CUTE DRAWINGS? THE DISCONNECT BETWEEN STUDENTS' PICTORIAL REPRESENTATIONS AND THEIR MATHEMATICS RESPONSES TO FRACTION QUESTIONS

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ABSTRACT

Third and fourth grade students' responses to open-ended questions requiring the modeling of fraction concepts were examined in order to determine the types and prevalence of difficulties students exhibit using pictorial representations in the problem-solving process. When developing pictorial representations, students experienced difficulties with model selection, partitioning, and comparison. Four specific difficulties students experienced in using pictorial representations to solve problems were: not answering the problem goal, incorrect model selection, failure to overcome whole number bias, and struggles with part-whole understanding.

Keywords: Fractions, Modeling, Representations, Elementary School.

RESUMO

As respostas dos estudantes da terceira e quarta classe as questões abertas requerem a modelagem dos conceitos de fração que foram examinadas a fim de determinar os tipos e a prevalência de dificuldades que os estudantes apresentam ao usar representações pictóricas no processo de resolução de problemas. Ao desenvolver representações pictóricas, os alunos experimentaram dificuldades com seleção, partição e comparação do modelo. Foram quatro dificuldades específicas que os estudantes experimentaram na utilização de representações pictóricas para resolver problemas, são elas: não responder o objetivo problema, seleção incorreta do modelo, falha em compreender o número inteiro, e resistência em compreender a parte-todo.

Palavras-chave: Frações, Modelagem, Representações, Escola elementar.

1. Introduction

Understanding fractions is the foundation for comprehending ratios, proportions, percents, and decimals. Both the National Mathematics Advisory Panel and the National Council of Teachers of Mathematics (NCTM) suggest that U. S. curriculum should provide in-depth coverage of rational numbers from fourth through eighth grades (Common Core State Standards Initiative, 2010; NCTM, 2006). One explanation for students' difficulties with fractions is their lack of visualization skills and their inability to use those visualization skills to create and interpret various fraction representations (Arcavi, 2003). As students learn to develop their own representations and learn to use the representations as problem solving tools, students develop a deeper understanding of fractions (Siegler et al., 2010). Although it has become more common for fraction instruction to use pictorial representations, the depth and breadth of their use varies greatly from classroom to classroom (Abrams, 2001).

At the simplest level, pictorial representations present the simplification of a mathematics concept into a single image (Ng & Lee, 2009). In this way, the representations themselves become the image students hold for specific symbols and terms, making the symbolic terms more concrete to the learner. However, pictorial representations "have often been taught and learned as if they were ends in themselves" (p. 67), and students do not receive guidance on how to develop and use the representation to aid them in problem solving (NCTM, 2000). Without this guidance, many students are not able to create representations that they can then use to accurately solve a problem. This creates a disconnect between the representations used in instruction and the students' ability to develop and use representations to provide accurate mathematical responses.

The purpose of this research was to examine the nature of the disconnect between students' development of representations and students' ability to provide accurate mathematical responses based on those representations in classrooms where teachers were using the part-whole approach to teaching fractions. We were particularly interested in whether students could create an accurate representation, whether students could use the representation to determine an accurate mathematical response, and what the representations revealed about students' thinking.

2. Mathematical Representations

According to the National Council of Teachers of Mathematics (NCTM), "The ways in which mathematical ideas are represented are fundamental to how people can understand and use those ideas" (2000, p. 67). Extensive research on fractions has revealed five sub-constructs of rational number knowledge: part-whole relations, ratios, quotients, measures, and operations (Kieren, 1980), and three partitioning schemes have been identified (Lamon, 1996): (1) halving – an early developed partitioning action (Pothier & Sawada, 1983), (2) dealing – a primitive form of partitioning which generates equal shares by distributing in a cyclic fashion until all shares are given out (Davis & Pitkethly, 1990), and (3) folding or splitting – where the number of pieces grow with the number of folds (Confrey, 1998; Kieren, Mason & Pirie, 1992). While some researchers (e g., Fuchs et al., 2013; Lamon, 2005; Torbeyns, Schneider, Xin & Siegler, 2014) have questioned the use of the part-whole sub-construct for developing understanding of fraction concepts, Charalambous and Pitta-Pantazi (2007) suggest that the part-whole sub-construct is necessary as a foundation for understanding the other sub-constructs. In this paper we provide evidence that supports researchers' concerns about the utility of the part-whole meaning of fractions as a foundation for developing fraction conceptions.

For many children, formal education concerning rational numbers begins with the use of concrete fraction pie or fraction square manipulatives and the drawing or "shading in" of part-whole representations. These representations play several functions in students' development of mathematical ideas. First, learning to represent fraction concepts in pictorial representations encourages students to mentally simplify the concept into a single image. Pictorial representations become placeholders for thoughts, allowing students to mentally work on one part of the model without being overwhelmed with the task of mentally trying to hold the whole picture in their minds (Woleck, 2001). The representations also become connecters for retrieving concepts from memory. When hearing or seeing the symbol 3/4, many students will instantly picture a square with three parts shaded in. This representation holds the part-whole meaning of the fraction 3/4. By supporting students' development of these visualizations teachers encourage students to make meaningful connections among different types of representations and to develop abstractions of mathematical concepts (van Garderen, 2006). Finally, representations become the tools that the students use to articulate, clarify, justify, and communicate their mathematical solutions.

2.1 Development and Use of Pictorial Representations

Students' representations are not static, but dynamic, and their development often follows a cyclical process. As students internally attach more elaborate meanings to a representation, they often simplify their external representations. In this way, the external representations become metaphors for complex mathematical understandings (Abrams, 2001; Kiczek, Maher, & Speiser, 2001; Woleck, 2001).

As students begin to manipulate pictorial representations, they experiment with the effects of their manipulations and modifications, thereby developing new understandings. This knowledge can then be applied to solving problems as students learn to select appropriate model types, develop their pictorial representations, and then use the representations to develop solution strategies (Ng & Lee, 2009). In a study investigating the problem-solving methods of proficient problem solvers, Larkin, McDermott, Simon, and Simon (1980) observed that proficient solvers tended to develop a complex representation such as a picture, diagram or table in order to solve problems. The pictorial representations became the organizational tool, which the solvers used to record and plan solution strategies (Whitin & Whitin, 2001). However, the effective use of pictorial representations for problem solving does not occur spontaneously, and students require guidance in their selection, manipulation, and interpretation of representations (Abrams, 2001).

2.2. Students' Difficulties in Drawing and Using Fractional Representations

Goldin and Shteingold (2001) explain that there are internal and external representations. Internal representations are not observable, but can be inferred from a students' development of external representations. As students interact with representations in their environment, they develop and change their internal representations. Conceptual understanding is deepened as students develop flexibility within and between their internal representations (Goldin & Shteingold, 2001). To develop conceptual understanding requires more than the student observing representations. Understanding requires that the student uses new information to adapt their own internal representations and, in this way, the representations developed by students become a unique reflection of a student's understanding (Lamon, 2001). In this manner, students' external representations become a mirror of their conceptual understanding.

