DOI: 10.20396/zet.v29i00.8661216

The specialized knowledge of a primary teacher working on subtraction

El conocimiento especializado de um maestro de primaria cuando enseña la resta

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Abstract

This study explores the specialised knowledge deployed by a mathematics teacher using the Open Algorithms Based on Numbers (ABN) method over a series of lessons on subtraction. The theoretical tool used for the analysis is the Mathematics Teachers' Specialised Knowledge (MTSK) model. The research took the form of a case study of a primary teacher working at a Charter School in Huelva, Spain. The data consisted of the transcripts of lessons involving a total of four activities on subtraction, plus a subsequent semi-structured follow-up interview. The transcripts were broken down into units of information, and analysed for evidence of specialised knowledge within several subdomains of Mathematical Knowledge and Pedagogical Content Knowledge. The results allow us to conjecture a relationship between this knowledge and the teacher's training in the ABN method.

Keywords: Subtraction; Specialized Knowledge; Open Algorithm Based on Numbers, Case Study.

Resumo

Esta pesquisa visa compreender o conhecimento especializado que um professor de matemática possui em relação à subtração, em cujo processo de ensino utiliza algoritmos abertos baseados em números (método ABN), utilizando o modelo Mathematics Teacher Specialized Knowledge (MTSK) para sua análise. Para o efeito, foi realizado um estudo de caso de um professor primário do segundo ano de uma Escola Concertada de Huelva. A partir das transcrições das gravações de quatro atividades em que a subtração é trabalhada, bem como de uma posterior entrevista semiestruturada, foram obtidas evidências e indícios que mostram que o professor possui conhecimento especializado em subdomínios de conhecimento de conteúdo e conhecimento didático conteúdo, o que nos permite presumir uma relação entre esse conhecimento e a formação docente no método ABN.

Palavras-chave: Resta; Conocimiento Especializado; Algoritmos Abiertos Basados en Números; Estudio de Caso.

Sent on: 14/09/2020 – **Accepted on**: 01/02/2021 – **Published on**: 27/05/2021

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DOI: 10.20396/zet.v29i00.8661216

Introduction

Given the difficulties that traditional methods of doing arithmetic has caused children over the years, a change is needed in the way that it is taught that goes beyond methods based on meaningless repetition and introduces new strategies and techniques (Fuson, 1992; Carpenter, Franke, Jacobs & Fennema, 1996; Brown & Burton, 1978). For this reason new forms of doing operations in class have arisen, such as the Open Algorithm Based on Numbers (ABN) method, which offer a variety of methods for tackling arithmetic and multiple techniques which take the needs of the pupil into account Bracho-López, 2013).

Although ABN is recent, the techniques underlying the method were foregrounded some time ago by authors (Castro, Rico & Castro, 1987; Maza, 1991) aiming to underline the importance of the decimal numeral system and the use of strategies which could facilitate calculations. Whatever name is given to these techniques, what is evident is that the teacher's awareness of them is fundamental to the effective teaching of subtraction.

This study explores the specialised knowledge of a teacher who expressed an interest in collaborating with the University of Huelva to carry out research into the possibilities opened up by the ABN method, and the issues it might raise within the ambit of mathematics teaching. In this respect, the teacher in question could be considered an ideal subject, as he had been using the approach for many years and frequently took the opportunity to attend conferences on it.

Various analytical models have been used for exploring teachers' knowledge, the majority of them deriving in some degree from the seminal work of Shulman (1986, 1987). As will be explained more fully below, this work makes use of the Mathematics Teachers' Specialised Knowledge (MTSK) model developed by Carrillo *et al*. (2018).

The research questions this study addressed are the following:

- What specialised knowledge is deployed by a primary teacher when they tackle subtraction with second year pupils?
- Which elements of this knowledge might be associated with their training in ABN?

In relation to these questions, the goals of the study were the following: (a) to identify the different elements of specialised knowledge brought into play by the teacher when working on subtraction with their pupils; and (b) to identify which of these elements are connected to their training in ABN.

Theoretical foundations

The study falls within the ambit of research into mathematics teachers' knowledge, and so draws its theoretical underpinnings from the Mathematics Teachers' Specialised Knowledge model (henceforth MTSK), which also constitutes the analytical tool, and the

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Open Algorithm Based on Numbers (ABN) method. With regard to the latter, the principles of the method, and the reasons for its emergence (background), are given consideration. In terms which elements of content to be included, those relating to the techniques of ABN regarding subtraction were selected as these would constitute the knowledge base for evaluating the teacher's specialised knowledge. The other pillar of this study centres on the teacher's knowledge. For the analysis of this content area, and as the theoretical foundation, the MTSK model was used, providing an analytical tool for examining specialised knowledge.

Background to ABN

Students needs have developed, and with them the need for a shift in methodological approach. With respect to mathematics, it has been half a century since Ablewhite and Paret (1971) questioned the effectiveness of the methods used to teach arithmetic at school after witnessing pupils encounter innumerable problems arising from the mechanical application of conventional algorithms.

According to Bracho-López, Gallego-Espejo, Adamuz-Povedano and Jiménez-Fanjul (2014), there is an urgent need to develop better teaching methods. Many of the gradual changes that have taken place are indicative of the need for a more far-reaching shift if we are to solve the problems inherent in current practices, which place an illogical emphasis on rote learning without any understanding of the processes involved.

