Children creating ways to represent changing situations: on the development of homogeneous spaces

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Abstract
This paper focuses on children creating representations on paper for situations that change over time. We articulate the distinction between homogeneous and heterogeneous spaces and reflect on children's tendency to create hybrids between them. Through classroom and interview examples we discuss two families of tasks that seem to facilitate children's development of homogeneous spaces: 1) making selected features directly visible, instead of requiring intermediate steps and calculations; for example, to be able to directly compare different sets of data combined in a single graph, and 2) exploring well-defined figural components that can be used in graphing, such as line segments or sequencing from left to right, that are introduced as a resource.
Key-words: *Situations; change over time; graph; hybrids; heterogeneous; homogeneous spaces.*

Resumo
Este artigo investiga crianças criando representações para situações que mudam com o tempo. Articulamos a distinção entre espaços homogêneos e heterogêneos e refletimos sobre a tendência que as crianças têm de criar espaços híbridos entre os mesmos. Usando exemplos de sala de aula e de entrevistas, discutimos duas famílias de tarefas que parecem facilitar o desenvolvimento de espaços homogêneos: 1) tornar características selecionadas diretamente visíveis, ao invés de serem necessários cálculos e passos intermediários; por exemplo, ser capaz de comparar diretamente diferentes conjuntos de dados combinados em um único gráfico, e 2) explorar componentes imagéticos bem definidos, que podem ser usados para traçar gráficos, tais como segmentos de reta ou seqüenciamento da esquerda para a direita, que são introduzidos como recurso.
Palavras-chave: Situações; variações no tempo; gráfico; espaços homogêneo, heterogêneo; híbrido.

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Introduction

This paper focuses on children creating representations on paper for situations that change over time. The theoretical framework that we aim to illustrate in this paper is the result of many experiments and previous analyses (Nemirovsky & Monk, 2000; Nemirovsky, Tierney and Wright, 1998, Noble et al., 2001). We have selected examples and episodes from classrooms and interviews that seem rich instances of issues associated with children's development of homogenous spaces. Homogeneous spaces are formed by points whose only property is their location in relation to a system of reference; because points are defined exclusively by their position, they can be arbitrarily re-located without affecting the space and its content. In contrast, heterogeneous spaces include zones with ad-hoc properties other than location; heterogeneous spaces often have inconsistent scales and different regions are devoted to different variables. Children tend to create hybrid spaces that combine elements of homogeneity with traits that disrupt their homogeneity by introducing irregularities and ad-hoc adjustments. This tendency, we think, reflects their interest in pre-serving all the aspects of the data that matter to them, in structuring the representations as a narrative, and in focusing on what is rather than on what could be. In the theoretical framework section we outline key ideas about the creation and use of homogeneous spaces which interrelate aspects of perception, the origins of persistent tensions between emptiness and fullness as well as between place and location, and children's motivations to create hybrid spaces. The review of selected literature highlights how these ideas relate to reported research on students' development of mathematical representations. After describing our research methodology, we discuss classroom and interview episodes that illustrate two families of tasks that seem to facilitate children's development of homogeneous spaces: 1) Making selected features directly visible, instead of requiring intermediate steps and calculations; for example, to be able to directly compare different sets of data combined in a graph, and 2) Exploring well-defined figural components that can be used in graphing, such as line segments or sequencing from left to right, that are introduced as a resource.

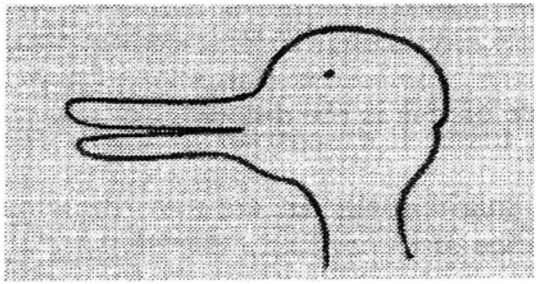


Figure 1 – Duck/rabitt from Wittgenstein (1953, p. 194)

Theoretical framework

Perceiving

At the core of the creation and interpretation of mathematical representations we see multiple perceptual shifts of the kind that Wittgenstein called "seeing aspects" or "seeing-as." In order to describe the nature of a seeing-as moment, it may be useful to recall the example of the rabbit/duck image:

> I might first look at the image and see a duck. This seeing is not an issue, it emerges as a natural unprepared perception: a duck is just what is drawn out there. Then someone else tells me that she sees a rabbit. Her assertion moves me from seeing the image for what it is, a duck, to seeing it *as* a duck. It is a moment in which I become aware that the image could be something else, and that what I was seeing is not what it is but one of the possible things that it could be. The duck that I still see becomes an appearance subject to change if only *I* could let it change. "How come you see a rabbit?" I might ask her expressing puzzlement. Her pointing at the drawing and saying "See the ears?" could be enough. A sudden shift takes place and I see the image as a rabbit. I am the same person and it is the same image, but our relationship has changed. Seeing-as is a particular type of experience. While one is crossing the street, one does not see an upcoming car *as* a car; rather, one sees a car. It is only when one sees the vehicle as an object of attention and against the possibility of it being something else (e.g. "*It* seemed a truck but now I recognize *it* as a car") that one momentarily sees it *as* a car, a truck, or something else.

A second perceptual experience central to mathematical symbol-use is what Casey has called "recognizing-in." Let us read one of his examples:

> When I was working on a summer job many years ago in my hometown, my employer remarked to me one day that he recognized my father in me. When I asked himhowthis was so, he said that I had "my father's walk" – his very gait, his style of walking. His perceiving of my walking was imbued with remembering; or rather, his perceiving me the way he did *was* his remembering. (Casey, 1987, p. 127)

His employer was not recognizing him *as* his father because he was fully aware that he was the son, not the father. But he saw his father *in* his walk, in a partial aspect of the son. Similar experiences happen to all of us; one might recognize in the drawing of an elephant the smile of a friend, or in the texture of a fabric the feel of a certain blanket. A distinctive element of recognizing-in is that one sees in-it something that is outside of-it.

Because both, seeing-as and recognizing-in, entail a confluence of differing points of view (e.g. the drawing as a rabbit or as a duck, the style of walking as his or his father's), they are dialogic experiences in the sense given to this term by Bakhtin (1981). One experiences the concurrence of two or more views of the same 'thing;' sometimes this co-existence appears to be convergent and overlapping and some other times opposed and displacing each other. Rather than oneself being 'inside' one of these perspectives or points of view (usually unaware of being so), through these experiences one feels to be in-between them grappling with the sudden emergence of branching paths. It is a co-existence in which each point of view enters into contact with other ones.

In the case of mathematical representations, seeing-as and recognizing-in are deeply interwoven because *seeing* it *as* a representation of such and such kind entails *recognizing in* it something that is outside of itself. For example, the shift from *seeing* a parabola *as* a graph of position vs. time for a moving object to *seeing* it *as* a set of points equidistant from a point and a straight line, could entail a related shift from *recognizing in* it an object moving in free fall to *recognizing in* it a focus and a directrix. Before going into more detail on these aspects, let us examine issues on how we conceive of space.

Figure 2 – The Cheshire Cat (Carrol, 1998)

Emptiness and nothing

'I wish you wouldn't keep appearing and vanishing so suddenly: you make one quite giddy.'

'All right,' said the Cat; and this time it vanished quite slowly, beginning with the end of the tail, and ending with the grin, which remained some time after the rest of it had gone.

'Well! I've often seen a cat without a grin,' thought Alice; 'but a grin without a cat! It's the most curious thing I ever say in my life!'

The difference between empty space and nothing is significant not only in philosophical discourse but also in the 'regular' use of symbolic spaces. If we are used to seeing a painting on a certain wall, removing it does create something new: it is now a different wall. Note that perceiving the wall as being different because of the painting's absence is based on a personal history with its presence, otherwise we tend to perceive 'nothing' on that spot. Similarly, noticing empty regions of a graphical space as actually forming what that space is, is also rooted in personal histories which children do not necessarily bring with them.