The form of external representations examined in this study were students' pictorial drawings of fraction concepts. A review of the literature identified some of the difficulties students have when developing fraction representations. One frequently discussed difficulty is partitioning. Partitioning is the sectioning of representations into equal shares (Lamon, 1996). Pothier and Sawada (1983) identify five stages in the development of students' partitioning skills: 1) partition into halves; 2) half the halves; 3) partition into even numbers; 4) partition into thirds, fifths, sevenths, etc; and, 5) partition into products of two odd numbers (ninths, fifteenths). Often children experience a disconnect between visualizing partitions and drawing those partitions. For example, they can visualize 13/15, but they struggle with drawing equal partitions (Smith, 2002).

Research also reveals the difficulty many students have in seeing and drawing fractions nested inside equal-sized partitions or, identification of parts and wholes. It is not until fourth grade that most students are able to visualize the nested equal-sized partitions and coordinate them with the concept of the whole (Grobecker, 2000). For example, students who do not have conservation of the whole will not see in a pie model that three-twelfths and one-fourth describe the same amount (see Figure 1). Instead, they focus only on the one-fourth and erroneously think of the three partitioned sections as three-thirds (Kamii & Clark, 1995).

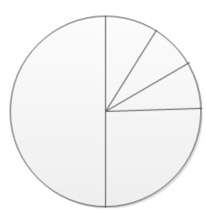


Figure 1. Pictorial representation for ¼ and 3/12.

Steinle and Price (2008) interviewed 41 school students enrolled in Years 3 to 10 on representations of ¾. Given a choice, the majority of students selected to draw circle representations. Although 80% of the students drew correct representations of ¾, subsequent responses to questions revealed that less than 25% knew that a circle divided into four unequal parts did not represent ¾. Only 66% of the students were able to correctly place ¾ on a number line which ranged from 0 to 1 and only 46% correctly placed ¾ on a number line which ranged from 1 to 4. The authors concluded that the students were not viewing fractions as numbers and suggested that teachers need to draw students' attention to the relative number (set), length (measurement), or area (region) of each type of representation used.

Revee and Pattison (1996) analyzed the drawings of 250 seventh- through ninth-grade students. Although instruction frequently used number line representations, over 95% of the students solved the assessment questions using only part-whole models and many of the representations suggested that the students were interpreting fractions as two quantities and not as a relationship of the parts to the whole. They found a positive correlation between the accuracy of students' representations and their problem solving ability.

The Ng and Lee (2009) study focused on the accuracy of students' representations. Eighteen mathematics educators identified three points of difficulty students had in developing pictorial representations: 1) lack of attention to accuracy in drawing, 2) not checking to see if all information needed is included in the representation, and 3) inconsistent use of the lines and shapes to organize the pictorial structure. A final difficulty identified by Ng and Lee (2009) is the failure of students to keep the problem-solving goal in mind. Even though some students were able to develop and draw correct pictorial representations, they failed to answer the problems correctly because they did not keep in mind the problem-solving goal for which the representation was constructed.

Bulgar (2009) tracked the changes in the representations of 13 fifth-grade students as they learned division of fractions concepts. She reported that the pictorial representations aided the teachers in understanding students' development of fraction division concepts and guided their selection of methods and tasks.

3. Methodology

3.1. Research Questions

As the research shows, students experience a variety of difficulties when using representations to learn the part-whole sub-construct of fraction concepts. Prior studies point to the disconnect between students' use of pictorial representations and students' ability to provide accurate mathematical responses based on those representations. The first two research questions in this study examined students' development of pictorial representations in classrooms where teachers focused their fraction instruction on the use of the part-whole sub-construct:

1. What portion of third- and fourth-grade students are able to create an accurate pictorial representation for a fraction task?

2. What difficulties do students have in developing accurate pictorial representations?

The third and fourth research questions sought to examine the disconnect between students' use of the pictorial representations and their ability to provide correct responses to the questions asked:

3. What portion of third- and fourth- grade students are able to use a student-generated pictorial representation to provide an accurate mathematical response for a fraction task?

4. What do students' pictorial representations reveal about the disconnect between the pictorial representations and the correct responses?

3.2. Methods and Procedures

This study was part of a larger research project in which third- and fourth-grade students from 17 classrooms participated in fraction instruction using text-based materials, physical manipulatives and virtual manipulatives. The results from the larger study indicated that there were no significant differences in overall achievement between the treatments using different manipulative types, based on pre-tests, post-tests, and delayed post-tests (Moyer-Packenham, Baker et al., 2013). However, learning and retention effects for students of different socio-economic status were equalized for students participating in the virtual manipulative treatment groups (Moyer-Packenham, Jordan et al., 2013). For more details on the larger research project see these two publications: Moyer-Packenham, Baker et al., 2013 and Moyer-Packenham, Jordan et al., 2013. In the current study, we focused only on students' use of writings and drawings as representations on open-response questions that appeared on the pre-tests, post-tests, and delayed post-tests in the larger study.

3.3. Participants and Setting

The 371 students in this study came from 17 classes in eight schools in two rural school districts. This included 162 third-grade and 209 fourth-grade public school students. Forty-two percent of the students participating were of lower socio-economic status (identified by qualification for schools' free/reduced lunch programs), and 54% were female. The number of participants for this study was slightly higher than the number in the larger study because the present study used data from students who had completed only the open-ended response items on the tests. Whereas, in the larger study, participant data were only analyzed for those students with complete data sets (i.e., the students had completed every test needed to conduct the statistical analyses).

3.4. Procedures

At the beginning of the study, students completed a pre-test that contained open-ended test items constructed for this study based on four standardized test databases: National Assessment of Educational Progress (NAEP), Massachusetts Comprehensive Assessment System (MCAS), Utah Test Item Pool Server (UTIPS), and Virginia Standards of Learning (Virginia SOL). At the end of the study, students completed a post-test immediately following instruction in the fraction units. Six weeks following instruction, students completed a delayed post-test which was used to assess retention. There were two open-response items on the post-test, and two open-response items on the delayed post-test. These tests were administered by the students' classroom teachers.

Between the pre- and post-testing, students participated in 45 minute instructional sessions focused on fraction content for nine to seventeen days (average 10.8 days), with the length of each unit determined by their classroom teachers. Topics for instruction included: understanding equal parts; understanding and using region, set, and number line models; naming and writing fractions; comparing and ordering fractions; and understanding equivalent fractions. Fourth-grade students also received instruction in the addition and subtraction of fractions. The teachers in the study focused most of their fraction instruction on the use of the part-whole model when teaching each of the fraction topics in the unit. The part-whole model was prevalent in their instructional materials and resources and therefore the teachers relied on the part-whole sub-construct as the focus of their fraction instruction.