The ABN method, amongst others, arose in response to these shortcomings of traditional calculation methods (relying on compressed algorithms) and other unresolved areas of difficulty for pupils, with the aim of providing a more learner-centred approach in the classroom (Martínez, 2011).

The Open Algorithm Based on Numbers (ABN)

The Open Algorithm Based on Numbers (henceforth ABN) was created in 2000 by Jaime Martínez as an alternative to conventional algorithms. After its inception, Martínez made modifications to the method aimed at improving it, and finally published the definitive version in 2010, along with explicit instructions for how to use it in lessons (Martínez, 2011).

According to Martínez (2011), the fundamental aim of ABN is to shift the focus of attention in doing calculations from the algorithms and to place it on the numbers (in the Decimal Numeral System, henceforth DNS) so as to make them more open (and comprehensible). This is achieved by allowing the pupil to decide how many steps to take in reaching the solution[3]. Among the benefits of the method are improvements in mental arithmetic, problem-solving and pupils' general attitude to learning mathematics. The method de-emphasises calculations using simply numbers and instead always provides a context for problems such that the DNS is foregrounded (Bracho-López, Adamuz-Povedano,

[3] In the purest sense of mental arithmetic.

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GallegoEspejo y Jiménez-Fanjul, 2014). According to Bracho-López et al., (2014), working with numbers in this way entails treating the numbers differently according to the relative value assigned to them by the DNS (units, tens and hundreds), as opposed to conventional algorithms which tend to promote mechanical operations in terms of the absolute values. In order to provide context, ABN recommends the use of manipulables such as sticks, money tokens and the like (Canto, 2014).

There is a great difference in terms of operations between ABN and traditional methods, which can be characterised as Closed Algorithms Based on Ciphers (henceforth CBC). According to Martínez (2018), subtraction can be carried out in four different ways, the choice of method depending on what the pupil wants to achieve (see Table 1). Further, there is no need to invoke the notions of carrying numbers and zeros, so it is not necessary for pupils to keep any information in their heads as all the steps in the calculation are explicit (Martínez, 2018).

Table 1 – Summary of subtraction procedures in ABN

	SUBTRACTION
Decrementation	The two numbers are written at the top of separate columns and an identical amount is subtracted from each. Further amounts are then subtracted (the amount can vary, but the same amount is always subtracted from both columns) until the lesser of the original numbers is zero. The other is the difference.
Comparison	This process identifies which is the greater of two amounts, and which the smaller.
Counting up	This strategy uses a grid with three columns and an indefinite number of rows. The final column records the larger amount (the desired amount), which is to be reached by incremental additions (recorded in the first column) to the smaller amount (recorded in the middle column). The size of each addition is decided by the pupil.
Counting down	This is the inverse of the above strategy. The starting point is the larger amount, which undergoes incremental subtractions to reach the smaller amount.

Source: adaptaded from Martínez (2011).

The key difference between ABN and CBC is that, as noted above, it is the pupil who determines the number of steps along the way by deciding how large each step should be (Martínez, 2011).

ABN could be considered a procedure for doing meaningful calculations, as opposed to the meaningless mechanical calculations which characterise CBC. If a teacher uses the system, they must necessarily understand the underlying calculations on which it is based, so there is a greater likelihood of it making sense to their pupils. Conversely, the method can be regarded as another resource available to the teacher. In either case, there are two elements of specialist knowledge involved, as we shall see below.

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Mathematics teachers' specialised knowledge (MTSK)

In recent decades there has been a growing interest in understanding the kinds of knowledge which teachers possess about the content they teach. In the research literature it was not until Shulman (1986) that the lack of research into this area was made explicit. Shulman coined the expression *Pedagogical Content Knowledge* (PCK) to put the focus on a paradigm that until then had been overlooked. The importance of PCK is that it constitutes the amalgam of knowledge that the teacher brings to their teaching. It is the application of pedagogy to content that enables the teacher to adapt content according to the understanding, interests and capabilities of the pupils (Shulman, 1986; Shulman, 1987). Shulman's (1986) paper is a vigorous rebuttal of G. B. Shaw's aphorism – "He who can, does. He who cannot, teaches" – which, Shulman says, is as insulting as it has been damaging to the profession. In its place he quotes Aristotle, "Those who can, do. Those who understand, teach" (p14), emphasising that understanding content is the key to its transformation.

Although Shulman (1986) stressed the importance of PCK, he did not develop an explicit model of analysis for its study. It was left to Ball, Thames and Phelps (2008) to pioneer an analytical model. This study, however, takes the Mathematics Teachers' Specialised Knowledge (MTSK) model as it focuses on the teacher's specialised knowledge for teaching mathematics (Carrillo, Contreras & Montes, 2013). The model was developed at the University of Huelva in response to the perceived difficulties in the MKT (Mathematical Knowledge for Teaching) model developed by Ball, Thames and Phelps (2008), especially with regard to the differentiation between what can be considered common and specialised knowledge. From an analytical perspective, for example, MKT appears to keep separate knowledge about students' difficulties and errors from knowledge about the reasons and mathematical foundations that enable teachers to understand these. MTSK overcomes this kind of problem by considering all teachers' knowledge as specialised, so that the model is specialised throughout (Carrillo, Climent, Contreras & Muñoz Catalán, 2013).