Imagine a Cartesian graph drawn on paper. Let it gradually vanish by itself. Before its complete disappearance, when, say, the curve drawn on it is already imperceptible but the axes are still outlined in your seeing, something quite extensive, stretching much beyond the boundaries of the paper could still be present: the space the graph had been inscribed in, a flat space with an origin and two separate linear dimensions. This entity is perplexing for reasons similar to the ones felt by Alice. Grins exist. Everybody can easily recognize a grin. But how could one see a grin without the grinning cat? Seeing the grin without the cat is like seeing something invisible. Analogously, how can we perceive an empty space? How could one sense a *boundless* extension with *no-thing* in it?

At different historical times philosophers and mathematicians elaborated diverse approaches toward these strange entities that can exist in complete emptiness; entities that offer space for things but remain unaffected by them.1 The emergence of modern conceptions of empty spaces took place in the 17 th century at the time in which the nature of the void and the possibility of vacuum were in the forefront of science and mathematics. Pierre Gassendi for example, asked:

> Let us imagine that the entire mass of elements included within the lunar sphere (..) has been destroyed by God and reduced to nothing so that absolutely nothing remains in its place. I ask whether or not after this reduction to nothing we do not still conceive the same region between the surfaces of the lunar sphere that had been there, but now empty of the elements and devoid of every body. (Gassendi, 1658/1972, p. 386)

Imagining cosmic disappearances and re-appearances, very much like the Cheshire cat that kept appearing and vanishing, Gassendi tried to prove that empty space does exist, albeit in an incorporeal fashion. The key argument for the existence of empty space was that it is *possible* for a body to occupy it; in other words, when every body is removed from it, what still remains are possible locations and regions to be in. His arguments opposed the contemporaneous view of Descartes for whom extension was inextricably attached to bodies. Descartes argued that what we call void is actually filled with ‘subtle matter’ that we cannot perceive. The issue was momentous during the seventeenth century; Evangelista Torricelli showed in 1643 that one can easily generate void with a column of mercury, and in 1654 Otto von Guericke generated such vacuum that several horses could not open a sphere sealed by atmospheric pressure. It was also the period during which Cartesian graphs were invented and maps generated with Mercator projection became widely used.

Place and location

We commonly experience the difference between location and place. If we changed the furniture in a room, say, the room is still in the same location but it is likely to have become a different place. We will describe how the use of homogeneous spaces entails two movements at once: 1) a reduction from place to simple location and 2) a constitution of a symbolic place that animates the homogeneous space.

- The reduction from place to simple location

Our views on the distinction between place and location are inspired by Casey (1997).2 Think of a typical apartment or house. Such dwelling is commonly organized in well-defined rooms, such as a kitchen, a bedroom, etc. Each one of these rooms is equipped to sustain certain kinds of activity. For a member of the culture in which these apartments are common, there are well-defined expectations as to what is to be found in them; one would know what is proper to do in each room and how to inhabit them. They surround us with a horizon of possible related things to do; furthermore, in pursuing some of them we enact the room as a habitat. We constitute each one of these rooms as a *place* to be in. Being in a place is being surrounded and contained by a situation. Among the qualities belonging to our experience of places, we must include that they locate and shape us; places locate us in the sense that being in a place is being positioned with respect to other places, and shape us in the sense that we act and move within its boundaries. Location and shape however, are far from characterizing a place as a whole. If we emptied all that is inside a kitchen so that cooking, preserving food, eating, etc. were not practical endeavors to pursue in there anymore, it would have become an entirely different place in spite of the fact that the location and overall shape of the room would have remained the same.

For a multitude of reasons we are often compelled to specify just the bare location of a place. This might be, for example, to designate a location on a map or a floor plan. Our culture has developed numerous techniques to represent locations built on an assumption that Whitehead (1926/1977) has called "simple location." The assumption of simple location is that spaces, regardless of what *takes place* in them, are formed by positions and nothing else. For example, that a point in a 2D Cartesian graph is at (3,2), is the unique property that distinguishes that point from any other point, or the latitude and longitude of a site is all that matters to the space underlying a certain map. The assumption of simple location turns these spaces into homogeneous ones because their points become interchange-able. Imagine that one takes all the points of a region and moves them to another region: nothing changes because the points have no other property than their location, they just become the points corresponding to the new locations without anything distinguishing them from the ones that used to be there.

Homogeneous spaces entail a reduction from place to simple location. To get a sense of how radical this reduction is, let us compare a few aspects:

• Each place is unique. Think of one of those buildings that have many apartments with the same layout. In spite of being built as replicas, each one of the kitchens is a unique place. They are populated by distinctive pieces of equipment and furniture. They smell differently as a result of particular cooking and eating practices. Certain conversations take place in one and not in others. Furthermore, their uniqueness is open ended because different people at different times will perceive it as being unique in a particular way. In contrast, homogeneous spaces that are defined consistently remain identical to each other. Nothing distinguishes, for instance, a graphical space for position vs. time with a certain scale and origin that one uses today from one used a year ago or from another that someone else is conceiving of oceans away.

• Being in a place means being bodily there. Places do not have fixed centers; they are centered by and through the living bodies of those who inhabit them. A thing is within reach not because it is at a certain distance from something else but because I can get hold of it by stretching my arm. What is within reach for me may not be for someone else, or it could be if she stepped on a chair. In contrast, homogeneous spaces are centered on conventionally defined points or regions. Points are judged to be more or less close or far from each other exclusively by a comparison of distances.

• In a place things are contiguous because they 'belong' there: the faucet is next to the sink, the plant goes into the soil, someone is sitting on a chair, and so forth. In a homogeneous space there is no reason for one point to be close to another, it just happens to be so. Points ignore each other.

• In contrast to homogeneous spaces that are often infinite and limitless, places are finite and bounded. Even if one is in an immense place, such as on a cliff in front of the sea, the horizon delimits it. Being the outer limit of our bodily space, the horizon – that circle pregnant with what is beyond – contains us in places that do not have physical boundaries.

In many circumstances we call for portrayals of place, or aspects of place, that could not be, or would only be indirectly preserved on a homogeneous space. This necessity meets its aim in heterogeneous spaces. Let us recall a famous example we are familiar with, the *Guernica*, Picasso's depiction of the Basque city under the bombardment of April 26, 1937 (Figure 3).

Let us do the vanishing exercise. Imagine that all that is in the painting gradually disappears; animals, extremities, faces, walls, all fades away. As opposed to the case of the Cartesian graph, there is no common origin in relation to which all these bodies are systematically positioned and seen. Objects are not located in a unifying space that extends beyond the painting and which we could sense in its emptiness. The knife at the center is not there by virtue of having a relative location but because it is piercing the horse. Contiguousness does not imply short distances. The light bulb/sun, suggesting the mixing of day and night, is on the upper edge because this is where they customarily are, not because it is a short distance away from the horse's head. Things are not seen from a unique observer's location; we see, for example, the two eyes of the bull as seen from the front, but also its body appearing from the side, and so forth.

Figure 3 – Guernica

• Animating Homogeneous Spaces

If we shift our attention from representations by themselves and look at their use by actual people in actual circumstances, we realize that *always*, even when the space being represented is purely homogeneous, the user brings a place to bear, a place for the representation and its content. In a remarkable set of case studies Ochs et al. (1996) investigated talk among physicists in a lab. The physicists used and discussed graphs for the phenomena they were investigating as if they were traveling on them and visualizing what would happen as they went along (e.g. "When I come down I'm in the domain state" as the speaker was talking/gesturing on a graph of magnetic intensity vs. temperature). This talk and gesture reflected their constitution of a graph as a place that included physical materials, magnetic fields, imaginary entities, and their own agency (e.g. moving around, comparing how I see things from here, how you see them from there, etc.). The places we create to animate homogenous spaces are at times populated by imaginary entities, such as mathematical structures or events. In other circumstances they include actual objects that have been measured or exist somewhere else, or mix real and imaginary ones.

The reader may feel a need to step back. All our previous examples for places suggested physical surroundings one can be in: a kitchen, a city, a cliff, etc.; now we are referring to places that might not even exist physically at all, and yet, we inhabit them. Being in these places entails using symbolic expressions, such as graphs, to move around and do things; it involves embedding simple locations in a lived context that animate homogeneous spaces – animating in the sense of experiencing them as having all the traits of places (uniqueness, being bodily there, things belonging, bounded by horizons, etc.). How is such animation achieved? We won't try to answer such complex question, but will offer pointers that illustrate how we orient ourselves to grapple with it.