The instructional sessions were taught by 14 public school teachers and 4 university researchers. The average number of years of elementary (K-6) teaching experience of the instructors was approximately 16 years. Mathematics instruction emphasized the use of a variety of pictorial representations. Researchers observed over 70% of the classroom instructional sessions. These observations indicated that pictorial representations were used by the participating students, individually and in a whole group setting, between 22 to 50 percent of the class time during the lessons. The pictorial models included virtual manipulatives, smart board technology, drawings and text based pictorial representations.

3.5. Data Analysis

The data analyzed for this paper focus on students' responses to five different open-response questions on the post or the delayed post-test (two questions from the third grade tests and three questions from the fourth grade tests). The two questions on the third grade test are labeled the Candy Bar and the Candy Cane questions. The three questions on the fourth grade test are labeled the Comparison, the Fraction Strings, and the Pizza questions. Two variations of the Pizza question were used on the post-test and delayed post-test.

Researchers developed a scoring rubric using an iterative process to assess students' performance on the open ended test items (Miles & Huberman, 1994). The rubric focused on determining the accuracy of students' pictorial representations for each problem and students' use of their pictorial representations to solve the fraction problem. Pairs of coders conducted the analyses for the open-response questions to ensure inter-rater reliability. During the first phase of the analysis, coders read through 10% of students' responses. Next, coders identified major categories of solutions and incorrect pictorial representations in student responses and used these to develop a scoring rubric. In phase two, this thematic rubric was used by the coders to independently score all of the student responses. Coders examined students' responses for types of correct and incorrect pictorial representations. After categories were developed for each question, the researchers created codes based on variations in students' representations and their strategies for using their representations to solve fraction problems. When there was not a consensus in the coding of a student's response, a discussion occurred to reach a consensus decision on the coding category. Finally, coders summarized and analyzed the results, focusing on trends in students' development and use of pictorial representations. Tables and graphs were used as tools to summarize trends.

4. Results

In the results that follow, each of the five open-ended questions are reported in three parts. The first part contains a short description of the problem and a table showing the distribution of students' responses. The second part, titled "Representation Development," contains an analysis related to the first two research questions which focus on the portion of students able to create an accurate pictorial representation and the difficulties students experienced when developing their representations. The third part, titled: “Representation Use”, contains an analysis related to the third and fourth research questions which focus on the portion of students who correctly used their representations to provide accurate responses and the disconnects between students’ representations and their responses.

4.1. Candy Bar Question: Grade 3

> *Jake broke a chocolate bar into four equal pieces and ate one piece. What fraction of the original chocolate bar is left? Explain using a picture that your answer is correct.*

The Candy Bar question asked third-grade students to develop a representation showing the fractional amount of candy bars remaining after an operation. In response, 17.9% of students drew incorrect or no models, 55.5% drew correct representations but incorrectly answered the question, and 26.2% drew correct representations and correctly answered the question (Table 1).

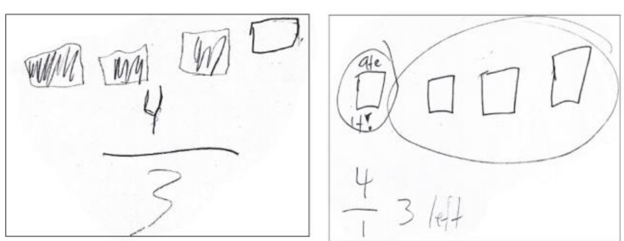
Table 1
Distribution of Post-Test Responses for Third Grade Candy Bar Question.

Response	Number of Participants	Percent
Incorrect drawing	29	17.9
Correct drawing and incorrect answer	90	55.5
Correct drawing and correct answer (3/4)	43	26.2

N=162

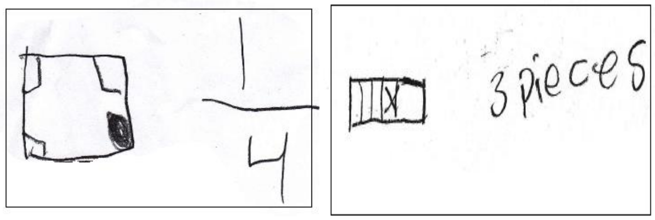
4.1.1. Representation development

On the post-test, 17.9% of the students did not draw a correct pictorial representation. The two most common difficulties students had when developing the pictorial representations were: not selecting the appropriate type of model, and not partitioning the drawn representations into four equal parts. Even though the problem was about a candy bar, a number of the students drew set models (individual pieces of candy) rather than region models. Although some children were able to use the set models to obtain a correct answer, the majority were not able to retain the part-whole concept and gave answers such as four-ones (four pieces with one eaten) or four-thirds (four pieces with three left). (See Figure 2.)



2a 2b
Figure 2. Difficulties of using set models.

Two types of difficulties were observed in students’ partitioning of region models into equal parts. Some students partitioned small parts of the whole and left large portions of the model not partitioned (see Figure 3a), and some students used four lines (the number of parts) to divide the region, mistakenly creating five parts (see Figure 3b). On the Candy Bar question, two difficulties limited students' correct development of a pictorial representation: incorrect model selection and incorrect partitioning.

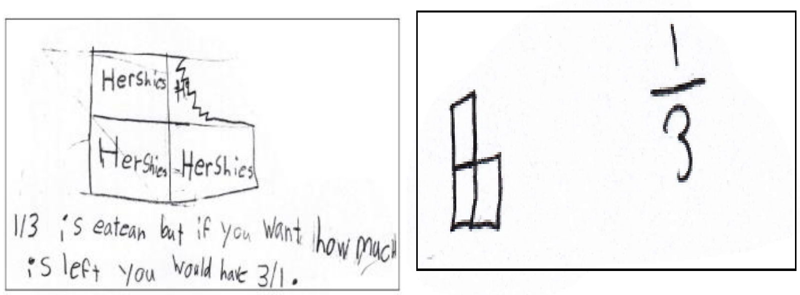


3a 3b

Figure 3. Examples of students' difficulties in region model partitioning.

4.1.2. Representation use

Although 82% of the students on the post-test developed a correct drawing, only 26.5% correctly answered the question. There were three types of incorrect answers: students did not respond to the yes-no question (8.6%); students incorrectly answered "¼" (39.5%); or, students answered 1/3, 3/1 or 3/3 (7.4%). The first two responses reflect a failure to retain the problem-solving goal. The responses of 1/3, 3/1, or 3/3 suggest that students did not conserve the image of the whole of the candy bar when answering the question. Their representations were typically region models that implied a whole partitioned into four parts, but the students had not outlined the part eaten and did not retain the image of the whole partitioned into four parts (see Figure 4a and 4b).



4a 4b

Figure 4. Examples of students' drawings with implied, but not outlined, wholes.

4.2. Candy Cane Question: Grade 3

Sally has 10 candy canes. Two-fifths of the candy canes are red while the others are white. How many of the candy canes are red? Draw a picture and explain your answer.

