EDUCATIONAL IMPLICATIONS OF SOME OF WITTGENSTEIN'S REMARKS ON MATHEMATICS: PROPOSITION, INFERENCE AND PROOF

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ABSTRACT

Inspired by Rousseau's ideas, the different tendencies of the new school have emphasised educational practices centred on the pupil's activity, presupposing in the learning process a natural rationality being developed as much as possible through observation and experimentation, and not by means of verbal instructions. The goal in this paper is to point out some misleading practices which arise from these empiric directives, by using some of Wittgenstein's remarks on the nature of mathematics, particularly those published in *Philosophical Remarks* (PR). As one of the major representatives of the linguistic turn, the Austrian philosopher presents a new conception of language where the senses are not developed from cognitive structures or abstracted from some extra linguistic reality; but rather they are formed *within* language, which has a multiplicity of functions. In particular, the mathematical propositions perform a normative function, very similar to rules, as they allow inferences and determine what makes sense and what does not make sense. From this perspective, we defend that learning mathematics involves essentially a *training*, process in which the student forms an *accepted* rationality, among many other effective and possible ones.

Keywords: meaning, rationality, teaching, new school, linguistic turn, Wittgenstein.

RESUMO

Sob a inspiração das idéias de Rousseau, diferentes tendências da escola nova têm enfatizado práticas educativas centradas na atividade do aluno, pressupondo no processo de aprendizagem uma racionalidade natural a ser desenvolvida, na medida do possível, através da observação e experimentação, e não por meio de instruções verbais. Recorrendo a observações de Wittgenstein sobre a natureza da matemática, especialmente as publicadas nas *Observações Filosóficas* (PR), tem-se como objetivo neste artigo apontar para algumas práticas enganosas que surgem a partir destas diretrizes empíricas. Sendo um dos principais representantes da virada lingüística, o filósofo austríaco apresenta uma nova concepção de linguagem em que os sentidos não

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são decorrentes de estruturas cognitivas ou abstraídos de alguma realidade extralingüística; mas são formados no interior da linguagem, a qual tem uma multiplicidade de funções. Em particular, as proposições matemáticas desempenham uma função normativa, muito semelhante às regras, uma vez que permitem inferências e determinam o que faz e o que não faz sentido. A partir desta perspectiva, defendemos que o aprendizado da matemática envolve essencialmente um *treinamento*, processo através do qual o aluno forma uma racionalidade *aceita*, entre muitas outras efetivas e possíveis.

Palavras-chave: significado, racionalidade, ensino, escola nova, virada linguística, Wittgenstein.

1. "Know that" *versus* "Know how"

Since Rousseau's *Emile*, the pedagogues of the new school have emphasised educational practices which are centred on the pupil's activity, where the teacher begins to play more a role of a guide (*gouverneur*) than that of a preceptor (*précepteur*) (Rousseau, 1999, p. 29). According to the Genevan philosopher, education must be negative, in the sense that "you shall not give precepts, but do so that they are found" (1999, p. 29). This maxim by Rousseau had repercussion on the educational milieu with such force that we can say we are still under its effect. While in his treatise on education he found space to reflect on the relationships between thought and language (1999, p.115) by recognising the force of the customs and habits in the constitution of meanings expressed by the different tongues, he affirmed that the same human reason was common to all of them and that it would be gradually formed in his ideal pupil (Emile) by providing the pupil with empirical situations appropriate for their development. Verbal teaching should be deferred as much as possible, in time to form an intellectual reason which would provide the child with discernment, capacitating him/her to distinguish the correct from the incorrect and good from evil.

In more recent versions of the pedagogical tendencies, which are heirs to Rousseau's ideas, the verbal contents continue to be seen just as *means* for the formation of capacities to be developed in the child, nowadays called *abilities* and *competences*, where the *know how* has priority over the *know that* [2]. New pedagogical practices are proposed as a reaction to what they denominated "traditional teaching", considered by them too attached to a verbal instruction. In spite of this pedagogical revolution after Rousseau's ideas, the linguistic turning point, which occurred after more than a century of the publication of *Émile,* seems to have passed unchanged in the educational milieu, to the extent that in the midst of so many changes language continues to be seen as the performer of a merely descriptive and communicative role. In particular, in the domain of school mathematics most of its directives are still anchored in a figurative conception of language, to the extent that its signs and propositions are viewed by a large part of

