DIDACTICAL REFLECTIONS ABOUT SOME PROOFS OF THE PYTHAGOREAN PROPOSITION

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This theoretical paper presents some dimensions considered in the literature to analyze proof in the teaching and learning of mathematics. In order to show how different types of proofs can be used with students of different levels, we use these dimensions to analyze four proofs of the Pythagorean proposition.

This essay consists of three main parts. First, we show some classifications of previous work on mathematical proof according to dimensions that we consider useful in the teaching and learning of the subject. The second part presents mathematical proof as an important but problematica issue in mathematics education. This is shown in different contexts. In the classroom, on the one hand, students have difficulties with the notion of proof and, on the other, teachers do not know how to teach proof. Finally, we show how some research results can help teachers in their task in order to improve students′ understanding of the Pythagorean proposition.

Classifications

Many studies on mathematical proof try to delimit, characterize, specify or define the term *proof*. We find different words with similar meanings such as *explanation*, *argument*, *justification*, *confirmation*, *verification* or *validation*. These meanings appear with no clear delimitation and sometimes even with quite confusing connotations. But, in spite of this and the differences between their theoretical frameworks, all of the studies consulted assign to mathematical proof a logical character that implies an unequivocal mathematical statement. Some researchers use the different terms to establish a rank between them. We have found two big groups of authors that use rigor to propose their classification. Rigor is a dimension, which conforms analytic dimensions that will help us in our later reflections. We have classified the meanings given to the notion of proof in some studies (see table 1).

In both groups we find different terms for the same kind of proof. They base their work on similar criteria to do their own classifications in terms of rigor. That is why heuristic argument, informal proof and so on are in the same stage. In the same sense, if we find a proof in which drawings or concrete numbers are involved and where inductive reasoning is predominant, all the authors who consider two levels would classify the proof in the less rigorous stage. On the other hand, as much deductive reasoning involves a proof, more formal it is considered.

Table 1

<table>
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+</td><td colspan="4">TWO LEVELS</td><td colspan="4">THREE LEVELS</td></tr>
<tr><td>Jaffe & Quinn (1993)</td><td>Movshovitz -Hadar (1996)</td><td>Martínez (2000)</td><td>Gutiérrez (2001)</td><td>van Dormolen (1977)</td><td>Blum & Kirsch (1991)</td><td>van Asch (1993)</td><td>Balacheff (2000)</td></tr>
<tr><td rowspan="2">Heuristic Argument</td><td rowspan="2">Informal Proof</td><td rowspan="2">Informal Argument</td><td rowspan="2">Empirical Proof</td><td>Level 0</td><td>Proof</td><td>Proof with drawings or examples</td><td>Explanat.</td></tr>
<tr><td rowspan="2">Level 1</td><td rowspan="2">Pre-formal Proof</td><td rowspan="2">Pre-formal Proof</td><td rowspan="2">Proof</td></tr>
<tr><td rowspan="2">Rigorous Proof</td><td rowspan="2">Formal Proof</td><td rowspan="2">Mathem. Argument</td><td rowspan="2">Deductive Proof</td></tr>
<tr><td>Level 2</td><td>Formal Dem.</td><td>Formal Dem.</td><td>Dem.</td></tr>
</table>

Other researchers who also consider the rigor dimension find more than three big groups. Miyazaki (2000) establishes six levels of proof in lower secondary school from inductive proof to an algebraic demonstration basing on three axis: contents of proof, representation of proof and students′ thinking (see table 2). We shall apply them to some geometric cases.