The Candy Cane question asked third-grade students to represent fractions of sets. In response, 21% of students did not draw a model or drew incorrect models, 60.7% drew correct representations of the two-fifths candy canes but answered the question incorrectly, and 17.3% created a correct drawing and correctly answered the question (Table 2).

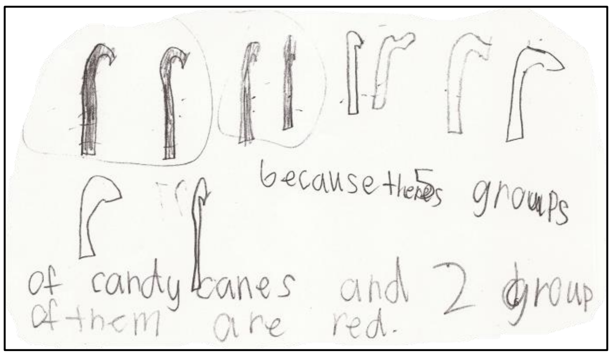
Table 2

Grade 3 Candy Cane Question Distribution of Post-Test Responses

Responses	Number of Participants	Percent
Incorrect drawing	34	21.0
Correct drawing-incorrect answer	100	60.7
Correct	28	17.3

N=162

Students who answered this question correctly modeled their answer in one of two ways. Some students drew five groups of two candy canes and colored two groups red (see Figure 5a). Other students drew two groups of five candy canes and colored two of the candy canes in each group red (see Figure 5b). Both strategies produced correct answers.



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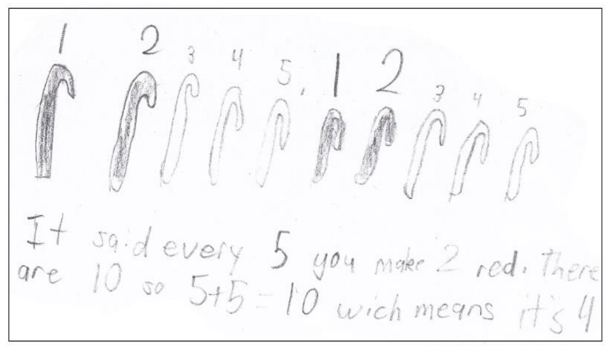


5b

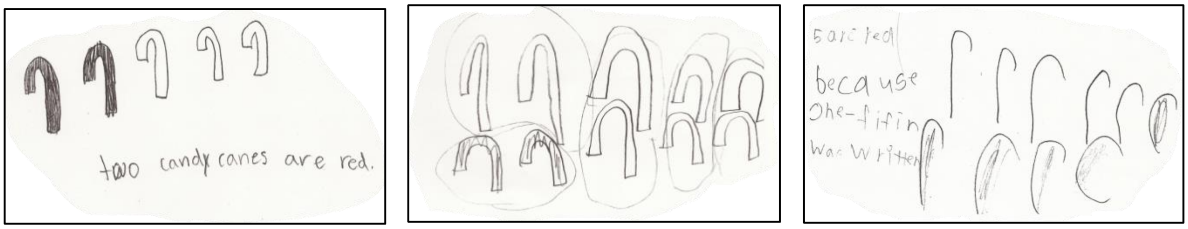
Figure 5. Students' correct responses to the candy cane question.

4.2.1. Representation development

A large number of students (21%) did not draw pictorial representations reflecting the beginning ratio of two-fifths. This included four response types: 1) candy canes with no indication of color; 2) more or less than either five or ten candy canes; 3) region models with each individual candy cane partitioned into five sections, with one section colored red; and, 4) no response. Thus, three difficulties limited students' correct development of pictorial representations: incomplete drawings, inaccurate drawings, and incorrect selection of representation type.

4.2.2. Representation use

Students drew pictorial representations reflecting fifths or tenths, but their representations revealed a lack of understanding of how to develop equivalent fractions when using set models. Three types of incorrect processes were identified: 1) 14.2% of the students did not expand the representation to reflect the ten candy canes and answered that two of five canes were red (see Figure 6a); 2) 32.7% of the students drew ten candy canes, but only colored two red, the numerator of the fraction in the original problem (see Figure 6b); and, 3) 14.8% of the students colored five of ten candy canes red (see Figure 6c). These difficulties centered around not understanding how to use multiplicative thinking for the set model.



6a 6b 6c

Figure 6. Examples of students' modeling difficulties in Candy Cane question.

4.3. Comparison Question: Grade 4

Mark says 2/3 of his candy is smaller than 3/4 of the same candy bar. *Is Mark right? Yes No* *Draw a picture to explain why you think Mark is right or wrong.*

The Comparison question asked fourth-grade students to draw a representation showing fractional amounts of two candy bars and to use the representations to determine which fraction was larger. In response, 37.3% did not draw a model or drew incorrect models, 23.9% drew correct representations but answered the question incorrectly, and 38.8% created a correct drawing and correctly answered the question (see Table 3).

Table 3

Grade 4 Comparison Question Distribution of Post-Test Responses

Response	Number of Participants	Percent
Incorrect Drawing	78	37.3
Correct Drawing/Incorrect Answer	50	23.9
Correct Drawing/Correct Answer	81	38.8

N=209

4.3.1. Incorrect representation development

Students had three types of difficulties in drawing correct representations: subtracting the whole numbers, comparing the whole numbers, and inaccurate pictorial representations (Table 4).

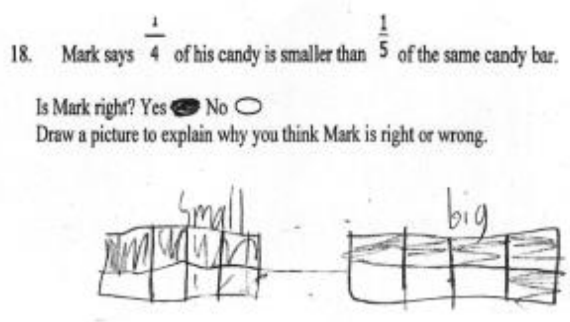
Table 4

Grade 4 Comparison Question Distribution of Incorrect Pictorial Representation Errors

Response	Number of Participants	Percent
Subtraction of Whole Numbers	4	5.1
Compared Whole Numbers	14	17.9
Inaccurate Pictorial Representations	60	76.9

Of the students who did not develop a correct model, 5.1% attempted to solve the problem by subtracting the whole numbers in the problem (e.g. 3/4 – 2/3 = 1/1, so 3/4 is not smaller than 2/3). Another group of students (17.9%) compared the numbers in the denominator only (e.g., 4 is greater than 3, so 3/4 is greater than 2/3). Students in these first two categories did not draw pictorial representations, but attempted to answer the question using symbols only. Students in the third category (76.9%) drew inaccurate pictorial representations.