[2] This is the terminology of the "pedagogy of competences" the Brazilian government adopted as a theoretical referential for the *Exame Nacional do Ensino Médio* [National Examination of Secondary Education] (ENEM), which is administered at the end of secondary school with a double function: to evaluate the pupils of secondary public schools, and to use the results of this evaluation as a criterion for admission to federal public universities.

educators to be representing its own objective reality, which could not be created or changed, but only perceived and described, as it can be noted in the following passage of the National Curricular Parameters[3] (PCN):

> (...) when studying numbers, students can *perceive* and verbalize relations of inclusion, such as that every even number is natural; but they will *observe* that the reciprocal is not true, because not every natural number is pair. On the study of forms, by *observing* different triangular figures, they can *perceive* that the fact that a triangle has angles with measures similar to the measures of the angles of another triangle is a necessary but not sufficient condition for the two triangles to be congruent. (PCN 1997, v.3, p.54, our italics)

Thus, inspired by Rousseau's ideas, who advocated that the master "must not give precepts but do so that they are found", contemporary tendencies of the new school in the domain of mathematics propose methodologies which start from observations and experimentations, as if its propositions would also have the character of hypotheses which should be formulated by the pupil on the basis of his/her previous knowledge and tested in empirical situations of daily life, in order to generalise his/her "discoveries" afterwards. Not only in this official document (PCN) one finds such an empiricist and, at the same time, mentalist epistemological conception, it is recurrent to hear the pedagogues use the slogan that "it is the child who constructs his/her own knowledge", supported by the affirmations of the Swiss epistemologist and psychologist Jean Piaget that the mathematical activity would be the consequence of the development of cognitive abilities, where modern mathematics[4], "through a notable convergence (...) reveals itself nearer to the subject's natural or spontaneous operations (child or adolescent) than the traditional teaching of this branch, excessively submitted to history (...)" (Piaget, 1998, p.217). From this cognitive perspective, it would be incumbent on the teachers to provide their pupil with "apprenticeship conditions" inserting them in determined contexts, such as solution of problems, "transversal themes", or even, only daily life situations, in order to enable them to discover these *a priori* entities and relationships *by themselves*.

> "(...) it is fundamental not to underestimate the pupils' mathematical potential and to recognise that they solve problems, even if they are reasonably complex, by using their knowledge of the topic and by trying to establish relations between the already known and the new (...) The attention given to the fact that the pupil is the agent of the construction of his/her knowledge through the connections established with his/her previous knowledge in a context of problem solution is relatively recent." (PCN, v.2, 1998, p.37)

While the emphasis on the pupil's activity (*know how*) has brought incontestable advances in the pedagogical field, thus rupturing a teaching model which was excessively centred on the teacher figure (*know that*), in the teaching of mathematics, the radical change of the pendulum for pragmatic proposals centred on the pupil's subjectivity runs the risk of naturalising knowledge of conventional nature and leads to confusions of conceptual nature with quite equivocal educational implications, which we intend to approach by having recourse to Ludwig Wittgenstein's philosophical

[3] *Parâmetros Curriculares Nacionais* (PCN) [National Curricular Parameters] are directives of MEC [Ministry of Education and Culture] for all public schools in Brazil. Although this official document has been written by many hands, one notices the strong influence of two main pedagogies: the piagetian cognitive constructivism and the "pedagogy of competences". Cf. Gottschalk (2002)

[4] Piaget refers to the theory of sets formulated by Bourbaki's group, beginning in 1935 with a series of publications which would have, mainly in the sixties, a great influence on mathematical education.

reflections on the concepts of proposition, inference and mathematical proof, which were published posthumously under the title *Philosophical Remarks* (PR)[5].