Table 2

<table>
<tr><td colspan="3"></td><td colspan="2">PROOFS</td></tr>
<tr><td colspan="3"></td><td>Inductive Reasoning</td><td>Deductive Reasoning</td></tr>
<tr><td rowspan="4">Students′ thinking</td><td rowspan="2">Concrete Operations</td><td>Other languages</td><td>C</td><td>B</td></tr>
<tr><td>Functional language of demonstration</td><td>D</td><td></td></tr>
<tr><td rowspan="2">Formal Operations</td><td>Other languages</td><td></td><td>B</td></tr>
<tr><td>Functional language of demonstration</td><td>D</td><td>A</td></tr>
</table>

The previous classifications lead us to think about the uses of proof in different contexts, particularly in mathematics education and in mathematics research. An explanation that can be considered as a proof in mathematics education, maybe not be a proof in mathematics research. And even if we focus on the field of mathematics education, we find differences depending on the educational level. These levels are going to determine the role of proof in teaching. The function of the proof shall be a dimension in our analysis. In this regard, we consider the classification proposed by Hanna (2000) (who presents a compilation from Bell (1976) and de Villiers (1996)).

She presents the following functions that she considers useful for thinking and doing research on the topic: *verification, explanation, systematization, discovery, communication, construction* of an empirical theory, *exploration* of the meaning of a definition or the consequences of an assumption, and *incorporation* of a well-known fact into a new framework and thus viewing it from a fresh perspective. (p.8) De Villiers (1996) criticizes the almost exclusive function of verification that traditionally has been given to proof. He argues that this is an important function but not the only one. We shall support this idea with our reflections.

We can consider other useful criteria to classify proofs. For instance, Ibañes & Ortega (1996) concentrate on the reasoning done and the technique used. They distinguish between methods, styles and modes (see table 3).

Table 3

Method	Syllogism
	Cases
	Reductio ad absurdum
	Complete induction
	Constructivism
	Analogy
	Duality
Style	Geometric
	Algebraic
	Of the coordinates
	Vectorial
	Mathematical Analysis
	Probabilistic
	Topological
Mode	Synthetic or direct
	Analytic or indirect

Teaching and learning of proof

We would like to start from two assumptions related to the teaching-learning procedure:

- We argue that proof is an important and useful process in mathematics teaching at all levels.
- However, teachers do not need to present all the proofs of a concrete theorem to their students.

We find studies on proof according to educational level: involving students, from primary school to university levels; and involving teachers, ranging from prospective mathematics teachers to practicing teachers. Inductive reasoning is the most common type of proof used with the young students. Based on this idea and trying to develop deductive reasoning in some elementary school pupils, Lobo-Mesquita (1996) highlights the role of visualization in the learning of geometry. Some studies show how proof is an obstacle in the educational process. Ibañes (2001) shows that pupils in the third course of high school have problems with proof outlines, recognition of procedures, and the use of some expressions. Almeida (1996) suggests that, at the university level, it may be desirable to use semi-formal or preformal proofs as a bridge to formal proofs. Rico, González and Segovia (1999) study different heuristics used by pre-service mathematics teachers trying to prove a geometric proposition.

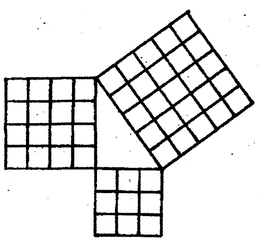
Proofs and the implied process is also a problem for teachers, who do not have a method to teach proof and sometimes don't know about the convenience of teaching a concrete proof (Cañadas, Nieto & Pizarro, 2001). Garnica (1996) analyzes answers given by mathematics teachers in terms of the modes of reading: technical and critical. Van Ash (1993; mentioned by Ibañes, 2001, p. 18) gives some ideas to help teachers to decide when to do a proof in class: to convince when we are not sure about a statement, to memorize a theorem, to learn useful algorithms, to end a search process, to expose a work method and to show meanings of definitions.

So proof seems to be an obstacle in the teaching-learning procedure. We are going to use the results of the mentioned studies to suggest ways to help both students and teachers in their daily task. In what follows, we try to connect the ideas related to proof mentioned before with a concrete case: the proof of the Pythagorean proposition.