The inaccurate pictorial representations were further examined and four difficulties were identified: a) students did not draw a representation, b) students partitioned incorrectly, c) students attempted to model both fractions on one model, and d) students selected inappropriate model types. Figures 7a and 7b demonstrate two types of partitioning difficulties. In Figure 7a, the student drew two partitioned candy bars and used the denominators to determine how many parts should be shaded. In Figure 7b, the student partitioned one candy bar by drawing one line for the value of each number in the two fractions (five lines for 2/3 and seven lines for ¾, for a total of 12 lines). Figure 7c shows a student's attempt to draw the comparison using only one representation. Students using this approach typically were able to accurately draw one representation, but were unable to show the magnitude of the second fraction on the same representation. As the student in Figure 7d wrote, "He (Mark) is wrong because, how can you add 5 more?"



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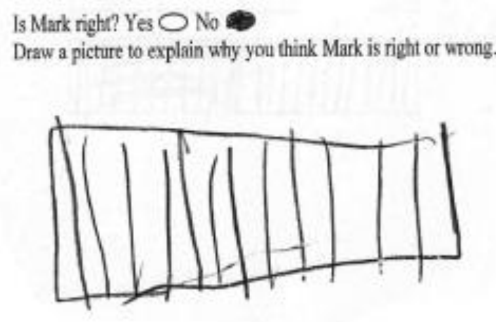

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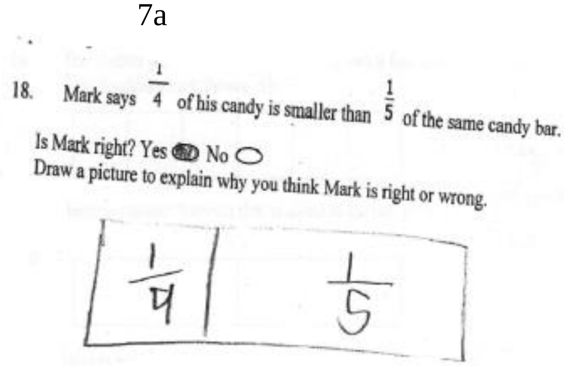

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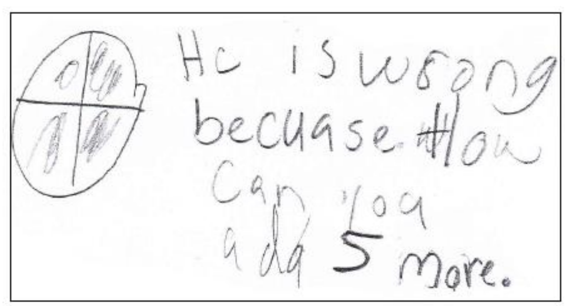


7d

Figure 7. Examples of pictorial representation comparison errors.

To better understand students' selection of model types, results of the pre-test were examined. Although a rectangle area model best reflects the context of this problem, students also used circle area models, set models, and measurement models. Table 5 correlates students' model selection with their success in answering the question correctly.

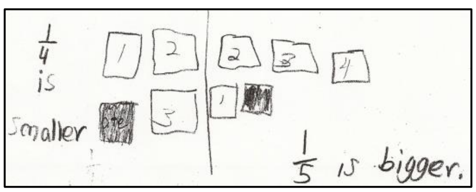
Table 5

Grade 4 Comparing Candy Bars Problem: Student Success Rates with Various Models

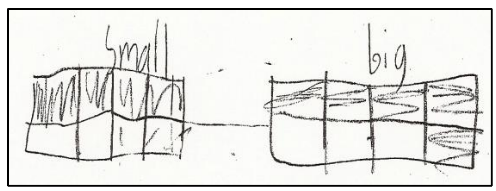
Model Type	N of all Students Using Model	Percent of all Students Using Model	N of Students Using Model Correctly	Percent of Students Using Model Correctly
Area – rectangle	29	47%	15	52%
Area – circle	11	18%	7	64%
Measurement	16	26%	12	75%
Set	6	10%	0	0%
Total	62		34	

N=62

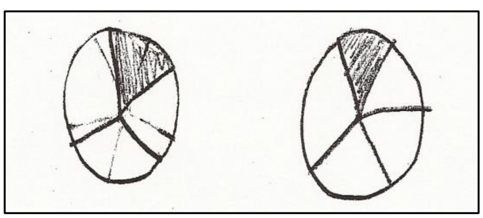
Of the area, set, and measurement models, the set model proved to be the least effective. None of the students who used this model answered the question correctly (Figure 8a). Although the rectangle area model (Figure 8b) is the shape of a candy bar, it was used successfully only 52% of the time. Similarly, only 64% of the students using a circle area model (Figure 8c) produced a correct answer. Many students had difficulties drawing the area models. In some cases, students tried to draw an area model, erased it, and then changed to a set model. The measurement model proved to be the most effective (75% correct). This higher success rate may suggest that a measurement model is easier to use when making comparisons between fractions with different denominators (Figure 8d).



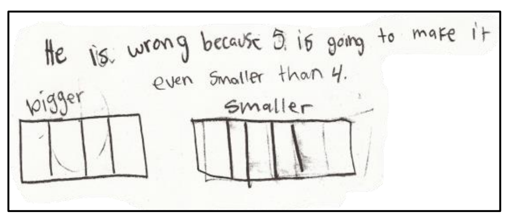
8a Set Model



8b Rectangle Area Model



8c Circle Area Model



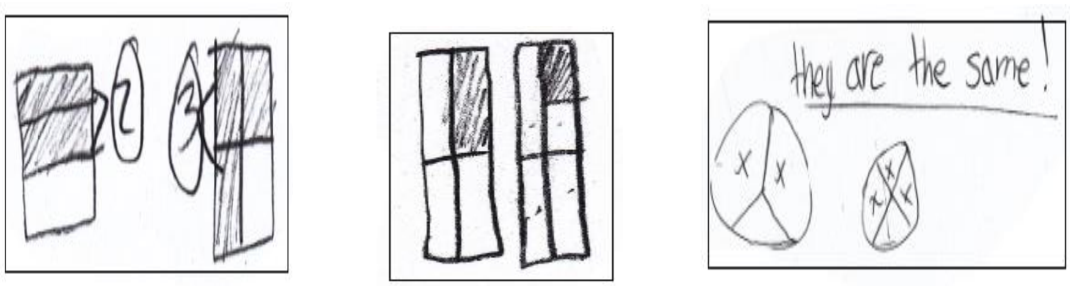
8d Measurement Model

Figure 8. Students' selection of pictorial representations for Comparison question.

Note: The student in example 8d understood that a larger denominator means smaller pieces. However, the student had difficulties producing precise drawings of the fifths. In this case the error in drawing did not influence the student's understanding.