• Constituting a place for symbolic expressions, including those that are built on homogeneous spaces, entails talking, acting, and gesturing without distinguishing between symbols and referents; we call 'fusion' this pervasive aspect of symbol use. It is by fusing symbols and referents that being 'here' opens up events and scenes different from the ones that would occur if one were 'there.'

• The places that we so constitute are newly occupied and expanded through seeing-as moments. Suppose that one is using a graph of position vs. time to ascertain distances at certain times and then one is asked to estimate the velocity at those times, such request might prompt us to see the same graph as a varying slope/speed; while the graph remains the same, we would now see another aspect which could change in radical ways what we see 'taking place.'

• As we inscribe or interpret symbols on a homogeneous space, we come to recognize outside events and entities in parts or elements of these inscriptions. For example, we might recognize in the vertex of a parabola of height vs. time the moment in which an object we have thrown upwards turns downwards. This ability to recognize outside events in a graph entails educating our perception so that we can *directly* perceive symbolic elements as showing occurrences and stories.

Whitehead (1926/1997) has elaborated on the utter dominance of simple location in Western culture and on its devastating impact. It is a view echoed by much of the 20 th century continental philosophy. Sophisticated technologies were created to communicate science as if it were produced by impersonal and placeless activities. Even in anthropology there was a time not long ago in which the task seemed to be to locate beliefs and customs within a space of binary oppositions. The constitution of places in the use of homogenous spaces went underground, as it were. We believe that children's fluency with homogeneous spaces does not need to be founded on enforcing the reduction of place to simple location, but on children's ability to animate and inhabit them. This is the topic of the last two pieces of this section.

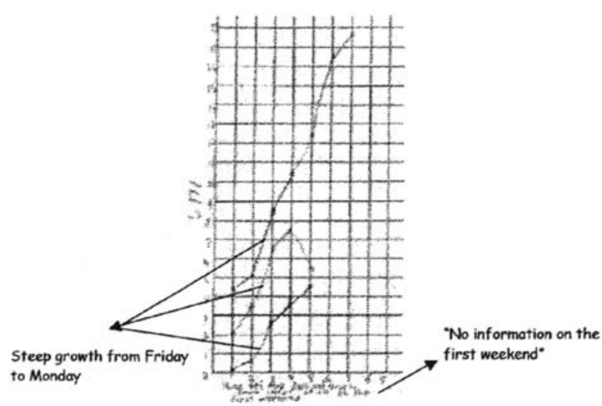


Figure 4 – Plant growth

Hybrids

Innumerable representations we encounter are hybrids between homogeneous and heterogeneous spaces, combining features of both to grapple with tensions such as those between consistency and emphasis or generality and uniqueness. Children also tend to create hybrids for the representation of changing situations. Often they use or adapt graphs in ways that lose homogeneity but bring forward aspects of the story they wish to tell. It is not difficult to recognize their motivations to hybridize: to preserve the information that matters to them and to create space for what *is* rather than for possibilities that do not make much sense to them. Often they are reluctant to make choices that either 1) eliminate space for actual events or features of the data they wish to depict or, 2) create space for data they do not have or for absent events. In both cases there is a trend to avoid empty space, possibly because it seems to add nothing.

One example of omitting space for data they did not have, came when students in fourth grade, who are typically 9 or 10 years old, measured plants they were growing at school and drew line graphs to show their heights over time. Many labeled the horizontal axis with the weekdays but left no space for weekend days for which they had no measurements. This omission generated what visually appeared as a high steepness of growth from Friday to Monday (Figure 4).

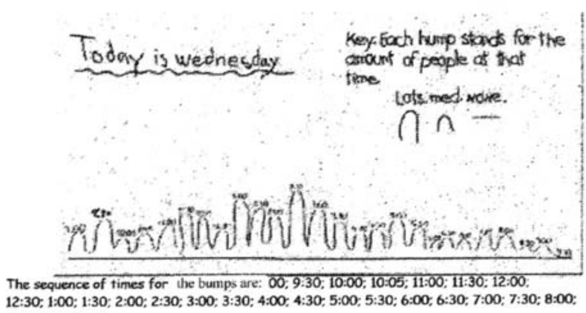
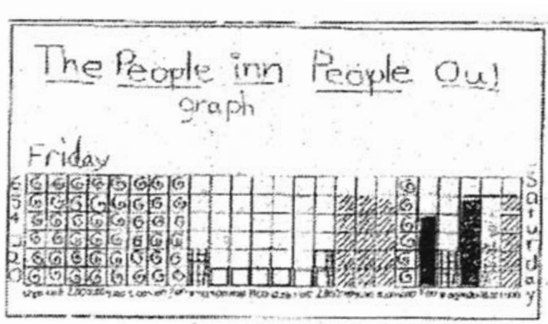


Figure 5 – Sequence of bumps

Others omitted space for data not because they didn't have the data, but because they knew that nothing significant had taken place over that period. For example, most children omitted evening and night hours on the graphs of changing population of their classroom when no one would be there, but included them on the graphs of changing population of their homes. When we asked some students why they had not listed the night hours, they inferred we meant there would be some people in the classroom then. One child went immediately to ask her teacher how late she stayed at school; another asked whether the janitor would be present. They only added the extra hours when they learned someone would be there (Figure 5).

Emy, whose graph of changing population in her home was otherwise quite conventional, used one block for zero persons home and two blocks for two persons home marking 0, 2, 3, 4, 5, up the vertical axis to identify the bar height showing the number of people at home (see Figure 6). When asked why there was no place for one person, she said, “Because there is never only one person home at my house”.

While omitting space for data that is ‘absent’ or whose value is zero, children build their representations to preserve all their data and to show it most clearly. One of the ways they do this is by providing keys or multiple ways to show an aspect of the data that can be read from the labeled axes. To make sure that the reader will not miss important features of their representations they often choose redundant marks. For example, in Emy’s population at home (Figure 6) she colored the bars on the graph with different colors and put different decorations in them to show different numbers of people. Other children added names and marks to identify who was at home or how many people had left or entered the house at a certain time (Figure 7).



Each column on the horizontal axis is marked by the hour, starting at 12.00 and ending at 11.00

Figure 6 – Emy’s graph

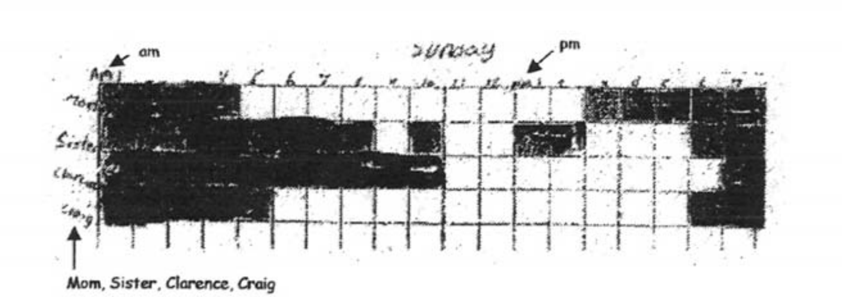


Figure 7 – Mom, Sister, Clarence, and Craig at home

Another way they preserve data is to label successive marks on the axis to match points for which they have data. In this way the specific data is preserved. For example, children asked to show the changes in population in their home or in their classrooms through a day wrote only the exact times when people came and went instead of a regular scale with hour or half hour intervals (Figure 8).

To prevent an empty space to be seen as 'nothing' children tend to mark empty spaces to indicate attributions of meaning. Emy, for instance, explicitly marked the blocks that indicated zero people at home (Figure 6). In Figure 9 the child wrote 'Nobody Home' on the empty row. Figure 10 shows an example of a time line in which the child invented an icon for 'nobody at home'.