DOI: 10.20396/zet.v29i00.8661216

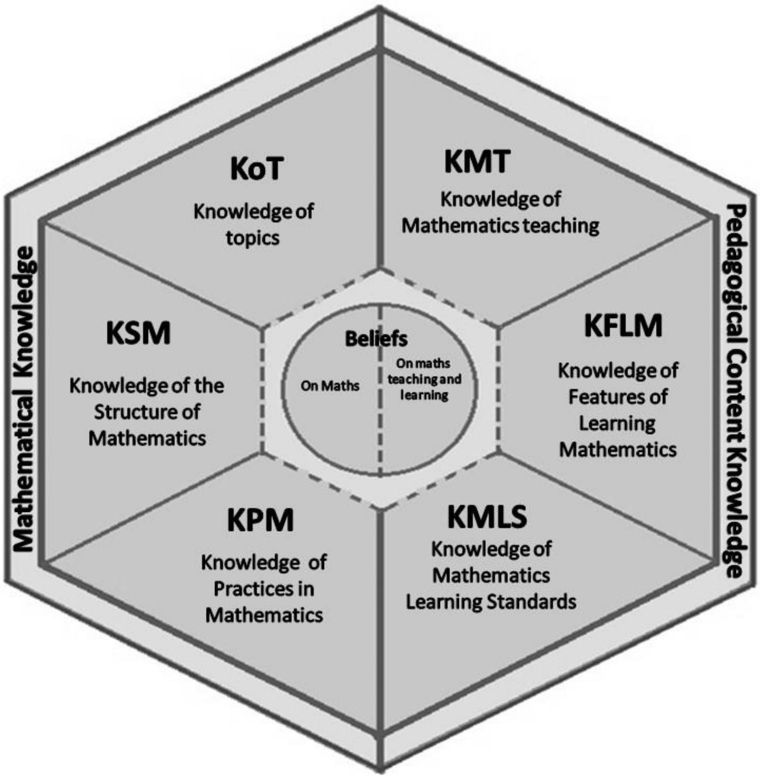

Figure 1 – MTSK subdomains
Source: (Carrillo et al, 2013).

In addition to the domain of beliefs, which interacts with all aspects of the teacher's knowledge and hence infuses all the subdomains (Carrillo et al., 2013), the model recognises two domains of knowledge: Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK). According to Carrillo, et al. (2013) these constitute two aspects which Shulman (1986) contemplated, and which are incorporated into the model with precise indicators to facilitate analysis.

In its turn, Mathematical Knowledge (MK) is divided into three subdomains, each with its associated categories for a more fine-grained analysis Carrillo et al., 2013; Aguilar, 2013):

- Knowledge of Topics (KoT): this concerns the teacher's knowledge of mathematical content. It includes deep knowledge about the meanings, definitions, properties and examples associated with any particular topic. Elements of knowledge in this subdomain can be analysed according to four categories: *Definitions, properties and foundations; Procedures; Registers of representation; and Phenomenology and applications.*
- Knowledge of the Structure of Mathematics (KSM): this concerns the connections that the teacher is able to make between content items. Connections can take four different form: *Connections based on simplification*; *Connections based on increased complexity*; *Transverse connections*; and *Auxiliary connections*.

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- Knowledge of Practices in Mathematics (KPM): this concerns knowledge of the procedures required in constructing or validating a mathematical outcome, and also knowledge about how to communicate mathematics. There are no categories in this subdomain, but there is a series of accepted conditions framing items within it, such as forms of validation and demonstration.

Within Pedagogical Content Knowledge (PCK) the following subdomains can be found (Carrillo et al., 2013; Aguilar, 2013):

- Knowledge of Mathematics Teaching (KMT): this encompasses the knowledge a teacher requires about different kinds of resources, examples and materials for teaching a mathematical item, and how to get the most out of these. It comprises three categories: *Theories of mathematics teaching; Teaching resources (physical and digital); and Strategies, techniques, tasks, and examples*.
- Knowledge of Features of Learning Mathematics (KFML): this concerns the teacher's knowledge of the features involved when students learning a particular item. There are four categories in this subdomain: *Theories of mathematical learning; Strengths and weaknesses in learning mathematics; Ways pupils interact with mathematical content; Interests and expectations.*
- Knowledge of Mathematics Learning Standards (KMLS): this involves knowledge about the relevant curricular specifications regarding expected learning outcomes at the various levels of study, along with knowledge of the research into mathematics education in this regard. This subdomain is made up of three categories: *Expected learning outcomes; Expected level of conceptual or procedural development; and Sequencing of topics.*

In summary, the MTSK model is an analytical tool designed for exploring in granular detail the knowledge accumulated by a teacher around specific mathematical content. In this study it is used to map out the network of knowledge on the topic of subtraction demonstrated by a primary teacher in the course of an observed lesson and a follow-up interview. In particular, we focus on the teacher's background in ABN in order to explore the influences that the method has had on his knowledge of the topic.