4.3.2. Representation use

Three types of inaccuracies were identified, which may have made it difficult to use the representation to visually compare fractions. First, some students drew two different model types or partitioned their pictorial representations in two different directions, making it difficult to compare the representations visually (Figure 9a). Second, pictorial representations partitioned into unequal parts limited students' ability to make visual comparisons. The most common unequal partitioning occurred when students partitioned objects into fifths (see Figure 9b). Students drew a model for one-fourth and then halved one of the fourths to partition the model into fifths. Third, some students' drawings of pictorial representations differed significantly in magnitude, making it impossible to visually compare amounts (Figure 9c). Thus, although students drew models with the correct fraction ratio of shaded to unshaded partitions, their inconsistent use of model types, the magnitude of partitions and the magnitude of the whole limited their ability to use the models as tools of comparison.

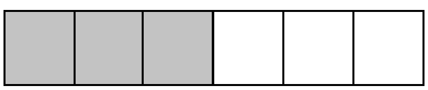


9a 9b 9c

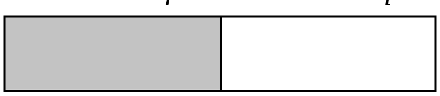
Figure 9. Inaccurate drawings for comparison question.

4.4. Fraction Strings Question: Grade 4

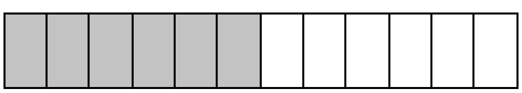
The shaded part of each string below shows a fraction.
This fraction string shows 3/6:

A. 

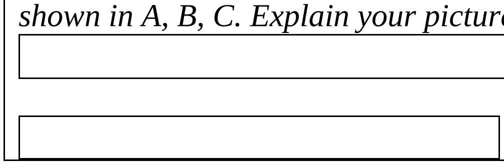
Here is another fraction that is equal to the one in A

B. 

Here is another fraction that is equal to one in A and B.

C. 

Shade in the fraction strings below to show two different fractions that are equal to the ones shown in A, B, C. Explain your picture.



The Fraction Strings question asked fourth-grade students to develop a representation showing two fractions equal to the target fraction. In response, 37.3% of students did not draw a model or drew incorrect models, 20.6% drew one string correctly and one string incorrectly, and 42.1% created two correct drawings (Table 6).

Table 6

Grade 4 Fraction Strings Problem: Distribution of Responses

Response	Number of Participants	Percent
Incorrect drawing	78	37.3
One string incorrect/One string correct	43	20.6
Correct	88	42.1

N =209

4.4.1. Representation development

Students who did not develop a correct representation made two types of errors: incorrect partitioning and rearranging parts. Students used incorrect partitioning when they drew the same number of lines as the denominator of the fraction (e.g., six lines for 1/6, thereby creating seven sections). The second type of error, rearranging parts, occurred when students drew three-sixths again, but with a different arrangement of shaded and non-shaded partitions so that the fraction parts were not placed together (Figure 10).

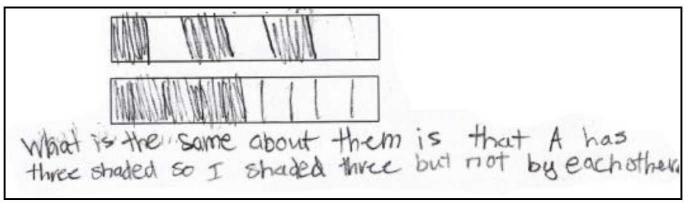
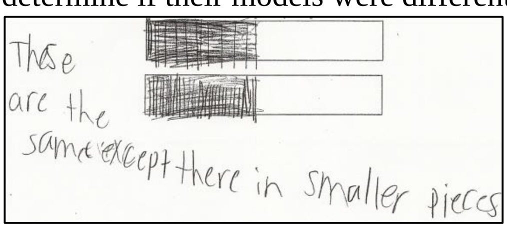


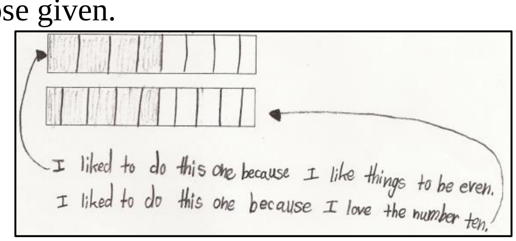
Figure 10. Student drawing of "Rearranged Fractions".

4.4.2. Representation use

Students almost exclusively used vertical partitions (rather than horizontal) to create their pictorial representations of new fractions, suggesting that most students probably first calculated the fractions and then drew the model based on their calculations (see Figure 11a and 11b). Thus, the model served as a reflection of other problem solving strategies, not as the primary means of solving the problem. In many cases, students gave solutions identical to the examples already given. Their models were correct, but they neglected to refer back to the original question to determine if their models were different from those given.



11a



11b

Figure 11. Fraction string problem: Student solutions.

Note: The student demonstrated understanding of equivalence by partitioning each string into two equal pieces. However, the model was not part of the student's problem solving process.

4.5. Pizza Question: Grade 4

Version 1: *José ate ½ of a pizza. Ella ate ½ of another pizza. José said that he ate more pizza than Ella, but Ella said they both ate the same amount. Use words and pictures to show that José could be right.*
Version 2: *A pizza is sliced into 10 equal pieces and Jose ate 4 slices of the pizza. Another pizza is sliced into 5 equal pieces and Ella ate 2 slices of the pizza. Jose said that he ate more pizza than Ella, but Ella said they both ate the same amount. Use words and pictures to show that Jose could be right.*

The Pizza question required fourth-grade students to consider the magnitude of the whole when comparing fractions. Two different versions of the pizza problem were presented to students on the post-tests. In response, 37.3% of students on Version 1 and 45% on Version 2 did not draw a model or drew incorrect models, 43.1% (Version 1) and 51.2% (Version 2) drew correct representations but answered the question incorrectly, and 19.6% (Version 1) and 3.8% (Version 2) created a correct drawing and correctly answered the question (Table 7).

Table 7
Grade 4 Pizza Question Post-Test Distribution of Responses

Response	**Version 1**		**Version 2**	
	Number	**Percent**	**Number**	**Percent**
Incorrect drawing	78	37.3	94	45.0
Correct drawing, Incorrect Answer	90	43.1	107	51.2
Correct	41	19.6	8	3.8

N=209

4.5.1. Representation development

The percent of students who did not draw correct representations was 37.3% on Version 1 and 45% on Version 2 of the question. Students exhibited two main representation development difficulties: drawing both amounts on one model and incorrect partitioning. Even though the questions stated that there were two different pizzas, 13.4% (Version 1) and 1.9% (Version 2) of students drew Jose's and Ella's portions on one pizza. This became a constraint on students' conception of the whole, limiting their ability to consider the possibility that the pizzas were of different magnitudes. One student wrote, "If someone takes one-half and another person takes one-half, then the pizzas gone" (see Figure 12).