2. From an educational experience to a radical linguistic turn

The change of thought from the author of *Tractatus Logico-Philosophicus*[6] to the author of *Philosophical Investigations* (PI) already appears in his 1929-30 notes, soon after a period in which the Austrian philosopher went through quite an intense teaching experience by having held classes in extremely poor villages in the inland of Austria to primary school children (from 1920 to 1926) during the Austrian educational reform, then led by Otto Glöckel. The ideal of this reform was to implement the principles of the new school in Austria, which had been recently destroyed by World War I; one of its objectives was to free the peasants and the working class from the fetters of the anterior regime and to turn them into participant citizens of a democracy (Bartley, 1978, pp. 77-8). By teaching his pupils in accordance to the principles of a pedagogy focused on work (*Arbeitsschule*), where the acquisition of knowledge should occur predominantly through the child's activities, Wittgenstein would realise that the language involved in these activities seemed to fulfil very different functions, since part of its utterances resembled those of the propositions of mathematics, expressing certainties, truths with a certain degree of necessity, different from the hypothetical utterances of natural sciences (Gottschalk, 2012).

As he formulated more precisely much later, at the end of the 1940's (Wittgenstein, 1979), when one enunciates propositions such as "I exist", "only I can feel the pain I have now", "every object is identical to itself", "this hand is mine"… which have the appearance of being empirical propositions, one is not describing some fact of the world which could verify them, attributing a value of truth to them (V/F); those are certainties which are beyond question and one cannot imagine their contrary either. The conviction with which we enunciate them is equivalent to our attitude when we affirm that $2 + 2 = 4$ in the field of arithmetic: "two plus two *must* be equal to 4". Like this mathematical proposition, those other propositions also seem to be absolute, *necessary* truths. Thus, all these affirmations appeared to be unlike the empirical propositions, whose value of truth/meaning derive from the facts they represent. Would they, then, refer to another kind of reality? If so, what would they represent? In his struggle for answering these philosophical questions without recurring to metaphysical answers, Wittgenstein realizes that the relationship between language and world was quite more complex than he thought at the time of *Tractatus*. The referential model of language, on which he had based his considerations in order to elaborate the theory of meaning in the first phase of his thought, did not seem to measure up to this and to so many other questions any more.

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[5] Wittgenstein's notes, organised and published by Rush Rhees and translated into English by Raymond Hargreaves and Roger White, pertaining to the period between February 2nd 1929 and the last week of April 1930. The commentators consider them as being part of a period of transition between his ideas of *Tractatus* and *Philosophical Investigations*.

[6] This was his first published work (1920), with great repercussion among the logic positivists of the Vienna Circle and also in the educational area, with strong influence on the work of analytical philosophers in Oxford and Cambridge, such as Gilbert Ryle and Michael Oakeshott.

Once back in Cambridge, now as a university professor, Wittgenstein presented his philosophical remarks as a research project to philosopher and friend G. E. Moore, then a professor at Trinity College. Shortly after Wittgenstein's death in 1951, Moore handed these observations to Rush Rhees, who made a selection of these manuscripts and organised them in the text which is known today as *Philosophical Remarks* (PR). Surprisingly, these writings already depict the fundamental themes Wittgenstein treated in the course of his posterior reflections on the nature of mathematical knowledge, with the advantage that, as early as 1929, we see the birth of the questions which afflicted him in intimate connection with his also nascent discussion on the possibility of a private language, which would lead him to reject a mentalist position on the constitution of meanings, concluding that there is no thought without language; any significant action itself would already be linguistic. According to some commentators of his work (Bartley III, 1978; Moreno, 2003), his past experience as an elementary school teacher had also contributed to the questioning of the language conception that presupposes that the meaning of a word "is the object for which the word stands" (PI, §1), rupturing, in this way, the dogmatic view of theories of the sense, which considered language as performing a merely figurative role, so it would be learned by children through ostensive gestures and definitions.

In the first part of his later work, *Philosophical Investigations*, organised by Wittgenstein himself, he claims that the "ostensive teaching of words" (the teacher's pointing to the objects, directing the child's attention to them, and at the same time uttering a word) is just an important *part of the training* of children which are brought up to perform certain activities, to use words as they do so, and to react in certain ways to the words of others (PI, §6). In other words, the understanding of the word is not limited to the *ostensive teaching*, which can have different purposes (to establish an association between the word and the thing, taking the thing to a certain place, just to repeat the word, to imagine it mentally…), so one must also enter it in contexts of *use*, leading to different understandings of the word, process which gradually constitutes the meaning of the word. Thus, the referential use is only an initial, preparatory stage, of the process of constitution of the senses in the ordinary language. In this Wittgensteinian perspective, the whole process is a product of a *work of language*; there is no extra linguistic reality which fundaments the senses one attributes to the facts of the world (Moreno, 2011).