Methodology

This study is located within the interpretive paradigm, and hence, as Muñoz-Catalán (2009) note, it aims to understand phenomena in the context in which they occur. In this particular instance, the objective is to gain an understanding of the specialised knowledge deployed by a teacher when giving lessons on subtraction. Further, as a study with a qualitative focus, it incorporates recordings and interviews which provide an interpretive and naturalistic perspective of the phenomenon (Denzin & Lincoln, 2000, cited in Muñoz-Catalán, 2009).

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With respect to methodology, the research took the form of a case study, defined by Bassey (1999) as “study of a singularity conducted in depth in natural settings” (p.47). Of the three types of case study noted by Stake (2000), this follows the instrumental model as it enables a deeper understanding of a topic to be achieved through the analysis of a particular instance, in this case, that of Bartolomé.

At the time of the study, Bartolomé was a primary teacher with 22 years of experience, during the last 8 of which he had been using the ABN method at a charter school. In order to improve his competence with the method, he had attended more than a dozen training courses. In addition, he was an active participant in the ABN community and had created his own blog, where he talked about his experiences in using the method and created various teaching materials.

Data collection was carried out through video recordings of mathematics lessons given by Bartolomé to second year primary pupils in the months of January and February (2020). From these, those sections dealing with activities involving subtraction were fully transcribed. In addition, an interview with Bartolomé was carried out to provide the opportunity to explore in more detail the specialised knowledge brought into play in the course of the lessons, and more especially, to obtain evidence of whether his knowledge, when brought to the surface, was connected to his training in ABN.

With regard to the analysis of the data, the instrument used for this purpose was the MTSK model, as this facilitates the recognition and interpretation of the specialised knowledge deployed by a teacher in the course of teaching a specific mathematical item (subtraction). As noted above, the instrument is comprised of different subdomains and categories which enable the analysis of data. It has been put to use since 2013 in a variety of research projects at different educational levels and for different curricular areas.

Once the recordings had been transcribed (from both the lesson observations and the interview), a selection was made of episodes which were particularly rich in data, which were then assigned to categories according to the MTSK model. Throughout the process of analysis, all interpretations were triangulated between the researcher and co-researcher.

Analysis of the results

This section describes the analysis of the Bartolomé’s specialised knowledge of subtraction deployed in four activities, from which a variety of evidence was collected. All four activities were drawn from the same textbook, published specifically with a focus on ABN. The analysis highlights the subdomains brought into play by the teacher during the teaching-learning process, and indicates the connection between this knowledge and his training in ABN techniques. Before going into the analysis, however, it is first necessary to describe the activities.

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ACTIVITY 1

Bartolomé asked the pupils to take out the materials which they would use to as support to help them understand the activity (circles, semi-circles and quarter circles made of card representing an hour, a half-hour and a quarter of an hour respectively). Using these materials and a grid (essentially a table with three columns and multiple rows for making a graphic record of the operations carried out in the activity), the pupils tackled a subtraction problem posed by the teacher: “I’m in my house at 15:30 and I go shopping. I get back home at 17:00. How long was I out?”

ACTIVITY 2

Bartolomé gave the pupils a problem involving two commercial products. He gave the price of one, and the difference in price between this and the other. Using this information, the pupils had to work out the price of the second item. The question text was as follows: “This mobile phone costs 684 euros. The mobile phone is 186 euros more expensive than the tablet. How much does the tablet cost?”

ACTIVITY 3

Bartolomé presented the pupils with a problem in which a man has a certain amount of money and needs a certain amount more in order to be able to buy a product. The ideal technique for solving the problem is that of counting up. The text was as follows: “You’ve got 4.38 euros. How much do you need to have 10?”

ACTIVITY 4

Bartolomé asked the pupils to carry out a task in which they started with a certain amount of money (represented in decimal notation: 2.82), and had to find out how much more they needed to arrive at another amount. In order to help the pupils with the concept of decimal places, which they found difficult, he used different colours according to the value, and coins made out of card. The text was as follows: “You’ve got 2.82 euros. How much do you need to have 10?” (Although the activity requires the same kind of thinking as in the previous problem, Bartolomé used different strategies and resources, which indicated a different kind of knowledge from that required previously, as shall be seen below).

Analysis of how these activities were carried out revealed knowledge pertaining to two subdomains of Mathematical Knowledge (MK) – specifically Knowledge of Topics (KoT) and Knowledge of the Structure of Mathematics (KSM) – and another two subdomains in Pedagogical Content Knowledge (PCK) – namely Knowledge of the Features of Learning Mathematics (KFLM) and Knowledge of Mathematics Teaching (KMT). The following section describes the teacher’s specialised knowledge according to these subdomains.

- **Mathematical Knowledge (MK)**

Analysis of Knowledge of Topics (KoT)

The information units revealed by the analysis and described below all pertain to the

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category Procedures. With regard to this subdomain, Bartolomé ('B' in the transcriptions) demonstrates his knowledge of the characteristics of subtraction. He knows that if one amount is subtracted from another, the result is the difference between the two amounts. In other words, he knows the meaning of subtraction, knowledge related to the descriptor 'How is something done?' in the Procedures category. He knows one of the ways to do subtraction, that of breaking up the subtraction.