The case of the Pythagorean proposition

The Pythagorean proposition establishes a relationship in a right-angle triangle: *the square of the hypotenuse is the sum of the squares of the legs*. It has hundreds of different proofs. Scott (1972) classifies many of them in four groups: 1. those based upon linear relations; 2. those based upon comparison of areas; 3. those based upon vector operation; and 4. those based upon mass and velocity. We will focus on the geometrical ones due to their didactical interest and their accessibility to students of different levels. The geometrical formulation of the proposition is as follows: *the square described upon the hypotenuse of a right-angled triangle is equal to the sum of the squares described upon the other two sides*. The common objective of all the proofs is to *verify* this proposition. We shall present four geometrical proofs of the Pythagorean proposition and shall comment on the dimensions we have mentioned: rigor, contents, representation, students′ thinking, method, style and mode.

(1)



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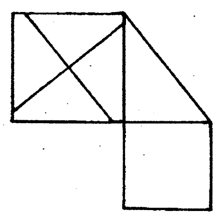

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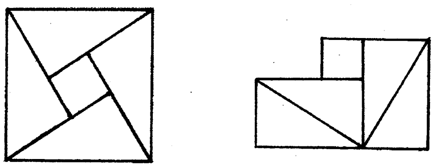

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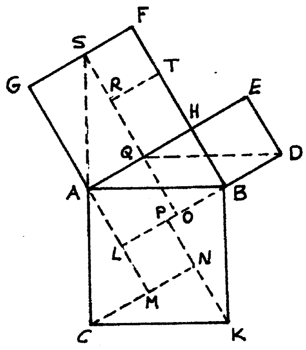


These proofs make evident the importance of visualization in geometry. Furthermore, they allow to *communicate* mathematical knowledge, to *construct* theory basing on empirical actions, to *explore* the meaning of the proposition and to *incorporate* a new knowledge into a framework.

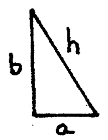
According to Ibañes & Ortega's classification, the proofs taken are direct proofs by cases, having a geometric style (although as we will see some of them involved other styles).

Focussing on rigor dimension (see table 1), most authors who consider two levels would situate these proofs as informal ones because they have been done with drawings and some of them do not need analytical reasoning. On the other hand, authors who consider three levels would situate proofs (2), (3) and (4) in the second rigor level because they are not the formal proofs we are used to in pure mathematics, but they are not just examples neither; and proof (1) would be classified in the lowest rigor level because is a concrete case that can be used as an example or an explanation.

We shall differentiate each proof from the others, pointing out differences and common points. Let us begin with proofs (1), (2) and (3), which follow the same strategy: to build the square upon the hypotenuse with the pieces given in the other squares.

In proofs (1) and (2) students just need to know the surface area concept. We have to avoid presenting the situation as a game and try to give meaning to the change of pieces. So we think that proofs (1) and (2) would be appropriate for primary school pupils, whose inductive reasoning is predominant, and for whom visualization and manipulation processes are very important. Let us remember that at this age, students' reasoning is mainly based in concrete and real objects. With the Ibañes & Ortega's classification, both proofs are purely geometric. We notice that there are differences between them. In proof (1), the pieces have the same area and the same shape, and they are situated in the same position regarding the big square. The proof follows easily from the comparison of the number of unit squares in the big squares. It consists of inductive reasoning and it is represented with numerals. So it is a "proof C" according to Miyazaki (2000) (see table 2). In proof (2), the pieces have different shapes with different positions as a consequence of having drawn parallel and perpendicular lines. This implies that students have to understand the meaning of the Pythagorean proposition instead of just solving a puzzle (Cañadas, 2001). According to contents and representation is a "proof B", according to Miyazaki (2000)) with concrete operations because it consists of consecutive actions on concrete objects and it includes deductive reasoning supporting these actions. So proof (1) could be used as an introduction to the proposition, inducing the *discovery* of the theorem. Proof (2) could be useful as an exercise of reasoning in order for pupils to show their understanding.