In formal language learning, like mathematics, this work of language appears in a paradigmatic way, since *all* its propositions have a transcendental role, in the sense that they are conditions of meaning to empirical propositions, as it will be later explained, producing new ways of organizing the empirical world. They themselves do not describe anything; they are only the conditions to empirical descriptions. However, today the exclusive referential way of viewing language still acts strongly on mathematicians, and it was also present in the different tendencies of the philosophy of mathematics predominant at the time of Wittgenstein. Most of his contemporary mathematicians had a platonic view of mathematics: that was the position, for example, of the great Austrian logician and mathematician, Kurt Gödel (1906-1978), and it was also defended by G. H. Hardy, a British mathematician, who would be often quoted in Wittgenstein's lectures about the foundations of mathematics, held in Cambridge in the 1930's, aimed at administering the philosophical therapy[7] of his realistic conception of

[7] Wittgenstein's philosophical therapy is a "method" proposed by him to elucidate conceptual confusions which lead to philosophical problems (Moreno, 2011).

mathematics (Gerrard, 1991), which often appears in many passages of Hardy's book, *A Mathematician's Apology*, such as:

> I believe that mathematical reality lies outside us, that our function is to discover or *observe* it, and that the theorems which we prove, and which we describe grandiloquently as our "creations", are simply our notes of our observations. (…) 317 is a prime, not because we think so, or because our minds are shaped in one way rather than another, but because *it is so*, because mathematical reality is built that way.[8]

The above affirmations (where a concealed mathematical reality which must be unveiled by the mathematicians is presupposed) reproduce the more general belief that every linguistic sign must refer to something outside language, which attributes meaning to it. In administering the therapy of the claims of Hardie as well of other contemporary mathematicians, Wittgenstein proposes that we *look* (and not *think)* (PI §66) as we actually use language by beginning to consider the multiplicity of the uses we make, which do not restrict to the referential use (PI §23). Therefore he will forge concepts such as those of "language games", "family resemblances", "following rules", "forms of life", among others, which will shed light on the *praxis* of language, where the words are intrinsically involved in different activities, thus constituting the meanings. From this linguistic and pragmatic perspective, the question is no longer to search for meanings outside language, in the way of the search for objects carried out by scientists of the empirical sciences, but to describe the linguistic processes which constitute them, where the mathematical propositions play a very interesting role. In particular, in the observations from *Philosophical Remarks* about the mathematical activity pointing to new perspectives for its teaching, as it will be later presented, Wittgenstein sees the mathematical propositions not describing a third reign to be gradually discovered, but performing a role very similar to *rules*.

3. Mathematical propositions as grammatical rules

In the course of his investigation on the constitution of meanings through language, published in *Philosophical Remarks*, Wittgenstein uses as recourse various examples of the mathematical activity, aiming to basically distinguish two types of propositions, the empirical ones and those he would denominate *grammatical*[9]. Many of the problems of philosophy would come from not making this distinction and taking normative propositions as descriptive ones, thus unduly generalising the function of the propositions of language. A way of dissolving these problems (and not of solving them) would be to only observe that a part of language has a descriptive function expressed by the empirical propositions, whereas the grammatical propositions would be *meaning conditions* to the other ones[10]. To draw our attention to these propositions which enable us to represent a fact of the world, Wittgenstein recurs to an analogy with the axioms of geometry:

[8] Apud Gerrard, 1987, p.17

[9] Wittgenstein would use the term "Grammatical" not to refer to the usual grammar of a language, but in the technical meaning, that is to say, as being a set of *uses* we make of our linguistic expressions that express the rules we follow in our different language games.