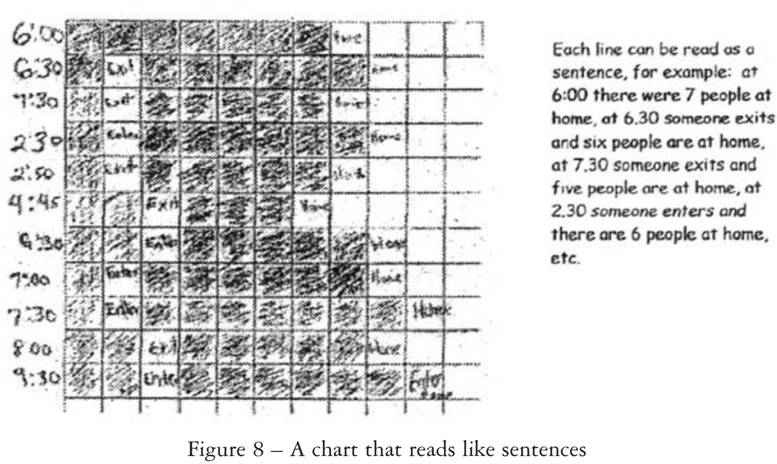


Figure 8 – A chart that reads like sentences

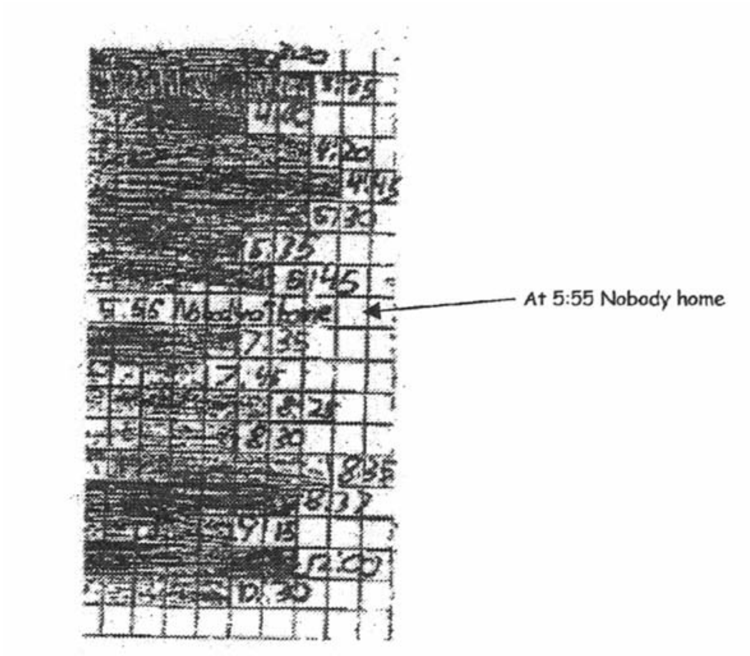


Figure 9 – Nobody home

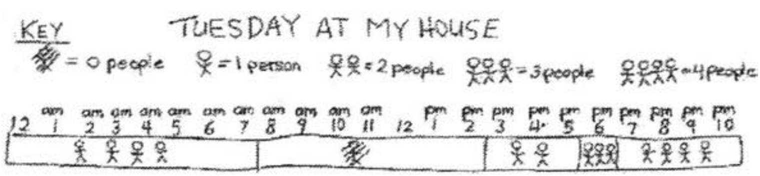


Figure 10 – Zero people

Direct seeing

Learning mathematics entails fluency with homogeneous spaces. But obviously there are countless kinds of homogeneous spaces and not all of them are equally accessible or valuable. Two features seem to have been historically significant in making some homogenous spaces more used. The first one is the possibility of seeing *directl*y, without intermediate steps and calculations, the aspects of most significance. For example, the choice of the Mercator projection became widespread in sea navigation because it allows one to directly see the direction one has to take in order to reach a destination: by following the line that connects the current location and the desired location anywhere on the map. A second feature can be described with words such as 'compactness' or 'density;' it refers to the ability to put on view multiple aspects. A homogeneous space that can combine in one what otherwise would require multiple displays is deemed advantageous, unless it becomes cumbersome and intricate. For instance, the motion of a harmonic oscillator in a graph of velocity vs. position is an ellipse. All the properties of such elliptical trajectory, like the ratio between its axes or their length, have definite meanings in terms of the motion's amplitude, frequency, and so forth; each ellipse gives a holistic and detailed glimpse into the characteristics of the represented oscillator. This ability to provide a unifying and simple depiction makes such space particularly compelling to those who have practice with it.

The pair of features we described above are related to two families of tasks or learning environments that seem to facilitate children's development from the use of hybrid spaces to their use of homogeneous spaces: 1) Making selected features directly visible, instead of requiring intermediate steps and calculations; for example, to be able to directly compare different sets of data combined in a graph, and 2) Exploring well-defined figural components that can be used in graphing, such as line segments or sequencing from left to right, that are introduced as a resource.

Both types of tasks have been discussed in the literature. Lehrer, Strom, and Confrey (submitted) for instance, describe how children 'invented' line graphs by trying to show families of similar rectangles in a direct way. Sherin (1997) points out the extraordinary flexibility of such a mundane piece: a line segment, and how children exploited its possibilities to convey complex ideas. Length, slope, thickness, color, and relative position are all features that can express attributions of meaning combined in compact and elegant ways. In the next section we review a selection of papers relevant to what we would call children's development of fluency with homogenous spaces.

Review of selected literature

The role of children's invented representations in learning mathematics is the subject of a growing literature (DiSessa, 1991; Bednarz, 1993; Sherin, 1997). In this literature we can recognize two big themes: 1) the resources and expectations that children bring to their use of graphs and 2) the processes by which their representational approaches change and get refined. Both themes talk to the overarching issue of how children's representations relate to conventional ones, an issue that is also part of a broader strand of work, which goes beyond mathematics education (Bamberger, 1988; Ferreiro and Teberosky, 1979; Karmiloff-Smith, 1979). In this review we will highlight ideas within the literature on research in mathematics education. Most of the work cited in this paper focuses on the representation of change and motion. Typical tasks that have been examined were the representation of motion (DiSessa, 1991; Sherin, 1997), and change of discrete quantities (Bednarz, 1993).

Resources and expectations that children bring to the inventing tasks

Perhaps the most comprehensive treatment of this theme can be found in Sherin (1997). He identified three "constructive resources relevant to the invention of representations of motion:" drawing, temporal sequences, and sensitivities to features of figural elements (e.g. features of a line segment). Drawing is a pervasive children's activity in our society. By drawing, children experiment with the use of a flat bounded representational space to combine socially established icons (e.g. the standard icon for a house) with their own idiosyncratic intentions. Temporal sequences refer to the use of symbols ordered along a linear dimension, so that the order in the sequence reflects the order in the time of reading (e.g. one might be supposed to read the symbol on the left before the one on the right) and the time of the represented events. Sensitivities to features of figural elements refer to children's playing with the possibilities of a visually displayed component, such as a line or a circle. These figural elements seem to act as 'seeds' from which new meanings associated with their multiple features (length, size, tilt, etc) emerge as the children refine their creations. Monk (2000) describes the centrality of these sensitivities – sensitivities to a curve being upward/

downward, smooth/sharp, and so forth – in students' interpretations of graphs.

In addition, children express in their invented representations critical, although mostly implicit, criteria and values regarding what a representation should be and accomplish. DiSessa (1999) reviewed what he has called 'meta representational competencies.' These are criteria and values that guide what is considered as an adequate and meaningful representation. Examples of these are:

- A representation should need as little explanation as possible.
- Representations that are spatially compact are, generally, better.
- The representation should allow determination of all relevant aspects.

Some of these criteria seem to reflect a context of communication (e.g. someone else should be able to understand it without additional explanations), others appear to be more related to what mathematicians often call 'elegance,' as in a certain proof being more elegant than another. Elegance seems to involve compactness, being graspable all at once, centered on what is essential, and so forth.

Another family of expectations that children bring to graphing emerges from their experiences with story telling and making sense of narratives (Nemirovsky, 1996). Children invent representations trying to be faithful to the story that matters to them. Their representations often break continuous events, such as the motion of a car, onto discrete episodes associated to keys or to a list of icons. They also make particular moments stand out because of their narrative significance or eliminate a space for others because they 'never happen.'

The processes by which their representational approaches change and get refined

One of the motivations for investigating children's representations is to design learning environments that allow for meaningful connections between their approaches and conventional mathematical representations. This poses the 'transition' issue, namely, what are the continuities and discontinuities between children's graphing and conventional graphing? Bednarz et al (1997) see this question at the center of what mathematics teaching and learning is. They base their 'socioconstructivist' approach on facilitating and stimulating students to enrich and extend their notational and representational inventions. DiSessa et al. (1991) highlight the extraordinary potential of children's inventions to evolve into sophisticated means of making sense of conventional systems. Other authors (Krabberdam, 1982; Lehrer, Schauble, Carpenter and Penner, 2000; Sherin, 1997) trace continuities and discontinuities between students' approaches and the fluent use of standard graphs.