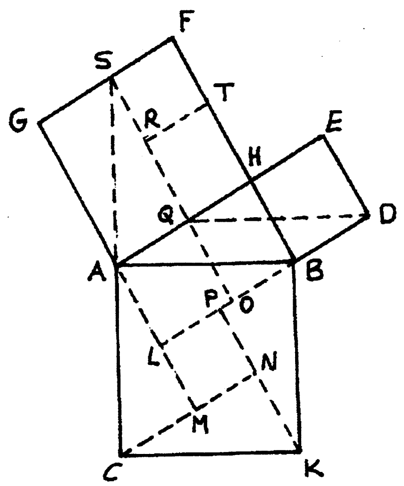
Proof (3) does not correspond to the geometrical formulation because we cannot visualize the right-angle triangle where we must build the three triangles to whose areas the proposition relates. In a Piagetian sense, this proof style follows "action strategies". This proof could be clearly situated in the second of the three rigor levels mentioned before. To finish the explanation process, we would have to draw and write as follows:



And comparing the areas of the two figures showed, we have that $h^2=a^2+b^2$.

It has an algebraic reasoning added to the general geometric position aforementioned. This is the characteristic that could differentiate "proof B with concrete operations" from "proof B with formal operations" (Miyazaki, 2000) in geometric proofs. So we think proof (3) is of the second kind. It is mainly the implicated reasoning with algebraic terms what allow to conclude that we should work with this proof at secondary level.

Proof (4) implies the highest level. The following reasoning is necessary to finish the proof:



Produce CA to S, draw SO parallel to FB, take HT = HB, draw TR parallel to HA.
Produce GA to M, making AM =GA. Produce DB to L, draw KP and CN parallel resp. to BH and AH. Draw QD.
Rectangle RH = rectangle UB.
Then: **Square AK** = triang. CKN (= triang. ASG) + triang. KBP (=triangle SAQ) + triangle BAL (=triangle DQO) + triang. ACM (=triang. QDE) + sq. LN (=sq. ST) = rectang. GQ + rectang. OE + sq. ST = rectang. GQ + sq. EB + rectang. QB + sq. ST = rectang. GQ + sq. EB + rectang. RH + sq. ST = **square BE + square GH.**

According to Ibañes & Ortega's classification, it is a pure geometric proof. Lines drawing and reasoning bear concepts like parallelism, perpendicularity, similarity and congruence. We add *systematization* of some results into a deductive system to the mentioned functions for the other proofs. According to Miyazaki (2000), proof (4) could be classified as "proof A" because it represents the most advanced level in a geometric proof. It involves deductive reasoning with a functional language of demonstration. So the appropriate level for this proof would be the end of secondary school or even the university level. Students at these levels are better used to abstract concepts and their deductive reasoning is more developed.

Discussion

We have shown how we can use different proofs of the Pythagorean proposition at different educational levels. For this task, we have used different dimensions drawn from the literature. These dimensions are: rigor, contents, representation, students' thinking, method, style and mode. This kind of analysis provides us with knowledge about which proof is more suitable for each educational level, keeping in mind students' characteristics in order to obtain better results in teaching and learning. The knowledge of the different proofs from this point of view help teachers to decide why (function), when (students' thinking, content and style) and how (representation) to do a proof in class. In this sense, teachers could be better informed about the convenience of teaching a concrete proof. We feel that this analytical process (that we have put into practice for the Pythagorean theorem) can be used for other propositions for which presenting a proof might be relevant.

References

Almeida, D. (1996). Proof in Undergraduate Mathematics in the UD: A case of Bridging from the Informal to Formal? In de Villiers M., Furinghetti, F. (Eds.). *Proceedings of the 8th International Congress on Mathematical Education*, Spain, 86-93.

van Asch, A. G. (1993). To proof, why and how? *International Journal Mathematics Education Science and Technology*, *2*, 301-313.