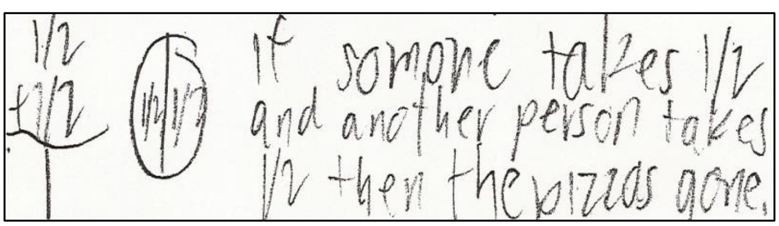
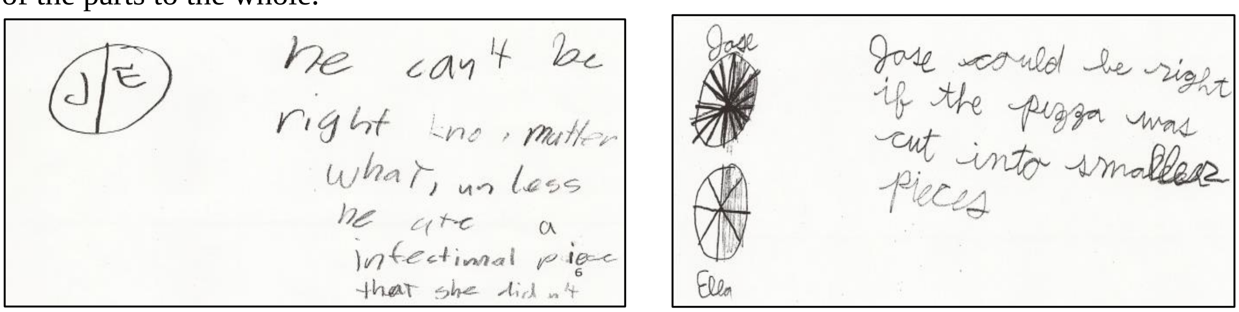


Figure 12. Drawing both portions of pizza on one model.

On the Version 2 question some students also had difficulty drawing pizzas divided into partitions of fifths and tenths, especially if they were attempting to draw both portions on one pizza. Some students partitioned the space into four parts and then partitioned one of the four parts in half to obtain five parts. This resulted in partitions of unequal magnitudes and led some students to incorrect answers.

4.5.2. Representation use

Three types of disconnects between the students' representations and the correct answers were identified. In all three, the students focused solely on the parts of the fractions and not on the wholes of the fractional relationships. In the first disconnect, 57.5% (Version 1) and 37.8% (Version 2) of students stated that Jose was wrong because the two fractions were equal. Their pictorial representations focused on showing that the two portions were equal, and therefore they concluded that Jose could not be right. In the second type of disconnect, 10% (Version 1) and 4.3% (Version 2) of students responded that Jose could have been correct if the pizza slices had been cut just a little bit unevenly. As one student wrote, "he ate a infentesmal piece that she didn't". They drew one piece just a little bit bigger or smaller than the other (see Figure 13a). These explanations suggest that students had not yet developed the concept that increasing the magnitude of one partition changes the ratio describing the part-whole relationship. They did not interpret the fraction as a relationship, but as two numerals describing amounts. In the third type of disconnect, 9.6% (Version 1) and 24.9% (Version 2) of students explained that if Jose's half were partitioned into smaller pieces, he would have eaten more pieces (Figure 13b). Their answers suggest that students focused on the number of the parts in the pizza and not on the proportional relationship of the parts to the whole.



13a 13b

Figure 13. Pizza problem responses.

4.6. Results Summary

To summarize students' ability to develop and use pictorial representations, the results were averaged across the five questions (Table 8). The average number of students who drew correct pictorial representations was 54.6%, and the average number of correct responses was 24.9%. These percentages reflect that almost half of the students had difficulty developing a pictorial representation. Even when students did develop an accurate pictorial representation, only 24.9% were able to use their representation to provide a correct response.

Table 8
Averaged Correct Responses to the Five Questions.

Question	Percent of Correct Representations	Percent of Correct Responses
Candy Bar (Grade 3)	67.3	27.8

Candy Cane (Grade 3)	17.3	17.3
Compare (Grade 4)	62.7	38.8
String (Grade 4)	62.7	42.1
Pizza Version 1 (Grade 4)	62.7	19.6
Pizza Version 2 (Grade 4)	55.0	3.8
Average	54.6	24.9

N=371

5. Discussion

The research questions in this study examined whether students could develop accurate pictorial representations for a mathematical problem situation and use their pictorial representations to provide accurate mathematical responses. As the results reveal, there is a large disconnect between students' pictorial representations and their abilities to use these representations to produce correct responses, even when they have created a correct pictorial representation. Students in this research study received daily exposure to a variety of pictorial representations for fraction concepts. Despite this exposure, a large number of students still had difficulties developing and using pictorial representations on open-ended response questions. Only 55% of students were able to develop correct pictorial representations, and only 25% provided correct answers to the open-ended questions. The results reveal weaknesses in students' abilities to develop representations of fractions and effectively use pictorial representations to solve problems. Synthesis of the findings suggests four main difficulties: 1) not answering the problem goal; 2) incorrect model selection; 3) failure to overcome whole number bias; and 4) struggles with part-whole understanding. (Results in this study showing difficulties with whole number bias and part-whole understanding were so closely intertwined that they will be discussed together.)

5.1. Not answering problem goal

In four different questions (Candy Bar, Candy Cane, Fraction Strings, and Pizza) a disconnect occurred because students did not check back to see if their solution answered the question. In the Candy Bar question, this disconnect caused half of the students who correctly drew a pictorial representation to answer the question incorrectly. These results are similar to findings by Ng and Lee (2009) in which students misinterpreted or failed to respond to all parts of the questions presented in algebra word problems. In the Candy Bar question, a disconnect between the representation and the question may have affected students' ability to keep the problem goal in mind. As explained by Hunting, Davis and Pearn (1996), students' schemes for solving mathematical problems consists of three parts: recognizing similar previously experienced situations, associating the new activity with the previous experience, and expecting similar results from the new activity. Typically, when shown an area fraction model students are asked to name the fraction shown. This is the activity they tend to associate with their previous experience. However, the Candy Bar questions asked students to name the fraction of the amount eaten. Students drew a correct model showing the remaining ¾ of a candy bar, but the image may have evoked a pre set frame of mind to name the part of the bar remaining and not, as the question asked, the part eaten. When the question asks for different information than the student has typically answered with the representation type, the student may have difficulty overcoming the information evoked by their pre set frame of mind. This shows the importance of increasing students' representation flexibility and teaching students to evaluate whether their answers match what the question is asking.