B: It's a subtraction. Let's see. If I take the 5 from the half past 3 what would I get? The time I'd been out. (A.1.2)[4]

[…] B: It was 684 and 186. Those who've got the notebook do it*. J, explain it to those who are listening.*

J[5]*: First, we take away 100 from the 186*. (He writes 100 in the first square.)

B: I put 100 there and I take that away from both of them. (A. 2.1)

Whole units are successively taken away from both amounts

684 - 186		
100	*584*	*86*
80	*504*	*6*
6	*498*	*0*

First column: amount to be taken away

When the smaller amount reaches zero they get the result

In order to investigate the possible connection between this knowledge and his ABN training, in the interview we asked Bartolomé what subtraction meant to him. He replied as follows: [...] *For me, the way I see subtraction changed after being trained in the ABN method. Subtraction for me now is nothing more than comparing one amount with another and seeing the difference between the two. Before that, the meaning it had for me was of something mechanical, now I subtract and now I have to carry I-don't-know-what. In this case you understand what you're doing a lot more, and there's no carrying things without knowing what you're carrying*.

[2] The units of information have been coded: 'A' stands for activity, the following number corresponds to the number of the activity (1.4), and the final number identifies the corresponding indicator.

[3] Student contribution identified by their initial

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With respect to the procedure for subtraction, the pupils started with one amount and needed to get to another (counting from or counting up). Carrying out the activities by breaking up the subtraction also provided evidence. The teacher showed that knows how to do the technique of counting up.

B: Very good, two cents. If I have 4.38 and I add 2 cents, what do I get to, I.? (I. thinks a while.) *I've got 4.38 and I add 2 cents. I get to… (A. 3.2)*

Table 2 – Summary of knowledge pertaining to the KoT subdomain

KoT		
INDICATORS OF ANÁLYSIS	CATEGORY	UNIT OF INFORMATIÓN
Knowing the meaning of subtraction	Procedures	A.1.2 A.2.1
Knowing the procedure for carrying out a subtraction		A.3.2

Analysis of Knowledge of the Structure of Mathematics (KSM)

With respect to the KSM subdomain, the analysis provides evidence of a 'Connection based on simplification'. In this instance, in order to help him reduce the difficulty involved in working on an item above the pupils' level (decimals), the teacher applies a simplification strategy, using the pupils' knowledge of the DNS to colour-code the decimal numbers in the monetary notation.

B: We've just heard J. being very honest and saying that he doesn't understand. Perhaps he's not the only one who doesn't understand. Look, I'm going to write these cents in red and these cents in blue, the ones from the gold coins. (In the top row of the grid, in which is written 'From 2.82€ to 10€' he writes the 8 in red and the 2 in blue.) (A.4.2)

*From 2.**82**€ to 10€*	
0.08	*2.90*
0.10	*3.00*
7.00	*10.00*
Total: 7.18€	

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Table 3 – Summary of knowledge pertaining to the KSM subdomain

KSM		
INDICATORS OF ANÁLYSIS	CATEGORY	UNIT OF INFORMATIÓN
Knowing a colour-coding strategy based on the DNS to reduce the difficulty occasioned by decimals	Connection based on simplification	A.4.2

- **Pedagogical Content Knowledge (PCK)**

Analysis of Knowledge of the Features of Learning Mathematics (KFLM)

Evidence was found in this subdomain of knowledge connected to the strengths and weaknesses that pupils might encounter in learning a particular item, and also knowledge concerning the ways pupils interact with mathematical content.

When Bartolomé breaks up the subtraction, he shows awareness of the difficulties the pupils might have with the new topic (time), and so decides to take away smaller amounts.

B: Pay attention and listen carefully to what I'm going to do everybody in your notebooks. I write 15:30 take away 17:00 and now I take away an hour. (On the board he draws the grid which will be used to do the subtraction, and writes '1h' at the top of the first column.) (A.1.3)

15:30 – 17:00		
1h		

We asked him in the follow-up interview whether he took advantage at any point of the freedom afforded by the open algorithm, to which he answered: "*Well, when it's their first contact with an item, something they don't know or which I know will give them problems for whatever reason, I prefer to start with small amounts to subtract so that the subtraction isn't so difficult for them, and they can concentrate on the procedure. Once they really know how to do it, they can subtract whatever amounts they want. Let's say, as always, the most important thing, particularly in the early stages, is to begin bit by bit so that they develop the ability*."

In relation to this and to his knowledge of subtraction, each time Bartolomé demonstrates a calculation he first checks that the pupils understand the procedure. In this case, he shows that he is aware of the difficulties they are having with it. The episode provides evidence of his knowledge of the kinds of difficulties that the pupils might experience, and is repeated over several activities.

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[…] *If I take away an hour, if it's taking away, what do I do J.? Which do I take it away from? Both times* [both amounts] *or just one? (A.1.4)*

B: Very good. You've taken away 100 from both, now how much do we take away?

