case where x=3 is the solution of either an equation or an inequality; then, if possible, to provide tasks to match a given solution, was not dealt with in either the Israeli or the Italian classes we investigated.

We expected the first part that related to the existence of a suitable equation to be easy, and the second part, where the students had to examine the existence of a case where x=3 is the solution of an inequality, to be problematic.

Task 9 was a standard task, similar to other tasks presented in Israeli and Italian classes. We assumed that most students would solve it correctly.

Procedure

In both countries, the mathematics teachers of the classes distributed the questionnaires, during mathematics lessons. The students in each of the countries were given approximately one hour to complete their solutions, which usually was enough time. The researchers analysed, categorised and summarised the different solutions. In two additional meetings the researchers decided on possible ways to present the data.

Results

The results will be presented in the following order. First, an analysis of Israeli and Italian students' responses to Task 1, then their responses to Task 9, to conclude with an analysis of the consistency in students' reactions to the two tasks.

Students' Reactions to Task 1

In both countries, none of the students had any problems in correctly responding that x = 3 can be the solution of an equation. Most of them accompanied their responses by an example, usually of a first-degree equation, such as 2x-6=0. This, however, was not the case with the participants' responses to the question whether x = 3 can be the solution of an inequality, in both Israel and Italy.

Table 1 – Frequencies of students' solutions and justifications to Task 1 (%)

	ISRAEL N=147	ITALY N=150
TRUE*	51.4	48.3
Valid explanation	5.4	2.0
A system of inequalities	15.5	0.7
X=3 belongs to the solution	3.3	3.0
Other**	27.2	42.5
FALSE	48.6	51.7
A solution of inequality is an inequality	19.5	22.0
Other**	29.1	29.7

* Correct response

** Irrelevant or missing justifications

Table 1 shows that in both countries, only about 50% of the students who responded to this task, correctly claimed that x = 3 can be the solution of an inequality. Still, most of them did not accompany their claims by any justification and only a few students, Israeli or Italian, gave valid explanations. These latter explanations were usually the presentation of the following example of the quadratic inequality $(x-3)^2$£ 0. More prevalently in Israel, but also in a few Italian cases, explained that the claim "x = 3 can be the solution of an inequality" is true, because x = 3 can be the solution of *a system* of inequalities. Such justifications were often accompanied by an uncomplicated, linear example, such as

$$\begin{cases} 2x-6\leq 0 \\ x-3 \geq 0 \end{cases}$$

Another type of interesting justification, given by a small number of Israeli and a small number of Italian participants was that "the claim is true, because x = 3 can belong to the set of solutions of an inequality." This justification was accompanied by illustrations, such as, $5x - 10 > 0$, further explaining that "the truth set (or solution) of this inequality is $\{x: x>2\}$, and 3 is one of the values that satisfies this condition, and therefore x = 3 belongs to the truth set of $5x - 10 > 0$."

Students' Reactions to Task 9

Only about 50% of both the Israeli and the Italian participants who responded to this task, correctly identified x = 0 as the solution of the inequality (see Table 2).

Table 2 – Frequencies of students' solutions to Task 9 (in %)

	ISRAEL N=128	ITALY N=168
X = 0*	53.3	51.2
$X \leq 0$	23.8	16.2
Phi	17.1	26.7
Other	5.8	0.9

* Correct solution

A substantial number of the participants claimed that the set of solutions was empty (Phi, or 'there is no solution to the given inequality'). Some of them volunteered the explanation that x^4 has an even power and thus it can never be negative, showing that they ignored the "zero-option". Another interesting phenomenon was the Israeli and Italian students' tendency to answer that the set of solutions of $5x^4 \leq 0$ was $x \leq 0$, which was further explained by a number of them, claiming, for instance, "I simply computed the fourth root of both sides of the inequality."

Examining the consistency in students' reactions to Tasks 1 and 9

As can be seen from Tables 1 and 2, and as mentioned before, about half of the participants from each of the two countries claimed that "x = 3 can be the solution of an inequality", and about half of the participants identified x = 0 as the solution of $5x^4 \leq 0$. That is, about half of the participating students pointed to the possibility of having x = a as the solution of an inequality, either in Task 1 or in Task 9. A question that naturally arose was, were these the same students? That is to say, did the students consistently express their understanding that x = a could be the solution of an inequality in their reactions to both tasks, by responding "true" to Task 1 and "x = 0" to Task 9? Table 3 shows that the answer to this question is no.