The issue of continuity and discontinuity is not resolved by choosing between having students inventing graphing or 'telling' them how to graph. As a matter of fact both aspects – inventing and adopting representational resources – are inseparable. One of the most important contexts that seem to facilitate the development of graphing is the communication of ideas. Knowing that others will have to understand what a representation means, as well as experimenting with how others interpret it, appears to be a rich component to stimulate refinement and to develop a ground for the appreciation/critique of conventional representations. Another component is to constrain the universe of possible symbols (e.g. "let's not use words") or aspects to be symbolized (e.g. "let's focus on speed, not on whether we jump or run"). With such constraints, the designed representations become more systematic and amenable to being compared with conventional ones. In some cases these constraints emerge from the children themselves – a tendency that diSessa (1999) interprets as a manifestation of their "meta-representational capabilities" – but in others the task, the tools, or the teacher may introduce them.

There are two additional references that we want to include in order to point at important directions that other researchers have taken to work on students' representations. The first one is Kaput (1998); Kaput elaborates on the contrast between computer-based and paper-based representations highlighting that the former ones can be dynamic in ways that drawings on paper cannot afford. He describes how this difference can play out in the learning and teaching of mathematics. The second one is Mesquita (1998); Mesquita uses Poincare's (1905/1952) ideas on the genesis and use of geometric space and how it radically differs from physical space as we bodily experience it. The complex relationship between the heterogeneous bodily space and the homogeneity of geometric space is the subject of work in contemporaneous neuroscience (O'Keefe and Nadel, 1978; Paillard, 1991).

Methodological approach

The interview and classroom episodes that we have selected for this paper were chosen from a larger body of work that we have generated over more than ten years. The interviews were conducted in 1989 and the classroom activities took place in 1994. Given such broad sources of data and reflections, we describe in this section our general approach toward the conduct of educational research.

The choice of a methodology is necessarily linked to what the research strives to get at. In order to make the case for our methodology, we will start by pointing out what exactly we want to investigate. Levine (1983) introduced the phrase 'the explanatory gap' to describe an open range of phenomena that cognitive science, at least in 1983, did not account for how the world is experienced by human beings. Cognitive scientists have postulated many types of mechanisms for, say, visual perception, but these mechanisms are largely silent when it comes to what it is to be conscious of an object in front of us, or attributing meanings to our surrounding space. The literature on the explanatory gap often cites Nagel's paper (1970) entitled "What is it like to be a bat?" The question "What is it like?" is a key to the explanatory gap. What is it like to be a student dealing with this or that problem? What is it like to use a graphic calculator to learn so and so? How do things look and feel like to a student?

In order to develop grounded descriptions of how things and events are experienced by students we conduct teaching experiments, sometimes with individual students and other times with small groups or in the full classroom. The teacher/interviewer comes to the session with ideas about activities and tools to ask the students to work on and with, as well as with goals regarding students' learning. These pre-planned questions and materials are to be used as resources and starting points. The session unfolds as an open-ended conversation and interaction, through which the teacher pursues the learning goals for her students. The teacher's ideas, surprises, and uncertainties are fully part of the experiment. As opposed to thinking of the session as a window into unaltered students' thinking, we strive to bring to the surface how students think in-this-situation with others and with such and such tools. We do not conceive of a learning environment as a set of tools and activities that pre-exist their use by students and are given to them (Noble, Nemirovsky, Wright and Tierney, 2001). Students and the teacher/interviewer constitute the learning environment on an ongoing basis. As Lewontin (2000) has asserted in relation to biological environments: “If one wants to know what the environment of an organism is, one must ask the organism” (p. 54). There is no way to ascertain what a learning environment is without examining what it is for the student. The relationship that is established between the teacher/interviewer and the student(s) is crucial. The teacher/ interviewer strives to express to the students through the ongoing interaction that she is trying to genuinely learn from them how things look to them, that this is not about withholding information to test whether they know something she knows, and that she is receptive to their contributions.

The analysis is based on the videotaped sessions and the students' work. The most important and difficult aspect of our approach to interpretation is to see and talk about the filmed events without “diagnostic” attitudes (this is good/bad, this child has such and such misconception, the teacher should have asked something else, this boy has x learning style, and so forth). Avoiding diagnostics demands a great deal of self-awareness. We sense a need to avoid these attitudes because they prevent us from learning anything new; they lead us to repeat ourselves. Diagnosing involves locating the object of analysis in pre-existing and often tacit taxonomies (e.g. students misconceptions, learning styles, etc.) by projecting our assumptions on the data; this is particularly inappropriate when we are still learning about the conceptual and empirical issues raised by the research questions. Instead of judging actions and utterances, our interpretive efforts strive to recognize what students experience. We try to make sense of how someone else makes sense, and understand something of what it is like to be in someone else's shoes at particular moments in particular contexts. The complex and momentous shift from the diagnostic to the interpretative attitudes is related to what Husserl has variously called “epoche,” “phenomenological reduction,” or “bracketing.” Bracketing is abstaining from seeing the world in terms of self-standing or absolute things and events, such as mechanisms or causal relationships, to seeing the world in how things and events are experienced by someone.

The analysis proceeds through cycles of examining and interpreting the data, which involve transcribing and writing interpretive notes. Through an iterative process, specific episodes are selected and overarching themes emerge; the themes become more and more connected to strands of literature that appear insightful. Remaining close to the data is of paramount importance because the validity of the results rests on such proximity. There is an inherent circularity throughout the process: episodes are selected because they help articulate emerging insights and insights get developed because certain episodes become focal. We think that this interpretive recurrence, often referred to as the "hermeneutical circle," can be generative and lead to fresh views to the extent that we bracket the data, avoid diagnosis, remain open to unexpected connections to diverse bodies of literature, and let ourselves become aware of tacit assumptions.

In the next section we discuss classroom and interview episodes. Due to space limitations we pursue transcript-based analysis only on the interview ones. The section is divided in two parts, each discussing a particular family of tasks that seem to facilitate children's development from the use of hybrid spaces to their use of homogeneous spaces: 1) Making selected features directly visible, instead of requiring intermediate steps and calculations, and 2) Exploring the use of well-defined figural components.

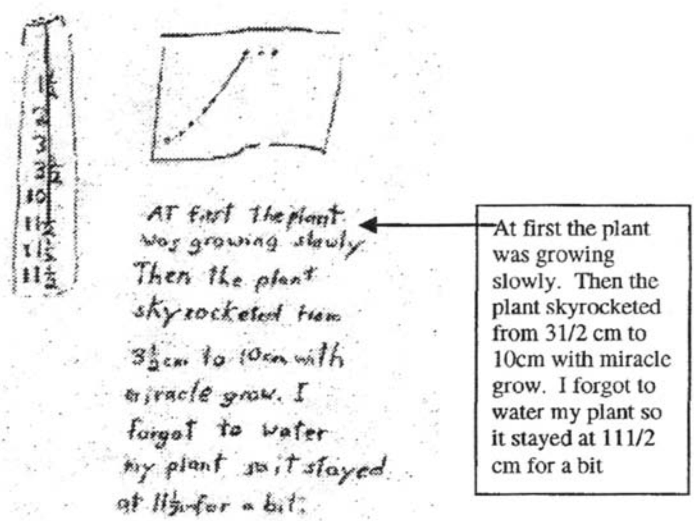


Figure 11 – I forgot to water may plant

Classroom and interview episodes

Making selected features directly visible, instead of requiring intermediate steps and calculations

Fourth grade: comparing graphs

The purpose of the fourth grade math of change activities (Tierney, Weinberg and Nemirovsky, 1994) is to support students' learning of the mathematics of change, focusing on their generation and interpretation of graphs, number tables and stories of events they are familiar with. Some of the activities involve growing plants. Figure 11 is an example of a number table, graph, and story generated by a student in a test toward the end of the unit.

Each child grew a plant for a couple of weeks and measured the height of the plant each school day and recorded measurements on a number table.