5.2. Incorrect model selection

On three different questions students selected models which were not congruent with the problem. On the Candy Bar and Comparison questions students selected set model representations to represent parts of a candy bar. This may reflect students' focus on the quantities of the whole numbers represented in the fractions. In past experiences of whole number understanding development, whole numbers have typically been associated only with discrete quantities. As observed by the researchers, evidence suggests that some students attempted first to draw region models, but erased them and drew set models. This suggests that students were not comfortable with how to develop the region model and reverted to their understanding of whole numbers being connected with discrete quantities.

In contrast, on the Candy Cane problem some students tried to represent two-fifths of a group of candy canes being red on each of the five candy canes in the set. The mostly likely explanation being that the students were associating fractions only with a region model. Both of these results suggest that students need more opportunities to manipulate multiple types of representations. Experiencing multiple representations of a numerical concept in multiple modalities may improve children's numerical abilities (Brannon, Jordan, & Jones, 2010; Jordan & Baker, 2011; Jordan & Brannon, 2009).

5.3. Failure to overcome whole number bias and struggles with part-whole understanding

For this study, results suggest that failure to overcome whole number bias and the struggles with part-whole understanding are interconnected. Whole number bias is "a robust tendency to use a single-unit counting scheme to interpret instructional data on fractions" (Ni & Zhou, 2005, p. 28). Examples of whole number bias are deducting that ¼ is greater than 1/3 because 4 is greater than 3 or that 1/2 + 1/3 = 2/5. Whole number bias limits students development of part-whole understanding (Lamon, 2005; Ni & Zhou, 2005; Stafylidou & Vosniadou, 2004). Part-whole understanding is built around the concepts of partitioning discrete sets or continuous models into equal parts (Charalambos & Pitta-Pantazi, 2007). Even though the instructional methods used to teach children in this study focused almost exclusively on part-whole instruction, many of the difficulties students had drawing and using representations reflect part-whole misunderstandings. Charalambos and Pitta-Pantazi (2007) in their review of the literature identified components to the mastery of the part-whole understanding of rational numbers. Four of the components are important understandings for third- and fourth-grade students and their development of effective representations.

1) Wholes must be partitioned into equal parts. Results suggest that many of students' difficulties in drawing representations were results of the students' whole number bias in that their focus was on the quantity and not on the magnitude of the drawing partitions. In developing representations for the Candy Bar question, some students drew partitions of unequal magnitudes and compared only the number of partitions. Others partitioned regions in this question and other questions by using the same number of lines as the number in the denominator, suggesting they

were focusing only on the whole numbers in the fraction and had not developed understanding of the denominator as an indicator of equal-sized partitions within the whole. Responses to the Pizza question suggested that, even when students appeared to have the concept and drew equal partitions in their models, the concept of equal shares, was for them, still negotiable. Some students suggested that each person in the question could still be getting one half, but one person's half would be a little more than the other. This willingness to negotiate the magnitude limited their ability to consider the wholes being of different magnitudes. These examples suggest that many of the students had not mastered the concept that a fraction is a representation of a whole partitioned into equal shares.

2) The parts must exhaust the whole. Several of the students drew representations in which only small parts of the whole were partitioned into the number of parts represented by the fraction. Again this suggests that the students were using whole number bias, focusing only on the quantities of the fraction digits and not considering the magnitude of the partitions making up the whole. Drawings in which the parts did not exhaust the whole were unusable as tools in developing understanding.

3) The more partitions, the smaller the parts become. An indicator of whole number bias is students' misconception that the greater the number in the denominator, the greater the magnitude of the fraction (Ni & Zhou, 2005). In the Comparison question, some students incorrectly answered the problem using only symbolic comparisons of the numbers. Others drew representations in which they drew partitions of fifths larger than partitions of fourths to show that 1/5 is larger than 1/4 . Others, similar to students in the Armstrong and Larson (1995) study drew correct models, but disregarded the magnitude of the fractions and focused only on the quantity, stating that ¾ was greater than 2/3 because three was greater than two. Armstrong and Larson (1995) suggested that exclusive use of visual comparisons using region models does not give students the need to develop more sophisticated strategies of comparisons.

4) The part-whole relationship is conserved regardless of changes in the size, shape, arrangement, or orientation of the parts. Misconceptions in conservation of the part-whole relationships were observed in all three equivalence questions. In the Fraction Strings question students suggested that two models in which shaded partitions were in different arrangements were different fractions. In the Pizza question, students suggested that partitions which had been subdivided were of a different magnitude than the same amount which had not been subdivided, because the quantity of partitions had been increased. In the Candy Cane question, only 17% of the students correctly conserved the part-whole relationship when doubling the set from five candy canes to ten. Most used one of the whole numbers in the original fraction to determine how many candy canes would be colored to represent the fraction of the doubled set. Guiding students through the process of proportionally changing both the whole and the parts of their representations can be powerful in the development of multiplicative understanding (Turner, Junk & Empson, 2007). However, as shown by the results of previous studies (e g., Armstrong & Larson, 1995; Kamii & Clark, 1995) it is not a process most students quickly grasp and requires multiple experiences with different types of fraction interpretations.

The underlying difficulty of these four areas was that students focused on quantity rather than magnitude when interpreting a fraction part-whole relationship. The results of this study suggest that students' lack of part-whole understanding limits the foundations needed for them to develop accurate internal representations which can serve for memory holders and as tools for developing new understanding. The students focused on quantity results in their representations which perpetuated their misconceptions. The results suggest that it cannot be assumed that because students are exposed to and work with correct representations students will develop correct representations. Repeated exposure to part-whole representations is not sufficient to help students master part-whole constructs or to overcome their whole number biases to the level that they can use representations to support their learning. Analysis of results from Torbeyns, Schneider, Zin and Siegler's (2014) study with187 sixth and eighth grade student from Belgium, China and the United States indicated that understanding of fraction magnitude was a strong predictor of students' overall fraction understanding. In the analysis, they also compared test results of United States' students, in which instruction is based almost exclusively on the part-whole interpretation with Belgium and China's students, in which instruction is based on a measurement interpretation. The Belgium and China students outscored the United States students on all categories of fraction magnitude and fraction operations. Two intervention studies, Gabriel et al. (2012) and Fuchs et al. (2013) successfully used instruction focusing on measurement to increase students with mathematical learning difficulties understanding of magnitude. Empson (1999) used fair share instruction with first-grade students and reported that at the end of the instruction sessions most of the students had developed an understanding of the relationship between the number of shares and the magnitude of the partitions and that many of the students could use this understanding to solve novel problems. These studies, and many other recent fraction studies suggest that an increased focus on the quotient and measurement interpretation of fractions can successfully help students to develop the necessary components of part-whole mastery and to overcome whole number biases (e g., Charalambos & Pitta-Pantazi, 2007; Steffe, 2004; Tzur, 1999).

5.4. Conclusion

Overall, the findings of this study are striking, indicating that even though third- and fourth-grade students had the opportunity to see and manipulate pictorial representations in virtual, physical and static forms, there was still a disconnect between students' ability to develop the representations and then use them successfully for solving problems.

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