Balacheff, N. (2000) *Procesos de prueba en alumnos de matemáticas*. Bogotá: una empresa docente.

Blum, W. & Kirsch, A. (1991). Preformal proving: examples and reflections. *Educational Studies in Mathematics, 22*, 183-203.

Cañadas, M. C. (2001). Demostraciones del Teorema de Pitágoras para todos. In Cardeñoso, J.M., Moreno, A.J., Navas, J. M., Ruiz, F. (Eds.). *Actas de las jornadas Investigación en el aula de matemáticas: Atención a la diversidad*, España, 111-116.

Cañadas, M.C., Nieto, E. & Pizarro, A. (2001). El valor de la demostración en la educación secundaria. In Berenguer, J., Cobo, B., Navas, J.M. (Eds.). *Actas de las jornadas Investigación en el aula de matemáticas: retos de la educación matemática del siglo XXI*, España, 167-172.

van Dormolen, J. (1977). Learning to understand what giving a proof really means. *Educational Studies in Mathematics*, *38*, 85-109.

Garnica, A. (1996). Fascination for the technical, decline of the critical: a study on the rigorous proof in the training of Mathematics teacher. In de Villiers M., Furinghetti, F. (Eds.). *Proceedings of the 8th International Congress on Mathematical Education*, Spain, 257-280.

Gutiérrez, A. (2001). Estrategias de investigación cuando los marcos teóricos existentes no son útiles. In Moreno, M.F., Gil, F., Socas, M., Godino, J.D. (Eds.). *Actas del V Simposio de la SEIEM*, Spain.

Hanna, G. (2000). Proof, explanation and exploration. *Educational Studies in Mathematics*, *4*, 5-23.

Ibañes, M. & Ortega, T. (1996). Mathematical Proof: Classification and Examples for Use in Secondary Education. In de Villiers M., Furinghetti, F. (Eds.). *Proceedings of the 8th International Congress on Mathematical Education*, España, 14-21.

Ibañes, M. (2001). *Aspectos cognitivos del aprendizaje de la demostración matemática en alumnos de primer curso de bachillerato*. Tesis doctoral, Universidad de Valladolid. (España).

Ibañes, M. y Ortega, T. (2001). Un estudio sobre los esquemas de prueba en el alumnado de primer curso de bachillerato. *Uno. Revista de Didáctica de las matemáticas*, *28*, 39-59.

Jaffe, A. & Quinn, F. (1993). "Theoretical mathematics": Towards a cultural synthesis of mathematics and theoretical physics. *Bulletin of the American Mathematical Society*, *29* (*1*), 1-13.

Lobo-Mesquita, A. (1996). Deductive reasoning in Elementary School. In M. de Villiers and F. Furinghetti (Eds.). *Proceedings of the 8th International Congress on Mathematical Education*, Spain, 221-225.

Martínez, A. (2000).*Una aproximación epistemológica a la enseñanza y el aprendizaje de la demostración matemática*. Tesis doctoral, Universidad de Granada. (España).

Miyazaki, M. (2000). Levels of proof in lower secondary school mathematics. *Educational Studies in Mathematics*, *41*, 47-68.

Movshovitz-Hadar, N. (1996). On striking a balance between formal and informal proofs. In de Villiers M., Furinghetti, F. (Eds.). *Proceedings of the 8th International Congress on Mathematical Education*, Spain, 43-52.

Scott, E. (1972). *The Pythagorean Proposition*. California: NCTM Classics.

Rico, L., González, M. J. & Segovia, I. (1999). Estrategias en la resolución de problemas geométricos. *La Gaceta de la Real Sociedad Española. 3 (2)*, 519-532.

de Villiers, M. (1996). Why proof in dynamic geometry? In de Villiers M., Furinghetti, F. (Eds.). *Proceedings of the 8th International Congress on Mathematical Education*, Spain, 28-42.