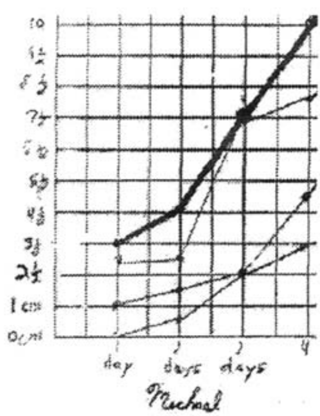


Figure 12 – Group graph for plant growth

Their individual graphs, constructed out of plotting points from the number table on grid paper, were intended to represent the growth of their plant. As mentioned barrier, it was common that they skipped in their graphs the weekend days on the time scale because they had not measured the plant heights on those days. Many children chose to mark the grid lines on the vertical axis with the specific heights they measured each successive day, rather than using a regular scale. In view of this, we decided to try asking them to create 'group graphs' by combining several individual ones into one. It was only then that the vertical scale became an issue for them. In some cases they solved the conflicts by adjusting their graph without regularizing the scale. For example, Figure 12 is a group graph in which the student lowered the beginning of the graph so that he could include data from other students' plant whose height started lower than his plant. On the other hand, Figure 13 is a group graph in which the student fully reorganized her graph, including a regular scale, to encompass the data from the other members of her group.

Skipping the weekend days became a subject of discussion when they noticed that someone might be misled by the slantiness of the line to think that the growth from Friday to Monday was abnormally high. Note in Figure 13 how the student also added the weekend days. In all cases the decisive aspects that prompted the children to homogenize their graphical space for plant growth were the need to create space for plants that might grow differently than their own and allowing for the viewers (including themselves) to directly recognize plant growth in the graph; 'directly' in the sense that it would not require viewers to compensate global appearances with local arrangements, such as the steepness from Friday to Monday with the absence of the weekend days on the time scale.

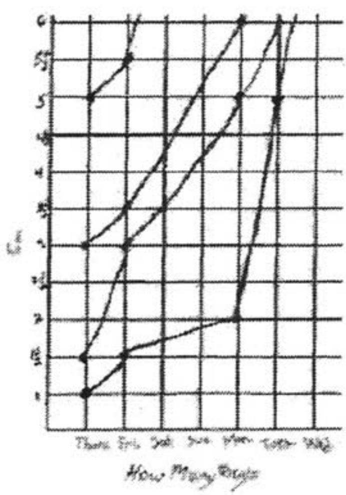


Figure 13 – A reorganized graph

Rose: showing changes

Rose is a 9-year-old girl. Our colleague, Mark Ogonowski, interviewed her. Mark and Rose had been discussing situations involving adding and taking away plastic blocks from a paper bag. We chose this episode because Rose is trying to create a graph that would show directly the *changes* in the number of blocks in the bag, as opposed to the amount of blocks in the bag. Mark had asked Rose to create a graph to show this sequence: start with three, add two, take away three, add 1 and take away two. Rose created the following graph (Figure 14).

Mark: Would you say that that figure [14] then is showing you what's in the bag at each time or the changes at each time? What would you say?.

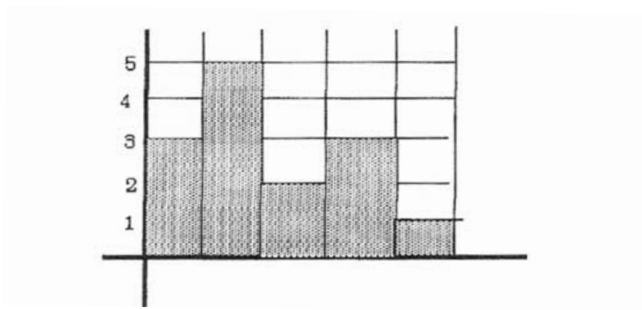


Figure 14 – Sequence of blocks in the bag

Rose: It could be showing you either of them, because it shows you what was changed. It doesn't really show you what was changed because you have to figure that out. (..) You could tell that two is the change because you could level it out, like go all the way across the three line, and say, oh look what a difference from 3 to 5. And ..Like you could go across the line that it was up to from the last one and see what happened, like if it's lower or higher than that line, and that's how much. You'd go across and you'd say, oh, there's nothing there [she moves her hand horizontally on the top of the '5' bar]. And you'd (..) fall down. And you'd say, ah, there we are. And you'd count the squares that you had dropped by.

In asking Rose what is being shown by her graph, Mark distinguished between the number of blocks in the bag 'each time' and the changes in that number *'each time.' Rose immediately recognized the difference that Mark was pointing out. First she said that it showed both, but immediately she added that it showed them differently: the former could be seen directly whereas the latter had to be 'figured out.' Such figuring out was described by Rose as a process of guiding one's own seeing of the graph: you extend the top of a bar across the one to the right and see what happens: you either fall down or move up; then you count the squares that it took to reach the top of the subsequent bar. She gestures an imaginary movement up or down to reach the next level, so that the length of this movement tells one the changes. We want to highlight Rose's eloquence regarding the difference between direct and indirect seeing because we think that this sensitivity is an important element in mathematics learning in general and in designing homogeneous spaces in particular*

Mark: (..) Could you make a second graph that shows you what each change was?.

Rose: What each change was. Hmmm. ...I'm not sure, because I have no way of I mean I could show the numbers each time, but I couldn't show whether it was plus or minus.

Mark: Why not?

Rose: Because I can't do that without showing what's in the bag at each time, I don't think. (..) you'd still have to figure it out a bit because you'd have to look to see whether it was higher or lower than the last time.

Mark: So you said you could do the numbers. What would the number be for this change? [pointing on Figure 14 from 3 to 5.]

Rose: The number would be two because you put in two.

Mark: What would the number be for this one [from 5 to 2]?

Rose: That would be three. (..) It would be like, let's see, whatever is not there, like three [indicating the 3 empty blocks from 5 to 2], and then there was or whatever is there [pointing to the shaded block up from 2 to 3] from the last one. (..)

The idea of creating a second graph to show the changes directly posed to Rose the difficulty of how to show whether the change was positive or negative. The graph that she had created was suitable to indicate numbers but there was no obvious way for her to convey in it the 'plus or minus' piece of information, which would require the hypothetical person to still look at the first graph to see if it was a downward or upward change. Note at the end a subtle shift in her noticing of the changes: it was a matter of 'whatever is not there' and 'whatever is there.' It is not so much an issue of movement anymore but of presence and absence, or emptiness *and fullness. These are for Rose 'seeing as' moments: she sees the same graph as either showing amounts or changes. Both ways of seeing the graph are available to her, and she switches back and forth between them. She will soon create a second graph that could also be seen as changes or amounts but in such a way that the changes could be seen directly and the amounts by 'figuring them out.'*

Mark: Can you think of any way ...You can invent a way if it comes to you, of showing those different kinds of changes. You'd have a hard time showing the minus ones, or whether it was the minus ones and the plus ones, but I'm just wondering if you can think up some way of doing that.

Rose: Well, I could ...This is kind of stupid, but for the plus, I could start it from the bottom and for the minus I could come down from the top.

Rose creates a new set of axis below the graph shown in Figure 14 (see Figure 15). She labels the left side with numbers from 1 to 5 increasing from the bottom, and the right side with numbers increasing from the top. Rose shaded the three initial blocks.

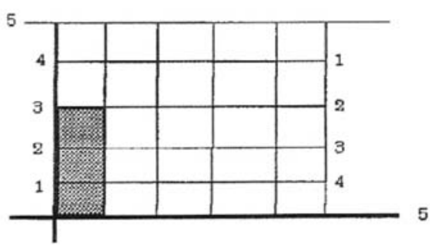


Figure 15 – Starting with three blocks

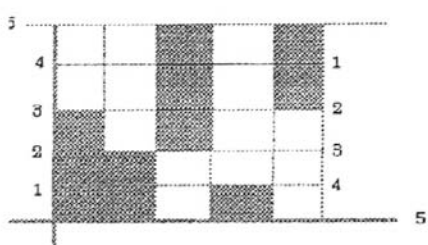


Figure 16 – The changes graph

Mark: So whathave youshown so far?

Rose: So I've shown three, and then you put in two {shades two blocks see bottom of Figure 16}. And then you take out three {shades three blocks counting down from the top}. Then you put in one shades one block . And you take out two {shades two blocks from the top}. That.