J: 80

B: OK then, both amounts, both of them, right? I write 80 there and I take it away from both. (A.2.2)

In order to be sure that such episodes were indeed evidence of the application of his knowledge, we asked Bartolomé the following question: "*The fact that you use four ways of doing subtraction might lead the pupils into getting the procedure wrong. Would you say that any of the ways are more difficult to understand*?" He replied as follows: "*It's true that it's in the second cycle – in third and fourth year – that they really get the hang of the four ways of doing subtraction. But in the second year of primary they find it challenging, especially when it is a new kind of problem or item, as we said before. It is more difficult for them to decide which one is the correct one. In my experience, since I've been using them with the pupils, the easiest are counting up and counting down, but one they get to grips with all four, this doesn't cause any problem. The thing is that in the second year of primary, and in the first year, they are still learning them*."

Finally, there is evidence of the teacher's knowledge of the ways his pupils interact with mathematical content, in this case doing calculations. Bartolomé shows that he is aware that the pupils have a personal strategy which helps them to do the calculation ("counting from").

[…] on page 98. Instead of counting up to one euro we're going to have to get to 10 euros. We're going to be given an amount, and we're going to complete that amount by counting up. Let's read the first part, the bit in the green box. What does it say J.? It says get to 10 euros by counting up, and then we get a T-shirt. ¿What does the problem say?

J: You've got 4.38 euros. How much more do you need to get to 10? (A.3.1)

[…] B: Very good, two cents. If I've got 4.38 and I add 2 cents what do I get J.? (J. thinks for a moment.) *I've got 4.38 and I add two cents. I get to ... (A.3.2)*

Table 4 – Summary of knowledge pertaining to the KFLM subdomain

	KFLM	
INDICATORS OF ANÁLYSIS	CATEGORY	UNIT OF INFORMATIÓN
Knowing subtraction is made easier with smaller amounts when introducing a new item Knowing pupils' difficulties insubtractinng from both amounts	Strengths and weaknesses	A.1.3 A.2.2 A.1.4
Knowing a strategy which the pupils use to help them do calculations	Ways pupils interact with mathematical content	A.3.1 A.3.2

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Analysis of Knowledge of Mathematics Teaching (KMT)

It is in this subdomain that there is most evidence of the teacher's knowledge, specifically in the category 'Teaching resources (physical and digital)', the significance of which will be considered in the discussion.

Regarding the category 'Theories of teaching', we can locate knowledge of an alternative to the traditional method of doing calculations (counting up) and its deployment to aid learning.

En relación con la categoría de Teorías sobre enseñanza situamos el conocimiento de un método alternativo de cálculo (la escalera ascendente) al cálculo convencional y su uso para facilitar la enseñanza.

[…] on page 98. Instead of counting up to one euro we're going to have to get to 10 euros. We're going to be given an amount, and we're going to complete that amount by counting up. Let's read the first part, the bit in the green box. What does it say J.? It says get to 10 euros by counting up, and then we get a T-shirt. ¿What does the problem say?

J: You've got 4.38 euros. How much more do you need to get to 10? (A.3.1)

[…] B: Very good, two cents. If I've got 4.38 and I add 2 cents what do I get J.? (J. thinks for a moment.) *I've got 4.38 and I add two cents. I get to ... (A.3.2)*

With regards to material and digital resources, the teacher makes use of various sorts (chiefly material) in his explanations in order to make learning subtraction accessible to his pupils. Hence, he is not only aware of such resources, but is also aware of their effectiveness in the teaching-learning process. Specifically, the following resources are used:

- Circles, semi-circles and quarter circles made of card (representing an hour, a half-hour and a quarter of an hour respectively).

B: Ok, now it's time to get out the hours [hour cards] and put them at the side of the board. Get three from the tables with the drawers, take three whole hours. Has everybody got three hours? (He draws three circles and a semicircle next to the grid.) *Take a half-hour as well* (He waits while the pupils collect the cards from the drawer). *Have all the groups got three hours and a half-hour?* (He asks for quiet). *Now R. What else do we need for the problem?* (A.1.1)

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15:30		

- Euro and cent coins of different denominations made out of card for the activities involving transactions.

J: Why is it written with a full-stop?

B: Because those are cents, J. Those are either copper ones or gold ones. (He picks up a card coin representing one euro and shows it to the pupil.) *This is one euro. The euro goes in front of the full stop. I've got three of them*. (He picks up three coins and counts them.) *One, two, three. And now I've got twenty-five*. (He picks up a 20 euro coin and a five euro coin, then puts them down.) *OK, I've got this, J*. (He holds up the five euro coin.) *I need another five cents. If I've got five and five, how much have I got*? (A.4.1)

B: I have 70 and 30 cents. I change them for one euro (He puts down the 50 cent coin, the two 20 cent coins and the 10 cent coins, and picks up a 1 euro coin, adding it to the other thee 1 euro coins.) (A.4.3)

- Colour-coding in the Decimal Numeral System

B: We've just heard J. being very honest and saying that he doesn't understand. Perhaps he's not the only one who doesn't understand. Look, I'm going to write these cents in red and these cents in blue, the ones from the gold coins. (In the top row of the grid, in which is written 'From 2.82€ to 10€' he writes the 8 in red and the 2 in blue.) (A.4.2)

De 2.82€ a 10€	
0.08	*2.90*
0.10	*3.00*
7.00	*10.00*
Total: 7.18€	

Finally, in the category of 'Strategies, techniques, tasks, and examples' we find the teacher's knowledge of technique by which he helps the pupils to understand that subtraction is an iterative process, in the course of which it is possible to see how much has been taken away and how much is left.