Table 3 – Frequencies of consistent and inconsistent reactions to Tasks 1 & 9 (in %)

		ISRAEL N=148	ITALY N=170
CONSISTENT		57.8	48.3
Task 1	*Task 9*		
True	Correct	29.36	23.9
False	Incorrect	28.44	24.4
INCONSISTENT		35.76	38.2
Task1	*Task 9*		
False	Correct	22	21
True	Incorrect	13.76	17.2
OTHER*		6.44	13.4

* Providing no response to at least one of the two tasks.

Only about 29% of the Israeli participants and about 24% of the Italian participants exhibited a general view that x = a can be the solution of an inequality and also correctly reached this type of a solution in reaction to the "solve" drill in Task 9. It is also notable that a similar percentage of each group rejected the option of x = a being the solution of an inequality, and did not reach the correct x = 0 solution in Task 9 as well.

More than 35% of the participants in each country were inconsistent in their reactions to the two tasks. Part of them correctly claimed that x = 3 could be the solution of an inequality, but did not identify $x = 0$ as the solution of the inequality in Task 9. More interesting were the inconsistent reactions of about 20% of both the Israeli and the Italian participants. On the one hand, they claimed that $x = 3$ can not be the solution of an inequality, usually explaining that "an inequality can only be the solution of an inequality". On the other hand, within the same questionnaire they reached an $x = 0$ solution to the inequality presented in Task 9.

Discussion

Our findings indicate that, as expected, all students in both countries were aware that $x = 3$ can be the solution of an equation, and that many of them encountered difficulties in identifying the possibility of $x = 3$ being the solution of an inequality. These findings can be examined by means of the Intuitive Rules Theory, formulated by Stavy and Tirosh (2000). Students expressed the views that "an equation-result can only be the solution of an equation task" or that "an inequality task must have an inequality-solution." These claims are in line with the intuitive rule Same A (equation / inequality relationship in the solution) – same B (equation / inequality relationship in the task).

Quite surprising were the findings showing students' difficulties in responding to the standard "solve" task. In both countries only about half of the participating students identified $x = 0$ as the solution of the inequality $5x^4 \leq 0$. It seems that, similar to previous studies, reporting "strange" solutions like Phi and R as problematic for students (Tsamir & Almog, 1999), this study identified that the $x = a$ type of solution is also problematic in cases of inequality-tasks, and should further be investigated. A wider analysis of students' reactions to this task, embedded in different theoretical frameworks (e.g., Fischbein, 1987; Arzrello, Bazzini, & Chiappini, 1993; Bazzini, 2000; Maurel & Sakur, 1998) will be provided in the oral presentation.

Most interesting were the findings related to the consistency of students' reactions to the two tasks. We should remember that both tasks were included in the same questionnaire and students were free to move back and forth among the different tasks. In this manner, students' correct solutions to Task 9 could have served as an example for correctly solving Task 1. Still, no student explicitly mentioned Task 9 when correctly responding to Task 1. Furthermore, a non-negligible number of students

(Italian and Israeli) responded to tasks 1 and 9 in a contradictory manner. They wrote, "an inequality can only be the solution of an inequality" (Task 1) and then, that $x = 0$ was the solution of the inequality $5x^4 \leq 0$ (Task 9). A possible explanation for this phenomenon could be that sometimes zero is regarded as a special number. Thus, students could accept $x = 0$, but reject $x = a$, when 'a' is other than zero, from being a solution of an inequality. Naturally, further research is needed to investigate such assumptions.

Fischbein Moreover, our findings call for interventions that deal with the specific issue of algebraic inequalities and with the general issue of consistency in mathematical reasoning. Questions that arise are, for instance, how to introduce inequalities? How to cope with inconsistencies in students' reactions to inequalities? and how to validate the correctness of specific solutions to inequalities? Suggestions for research based instruction will be presented and discussed in the oral presentation. Clearly, the impact of such interventions should be further investigated.

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Recebido em maio/2000; aprovado em jul./2000