Mark: So, I see what you mean now. If it goes up from this {bottom} line it's what you put in. If it comes down from here ..{top line}

Rose: That's what you take out.

At the beginning she does not feel secure about her idea ("this is kind of stupid"). Note that Rose did not start by adding bars to an undefined a graph but by designing a new space notably different from the previous one with two different scales: one starting from the bottom, another falling from the 'five' which was the highest amount in the sequence. The main element that made the changes indirect in the previous graph was that it was required to compare successive bars; in the new graph the need for comparison is removed: each bar tells of a change in itself regardless of the previous or subsequent ones. The exception is the initial '3' bar. Rose reflected the difference between the initial bar and the subsequent ones in the way she described them: the three is 'shown' whereas the subsequent two and three are 'put in' and 'taken out' respectively. It is likely that. Rose included the initial '3' in the changes graph because she wanted to preserve this information; or perhaps she would argue that the initial number of blocks is a change from the 'real' start with the empty bag. In any case it is significant that without the initial 3 the number of blocks in the bag could not be 'figured out' from the changes graph, and something essential would be missed. It is remarkable how similar Rose's approach was to the conventional way of graphing negative changes 'coming down' from the zero line. Perhaps the indeterminacy of a change of five blocks in Rose's graph would motivate her to value this convention.

Exploring new figural components

Third grade: from left to right

A series of activities in a 3rd grade unit (Tierney, Nemirovsky and Weinberg, 1994) are based on the idea of an imaginary elevator inside an indefinitely tall building with the particularity that its buttons are 'change' buttons (e.g. the +3 button moves the elevator three floors up from whatever floor the elevator is at). Most of the students were 8 years old. In one of these activities, children are asked to create graphical ways to show how the elevator moves following a sequence of change buttons. As we piloted the unit in several schools, we noticed that children typically tended to use sequences of lines or arrows imposed on a diagram of an elevator (Figure 17).

After exchanging representations with other children who tried to read the story from the drawings, one issue children thought important to fix is that one should be able to distinguish between the start and the end floor. They added marks to this purpose (e.g. 'S' and 'E'), and some, as in Figure 18, labeled each step. Seeing that it was possible to recognize in the same graph different elevator trips, they added marks to solve ambiguities.

This was a typical moment in which the 'transition' question came up: short of just asking children to leave behind their segmented paths and to teach them to use graphs of height vs. time, how could we encourage them to see the conventional graphs as connected to their inventions and helpful to express their own ideas? One approach that ended up being enormously fruitful was to ask the children to explore the idea of organizing their lines from left to right. This prompted them to reorganize their drawings in ways that they deemed advantageous. Their revised graphs were close to how graphs of height vs. time are organized. It meant for them to begin using a space in which being on the right meant 'after.' Figure 19 include examples of children's new representations.

Initially the children saw the element of moving to the right as an alternative way to solve the problem of the graphical overlap between arrows, but as they used it they began to deal with the sequence left-right as indicating the course of time and perceiving before/after relationships inscribed in visual patterns extended from left to right. The whole process exemplifies two simultaneous "movements" that we articulated in the theoretical framework: 1) Constraining the graphical space to simple location and eliminating ad-hoc properties (such as 'S' and 'E'), and 2) Animating the homogeneous space with stories and events that took place in the imaginary building.

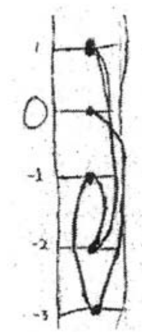


Figure 17 – Overlapping arrows

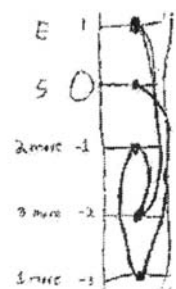


Figure 18 – Marking Start and End

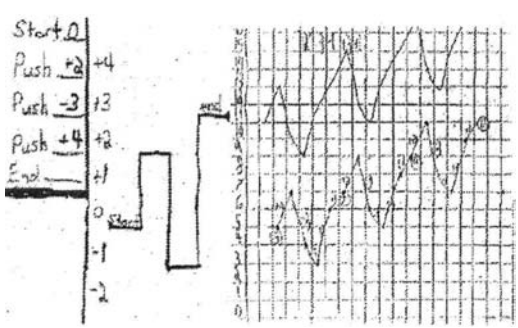


Figure 19 – Elevator graphs going to the right

+ + + + - - - -

Figure 20 – Sequence of adding blocks and taking away blocks

Carol: the curvy line

The next interview example is of Carol, an 8-year old girl who had just ended 2 nd grade. This episode was chosen because Mark introduced a new figural element for Carol to use: a curvy line with start and end points. This suggestion led Carol to create a continuous curve to show a sequence of changes from left to right. The whole interview had been based on problems of additive change as in the interview with Rose, in which plastic blocks are added or taken away from a paper bag. At this time Carol and Mark had been representing addition of blocks by a '+' sign and subtraction by a '–' sign; for instance, a sequence of four additions and four subtractions, each one including an undefined number of blocks, would have been shown by Figure 20.

They had used this notation to discuss problems of maxima and minima (e.g. "When would the most number of blocks had been in the bag?"). Then Mark asked:.

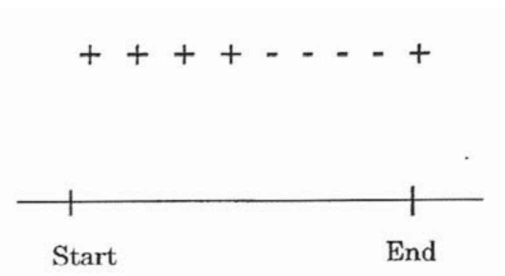


Figure 21 – Start and End points for a curvy line

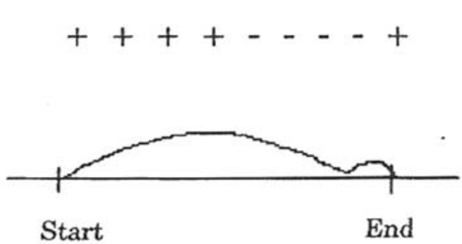


Figure 22 – A curvy line

Mark: Have you ever done something called a time line?

Carol: No.

Mark: It's a way of taking a line and showing what happens between two different times. So say this was at the start of this bag problem [he marks 'start,' see Figure 21], OK, and this is the end of it [he marks 'end,' see Figure 21]. This bag problem right here, we just see the pluses and minuses:

Mark: Now suppose there's a situation in which these [++++] are really big, these [- - - -] are pretty small, and this [the last plus] is kind of medium. Can you draw a line going from here [start] to here [end], it could be like a curvy line or any kind of shape that shows how many are in the bag at each point as you go from here to here [along the sequence of changes]?

Carol draws a curved line see Figure 22

Mark: Now that's an interesting shape. Could you explain it to me?

Carol: Well, here [the increasing portion of the first bump] it's getting higher and higher and higher. And when you get to here [the highest point, also pointing to the change from the fourth plus to the first minus in the sequence] it starts getting lower and lower and lower and then it goes up a little more.

Carol immediately curved the line in ways that show both, the size and the sign of the changes. Her bending of the 'time line' reflected her ability to recognize size and sign in a curve and to make use of this recognition for the purpose at hand. This remarkable move that seemed so natural to Carol showed the fluency with which she saw in aspects of a symbol (the slantness of a curve, its height, going from left to right, etc.) events that were outside of it (the sequence of discrete changes), this is the inside/outside process that characterizes the phenomenon of recognizing in. Such seeing overcomes dissimilarities as prominent as the fact that the curve is smooth and continuous and the pluses/minuses are uneven and discrete.

Mark: It goes up a little?

Carol: Yeah.

Mark: And why did it go down at the very end?

Carol: Because it's easier just to end it right here [on the horizontal line], you don't really have [to go down at the end]. It would just like end up here [above the horizontal line].

Mark: Oh, I see, the real end would be right here at this little top {the highest point of the second bump}.

Carol: Yeah. But it's easier just to make it go down like that.