DOI: 10.20396/zet.v29i00.8661216

[…] B: It was 684 and 186. Those who've got the notebook do it. J, explain it to those who are listening.

J: First, we take away 100 from the 186. (He writes 100 in the first square.)

B: I put 100 there and I take that away from both of them. (A. 2.1)

684 – 186		
100	*584*	*86*

Table 5 – Summary of knowledge pertaining to the KMT subdomain

	KMT	
INDICATORS OF ANÁLYSIS	CATEGORY	UNIT OF INFORMATIÓN
Knowing and using an alternative method for doing calculations	Theories of mathematics teaching	A.3.1 A.3.2
Knowing the effectiveness of material resources for improving understanding of an item	Teaching resources (physical and digital)	A.1.1 A.4.1 A.4.2 A.4.3
Knowing the technique of subtraction through ABN to understand that subtraction is an iterative process in which it is possible to see how much has been taken away and how much remains	Strategies, techniques, tasks, and examples	A.2.1

Assessment of the analysis

This study has identified evidence of a primary teacher's specialised knowledge across several subdomains of the MTSK model. It should be noted that the teacher employed the ABN method for doing subtraction, inconsequence of which the discussion below will consider whether the various areas of knowledge identified can be deemed to have derived from the teacher's training in the method. To do so we draw on the follow-up interview with the teacher. It has not been possible to compare our assessment of the results with previous research as no studies into the ABN method using the MTSK model are available.

With respect to the knowledge of subtraction (the KoT subdomain), the teacher not only has a clear understanding of what subtraction means, but he also bases his work on the ABN method. In *(A.1.2)*, he demonstrates that, in this procedure, the position of the minuend is not important. The subtraction is performed according to which is larger and which

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DOI: 10.20396/zet.v29i00.8661216

smaller, such that in the operation of subtracting, the lower amount is preceded by the larger. On being asked how long he had been using this method of subtraction, Bartolomé replied:

> *It's certainly true that with the traditional method of subtraction I never thought of placing the lower amount before the larger. Far less did I think that you can subtract in four different ways. Although in the ABN training they tell you that it is better, or more advisable, to subtract with the larger amount first, once the pupils understand what they're doing – they're comparing amounts – they know how to do it either way.*

This illustrates the influence of his training in general, and, it can be reasonably conjectured, with ABN in particular, given his extensive experience using the method. Naturally, it could also be argued that his knowledge of ABN forms a part of his specialised knowledge.

Further, in this excerpt, as in others *(A.2.1)*, the teacher breaks the subtraction down, a feature which is characteristic of ABN, as the algorithm is designed to allow pupils to subtract the amounts they choose according to how difficult they perceive the problem to be. Once again, when asked about this feature of the method in the follow-up interview, he replied as follows:

> *With the traditional method, which is the one I used to use before this one, I did the calculation mechanically, just the way you're supposed to with that method, and the possibility never occurred to me. It's true, though, that when it comes to correcting activities, for example, it can be a bit problematic since you rarely find that the pupils subtract the same amounts. Because it's an open algorithm, they take the number of steps they want, and subtract amounts that they find easiest to handle.*

Once again we can see the source of his knowledge.

The importance conferred on the DNS by ABN is illustrated in both the procedure the teacher adopts in *(A.3.2*), where he guides the pupils to round up an amount to the nearest whole unit (the category of 'Procedures'), and the strategy in which he colour-codes the decimal places in the prices to help the pupils deal with an area a little beyond their level (decimals) (the category of 'Connections based on simplification'), thus enabling them to do calculations involving money *(A.4.2)*. Both cases underline the importance of contextualising figures, regarded as an important element of ABN. Again, we took the opportunity in the interview to ask Bartolomé about this latter activity, and in particular why such importance was accorded to the DNS, and received the following answer:

> *Well, it would normally never occur to me to do decimals with second year primary pupils, but if you stop to think about it, they all know what money is. It's part of their everyday lives, they know that there are euros and cents. They know how many cents are in a euro, and in effect, they have been learning about the decimal numeral system since virtually infants. Me, because of my training and because of the videos about ABN I've watched, initially I thought, 'But how are they going to be able to understand decimals?' Well, they do understand them because they are euros and cents, things they see every day. So if you draw on the decimal numeral system it can help you to get them to understand because it's something they know, and something that with this method they've been working on since they were little.*

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With respect to ABN methodology within the KFLM subdomain, the teacher carries out subtractions by taking away small amounts, carrying out the operation little by little in order to minimise the difficulties which the pupils might encounter on first contact with a new content item *(A.1.3)*. Of the four methods for doing subtraction in ABN, one (decrementation) requires the same amount to be subtracted from two given figures. Being sensitive to the potential problems the pupils might encounter when carrying out this kind of subtraction, the teacher places significant emphasis on the procedure, providing evidence of his knowledge of the difficulties faced by pupils *(A.2.2)* y *(A.1.4)*. The final evidence of the deployment of knowledge pertaining to this subdomain is the teacher's awareness of the way the pupils interact with mathematical content. This can be seen in the "counting from" strategy suggested by the teacher when the pupils are working on the "counting up" method, one of the four subtraction methods proposed by ABN *(A.3.1) y (A.3.2)*. Given that the strategy of "counting up" antedates its inclusion in ABN methodology, we asked Bartolomé how long he had been familiar with it, to which he replied: "*Until I was trained in ABN I didn't know about these techniques, although I had heard of the method and knew it involved other techniques. Before then I had what you'd call the basic training any teacher would usually get in mathematics*." This again gives a clear indication of the provenance of his knowledge.

With regard to the final knowledge subdomain relating to Bartolomé's teaching – KMT – there is evidence of knowledge of strategies for teaching *(A.3.1)* and *(A.3.2)* when he activates the "counting up" method[6]. Indeed, both this and the other three ABN methods require the teacher to give detailed and explicit instructions for using the grid to perform the subtraction. Consequently, we can say that Bartolomé has a technique which enables him to help the pupils understand the process of subtraction, and which allows them to see how much has been taken away and how much remains *(A.2.1)*. We wanted to know how long he had known about this technique, and whether he had used the technique of breaking down subtraction before coming to ABN, or any strategy for doing decrementation in class. His answer was the following: "*When I used the traditional method I was fairly basic. I stuck to the activities given in the textbook and little else. It was when the whole range of possibilities fr teaching maths was opened up to me that I began to use teaching resources and become more fully trained*."

Also of note is the large number of resources which Bartolomé employs in carrying out these activities, using different contexts and topics (hours and coins), and placing emphasis on tactile experiences and contextualisation, precisely as ABN advocates. He also gives great importance to material resources: circular cards to represent hours *(A.1.1)*; coins made from card to represent various denominations of euros and cents *(A.4.1)* and *(A.4.2)*; and colour-coding to aid comprehension of unfamiliar content *(A.4.3)*. He shows awareness,

[6] Teaching strategy deriving from his knowledge of the strategy "counting from" as is typical of pupils at this stage (Maza, 1991), which again demonstrates Knowledge of Features of Learning Mathematics (KFLM).

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too, of the limits of these aids as well as their potential. Asked about the importance of using these kinds of teaching aids, and of contextualising learning, he said:

> *Well, I've learned a lot from material resources and they've been great use to me, because if the pupils make a mistake they don't have to think mentally about where they went wrong, it's there in front of them, so they can see the mistake. What's more, with ABN they've been learning through handling things since infants and that helps me to continue working that way. It's true that as you move up the school you tend not to use tactile things so much, but in the early stages of learning mathematics, you can learn a lot by having things you can hold in your hands. It helps a lot.*

As can be seen, his training in ABN has contributed a significant amount to his knowledge.

Conclusions

This research has identified different kinds of specialist knowledge brought into play by a primary teacher in the course of teaching subtraction. The knowledge was analysed and classified according to the corresponding categories within several subdomains of the MTSK model. In addition, the study also clarified which of these elements of knowledge derived from the teacher's training in the ABN method, drawing on the information presented in the theoretical framework to do so.

The results bring together evidence which lead us to think that Bartolomé has a strong grasp of teaching subtraction through this method. Specifically, we found evidence of knowledge pertaining to the subdomains Knowledge of Topics (KoT), Knowledge of the Structure of Mathematics (KSM), Knowledge of Features of Learning Mathematics (KFLM), and Knowledge of Mathematics Teaching (KMT). Of particular note in this latter subdomain is the wide range of resources he is familiar with, and which he uses as aids to understanding the concept of subtraction. We conjecture that this area of knowledge derives from his training in a method which emphasises the importance of using materials to contextualise content.

We can also state that his deep understanding of the principles and procedures of ABN can account for the interconnections that were identified between the different knowledge subdomains. With respect to the topic of subtraction, these interconnections can be seen in Bartolomé's knowledge of, and expertise in, various procedures for carrying it out (KoT). They can also be found in his knowledge of the features of teaching subtraction, including the use of resources, techniques and theories which facilitate understanding (KMT), and in his knowledge of the difficulties which pupils might encounter, and the way in which they interact with content KFLM). His knowledge also includes interconnections between content, which he makes use of to facilitate the learning process (KSM).

In this respect, we consider the original research questions to have been answered. These were aimed at exploring the specialised knowledge of a primary teacher working on the topic of subtraction with second year pupils, and to identify which elements of this

DOI: 10.20396/zet.v29i00.8661216

knowledge derive from his training in ABN. In short, he demonstrated considerable expertise in the area, and his approach drew on his training in the method.

It should also be noted that evidence of specialised knowledge was found in four subdomains (as given above), and that no evidence was found of the other two, KPM and KMLS. This may simply be a result of the data collection tool that was selected (observation). It would be interesting to run another comparative case study in a similar context with another primary teacher working with second year pupils, in this case unfamiliar with ABN, in order to further investigate the potential contribution of ABN training to the configuration of specialised knowledge.

Clearly, such knowledge can come from multiple sources, including the teacher's own experience, but it is no less certain that Bartolomé explicitly recognised the role of ABN in developing his knowledge.

Acknowledgements

This research was supported by Spanish Government (EDU2013-44047-P and EDU2016-81994-REDT), and the research centre COIDESO (Universidad of Huelva, Spain).

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