By ending the line on the 'End' point, Carol seemed to sense that the line had to end at the point marked by Mark as 'end;' however, she became aware of a conflict ("you don't really have {to go down at the end}"). This awareness of the dual role of that point (end of the line and number of blocks at the end) was likely to be prompted by Mark's question about the line going down *at the very end while it was clear to her that at that time there were still blocks remaining in the bag.*

Mark: I see. How would that change ..this is interesting. I'll give you a different color. Suppose the start numbers were pretty big {++++} and these {- - - -} numbers were about the same size ..the minuses were about the same size as the pluses. Then what would it look like? And this {the last plus} was in the middle.

Carol: It still goes like this {the increasing part of the first bump}, but then it goes like that {all the way down to the horizontal line} (Figure 23):

Mark: You end up where? Down lower? {when touching the horizontal line}

Carol: And then you go up here {top of the little bump}.

Mark: I see. So the only difference is it seemed like you went lower here {when the new line touches the horizontal line}.

Carol: Yeah.

Note the implicit use of a vertical dimension with the horizontal line indicating zero blocks. Carol developed a graphical space departing from the horizontal line upwards in which being higher meant 'more,' on the line 'zero,' to the right 'after,' and slanted up 'increase.' By being constrained to a single 'curvy line', Carol found new ways to express meaning with it: if the pluses are equal to the minuses the curve must touch the horizontal line again. In her second curve, Carol tried to maintain the sense that there are blocks remaining in the bag at the end of the sequence, by finishing the curve a bit before the point marked as 'end' which in her earlier curve seemed to force it to go down to zero.

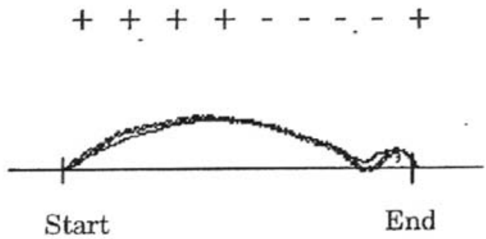


Figure 23 – Adding more blocks at the end

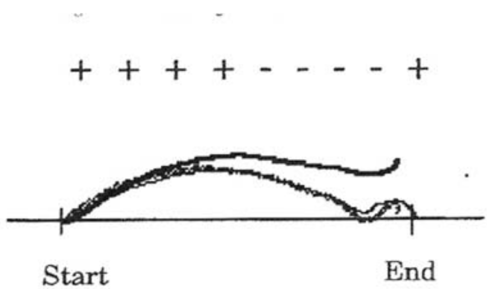


Figure 24 – The minuses are medium size

Mark: If I made one that went like this, sort of was the same as this {the up part}, but then it stayed almost as high

Mark draws a third line in another color see Fgure 24

Mark: What do you think, what would that tell you about the changes that I actually made?

Carol: ...Those {++++} were big. That {——}waslittle and that {the last +} was big.

Mark: OK, and how did you know, or what made you think that these changes {— -} were small?

Carol: Because this {the down part of the graph} doesn't go down very far.

Mark: So the bigger the change {pointing to the minuses} the more it's {the curve} going to go down.

Carol: yeah..

By reversing the task, from the curve to the sequence of changes, Mark prompted Carol to assess the slope of the curve. Note that Carol estimated that the last plus is big: it seems by itself to 'undo' all the minuses and it turns sharply upwards. In these interactions Carol shows a sophisticated recognizing of the size and sign of discrete changes in a continuous curve.

Discussion

Homogenous spaces help to open up unanticipated possibilities. They are useful to extend what is into the realm of what could be. This has a cost. Specifics of what is maybe lost, choices may ignore something that is significant to the symbol-user, and some of the stories that one wants to preserve may become difficult to tell. We showed examples of children's hybrid graphs that express such trade-off's. Then, through classroom and interview examples, we illustrated two families of tasks that seem to support children's development of homogeneous spaces: 1) Making selected features directly visible, instead of requiring intermediate steps and calculations; and 2) Exploring well-defined figural components that can be used in graphing. The feature to make directly visible in the 4 th grade classroom example was the *comparison* between the growths of different plants. In the interview example with Rose it was the *changes* in the number of blocks. The new figural element in the 3 rd grade classroom example was the *ordering* of the successive changes from left to right to show elevator movement. In the interview example with Carol it was a *curvy line* with start and end points.

Why would these tasks facilitate the development of homogeneous spaces? These are some of the conjectures that seem plausible to us:

• Making selected features directly visible builds on the difference between something that can be recognized directly, and that which requires an intermediate process of inferences and calculations. Comparisons within 'group graphs' without consistent scales were indirect and complicated, if not impossible, whereas regular scales enabled comparisons to become directly available to children's sight. As Rose's explanation ("It doesn't really show you what was changed because you have to figure that out") suggests, children are sensitive to this difference. To achieve this directness entails choosing which features are to be highlighted overall. Making one of them directly visible often implies that another has to be 'figured out;' Rose invented the changes graph fully aware that the numbers of blocks would have to be inferred from it; in other words, that one has to choose whether the graph directly 'tells' the changes or the number of blocks. Choices about what is to be in the foreground appear to be essential to the use and design of homogenous spaces.

• Experimenting with figural elements can be a playful activity based on exploiting the possibilities of a highly constrained but simple, familiar, and well defined figural element. Instead of adding icons, symbols, and other additional components, these explorations seem to prompt the symbolizer to identify ways in which the element itself can vary and to attribute meanings to these variations. By ordering the lines from left to right, children started to identify the slant of a line or the height of a step as reflecting the sequence of 'changes button's' that had been pressed, the repetition of a graphical shape as indicating that the same sequence of buttons had been pressed repeatedly; the need to add-on symbols to show which movement came first or last disappeared. As Carol experimented with the curvy line, she began to use its curvature to indicate relative size of the changes, to touch the bottom line as showing emptiness in the paper bag, to imagine the drawing movement to the right as expressing the passage of time, and to merge discrete changes with a continuous line. It is as if these tasks stimulate the symbolizer to articulate multiple meanings coordinated in a single visual feature, rather than splitting them across many co-existing signifiers. This need to integrate multiple meanings onto single shapes or trajectories makes homogenous spaces especially useful: no keys or ad-hoc icons external to the figural element are required to clarify what different locations represent.

These reflections attempt not only to inform instructional design, but also to contribute toward a conceptual framework that we strive to advance. Learning graphing, we think, entails developing the capacity to 'direct seeing' (i.e. without intermediate inferences and calculations) events and qualities dwelling in symbolic expressions; a development that involves intricate experiences of seeing-as, recognizing-in, interpreting emptiness, and animating homogeneous spaces. Rose's seeing-as involved her shifting back and forth between seeing the graph in Figure 14 as a graph of number of blocks in the bag at different times and as a graph of changes in the number of blocks in the bag at different times. Carol's use of a curvy line expressed her ways of recognizing-in it sequences of discrete changes. Empty space became significant as children strived to widen the *possible* plant growth patterns amenable to be represented in their graphs and to increase the directness with which plant growth can be recognized in them. The elevator graphs evolved from sequences of arrows to height vs. time graphs in ways that involved at once a restricted use of simple location and the animation of the homogeneous space with numerous events and stories *taking place* in them and in imaginary buildings.

The phrase 'direct seeing' evokes the ecological views of Gibson (Gibson, 1986), which are often referred to with the heading of 'direct perception'. There are significant similarities and differences with the Gibsonian perspectives that might become the subject of a future paper. Similarities stem from the view that what is perceptually salient are holistic, complex, and action-oriented features not necessarily related to the mathematical/geometric description of the perceived entities (Runeson, 1977): in the same way that we do not sense how to catch a ball by detecting its position in 3D space and performing calculus operations, we do not recognize a pattern of speed in a position vs. time graph by registering its values and computing derivatives. Differences arise from the notions that most of the 'affordances' that we perceive in symbolic expressions are cultural constructs rather than biological constraints, and that the symbolizer actively constitutes them on the basis of current aims and past experiences.

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Notes

1. This asymmetry between space and bodies was dissolved by the general theory of relativity, but that is a different story. When we think of a Cartesian graph we do not see its space distorted or shifted by the entities that populate it.
2. Casey (1997) focuses mostly on a distinction between place and space. We choose rather not to call this an opposition between place and space because our object of study are all the spaces created to represent things and events, including graphs, maps, and layouts, which constitute location-oriented homogeneous spaces, as well as informal ones (e.g. drawings) which often are heterogeneous and place-oriented.

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