[10] Wittgenstein also uses the metaphor of Heraclit's river in order to refer to this distinction between empirical and grammatical propositions. The first would be part of the flux of the river, whereas the second would constitute its conductor channel. Cf. Stern (1991)

> The axioms – for instance – that a straight line can be drawn through any two points has here the clear sense that, although no straight line *is* drawn between any 2 arbitrary points, it is *possible* to draw one, and that only means that the proposition 'A straight line passes through these points' makes *sense*. That is to say, Euclidean geometry is the syntax of assertions about objects in Euclidean space. And these objects are not lines, planes and points, but *bodies*. (PR, XVI, 178)

In this Euclidean context, the axiom itself has no meaning, but it has a *transcendental* function: it is used by the mathematician community as a rule one follows in order to judge what makes sense doing or what does not. For example, if I ask a pupil to draw a red straight line going through points A and B, while he can eventually draw one straight line through point A and another one through point B, as a mathematics teacher I will only consider the pupil's action as meaningful if he/she *actually* follows the rule expressed by the axiom, that is to say, if he/she only draws *one* straight line which contains the two points. By following that Euclidian axiomatic rule, the proposition, "I drew a red straight line through points A and B", can now be verified as being true or false, correct or incorrect, that is to say, this turns into an empirical proposition with meaning. According to Wittgenstein, ordinary language would also have its "axioms", or, in his own words, a kind of phenomenological language fundaments our explanations:

> A recognition of what is essential and of what is inessential in our language if it is to represent, a recognition of which parts of our language are wheels turning idly, amounts to the construction of a phenomenological language.
> Physics differs from phenomenology in that it is concerned to establish laws. Phenomenology only establishes the possibilities. Thus, phenomenology would be the grammar of the description of those facts on which physics builds its theories.
> To explain is more than to describe; but every explanation contains a description. (PR, I, 1)

Thus the philosopher's concern to distinguish a "grammar" of conventional and *a priori* nature from that which is represented in the linguistic description of a fact is not restricted to the field of mathematics, because, in a general way, one would have two types of proposition, the empirical ones (whose objective is to describe and to explain the world, like the propositions of physics) and the grammatical ones, rules which allow to attribute meaning to the represented fact.

> I do not call a rule of representation a convention if it can be justified in propositions: propositions describing what is represented and showing that the representation is adequate. Grammatical conventions cannot be justified by describing what is represented. Any such description already presupposes the grammatical rules. That is to say, if anything is to count as nonsense in the grammar which is to be justified, then it cannot at the same time pass for sense in the grammar of the propositions that justify it (etc.). (PR, I, 7)

By means of these observations, Wittgenstein emphasises the nature of the foundations of our explanations in the different activities which use language (which he would later denominate *language games*[11]), foundations *without foundation*, that is to say, without justifications other than language. These would play the role of rules, being constituted by *internal* relationships (sense relations), meaning conditions for any linguistic

[11] Term coined by him in the 1940's, by systematising part of his manuscripts in his work *Philosophical Investigations* (PI), which would only be posthumously published. Wittgenstein uses the expression "language game" to refer to the activities involved in the words and which, along with them, constitute the meaning of our linguistic expressions. The analogy with a game has to do, on one hand, with the rules one follows playing it and, on the other, with the grammatical propositions which allow the inferences one makes within our different linguistic activities.

representations about the world. In other words, the grammatical conventions are *arbitrary* and, at the same time, *necessary* in our forms of life.

> I should like to say, if there were only an external connection no connection could be described at all, since we only describe the external connection by means of the internal one. If this is lacking, we lose the footing we need for describing anything at all – just as we can't shift anything with our hands unless our feet are planted firmly. (PR, III, 26)

The propositions of mathematics are also part of what Wittgenstein calls the essential in our language, they are some of our grammatical conventions and, consequently, *they cannot be justified by a description of the represented*. On the other hand, they are also used to represent facts of the world. So how do we know if it is being used in one way or another (grammatical or empirical)? Wittgenstein answers this question by means of simple examples of combinatory calculus, in which he presents the pragmatic criteria one disposes of in order to distinguish its meaning:

> If you want to know what a proposition means, you can always ask 'How do I know that?' Do I know that there are 6 permutations of 3 elements in the same way in which I know there are 6 people in this room? No. Therefore the first proposition is of a different *kind* from the second.
> Another equally useful question is 'How would this proposition actually be used in practice?'; and there the proposition from the theory of combinations is of course used as a law of inference in the transition from one proposition to another, each of which describes a reality, not a *possibility*.
> You can, I think, say in general that the use of apparent propositions about possibilities – and impossibilities – is always in the passage from one actual proposition to another.
> Thus I can, e.g., infer from the proposition 'I label 7 boxes with permutations of a, b, c' that at least *one* of the labels is repeated. – And from the proposition 'I distribute 5 spoons among 4 cups' it follows that one cup gets 2 spoons, etc.
> If someone disagrees with us about the number of men in this room, saying there are 7, while we can only see 6, we can understand him even though we disagree with him. But if he says that for him there are 5 pure colours, in that case we don't understand him, or must suppose we completely misunderstand one another. This number is demarcated in dictionaries and grammars and not within language. (PR, X, 114)

Thus, when one uses combinatory calculus in order to organise an empirical situation, such as labelling 7 boxes with the permutations of three letters, the calculus pertains to the internal connections of that which is being represented, it is the meaning condition which enables us to affirm that one of the boxes will have a repeated label. The calculus in itself does not have a meaning; it is only rules which we follow for the representation of a fact. These rules, in turn, enable us to go from an empirical proposition to another one, as in the example above: we go from the empirical proposition that we "labelled 7 boxes with permutations of the letters a, b and c" to the empirical proposition "at least one of the labels repeats itself". And what allows this passage is to have followed the mathematical rule which postulates that there are only 6 possible permutations of 3 elements. Thus, one criterion to know if a proposition describes a reality or a possibility could be to verify how it is being used, as a description (V/F) or as an inference. But in the case of the latter, would the use of mathematical propositions mean that they themselves are somehow describing something, as if the computation were a type of experiment that allow us to predict empirical facts?

4. Inference and mathematical proof: discovering a third reign?

Wittgenstein answers the above question by proposing the following thought experiment. Let us suppose that you present the following problem to a little child: to distribute 11 apples among some people so that each person is given three apples. Then you ask the child to discover how many people there can be, expecting that he/she will give the answer 3. Consequently can you interpret that the calculation would be a type of experiment, as in empirical sciences, in which you foresee something which you will be able to verify? Wittgenstein goes on:

> (...) Now, suppose I were to go through the whole process of sharing and at the end 4 people each had 3 apples in their hands. Would I then say that the computation gave a wrong result? Of course not. And that of course means only that the computation was not an experiment.
> It might look as though the mathematical computation entitled us to make a prediction – say, that I could give 3 people their share and there will be two apples left over. But that isn't so. What justifies us in making this prediction is a hypothesis of physics, which lies outside the calculation. The calculation is only a study of logical forms, of structures, and of itself can't yield anything new. (PR, X, 111)

In fact, one can imagine a situation in which some of these people already had other apples, and when the child counted the number of people with three apples in their hand she/he could eventually have got to the result 4. Nevertheless, this does not invalidate the mathematical rule that 11 divided by 3 has 3 as a final result and not 4. It is not the calculation which foresees how the distribution will be, this prevision derives from the laws of physics, which authorise us to say that apples do not disappear nor do they appear out of nowhere either. The calculation only gives a determined *form* to the empirical distribution among the people who are present. It is in this sense that Wittgenstein peremptorily affirms that mathematics is *not* an empirical science. Independently of the concrete situation, in which eventually 4 people have 3 apples in their hands instead of only 3 people, one proves that the correct solution of the problem is 3 by means of a proof, that is to say, by making the mathematical division. This proof does not need empirical evidences. There can even be two mathematical proofs which demonstrate this affirmation and one of them can be translated into the other (PR, XIII, 149), but this is different from having two empirical evidences which confirm a determined event:

> Proofs proving the same thing may be translated into one another, and to that extent are the same proof. The only proofs for which this doesn't hold are like: 'From two things, I infer that he's at home: first his jacket's in the hall, and also I can hear him whistling'. Here we have two independent ways of knowing. This proof requires grounds that come from outside, whereas a mathematical proof is an analysis of the mathematical proposition. (PR, XXIII, 153)

Once again, Wittgenstein points out the difference between a mathematical proof (accomplished independently of causal relationships) and an empirical experiment (confirmed by external evidences). Let us suppose that a mathematics pupil is asked to find the solutions which satisfy the equation $x^2 = 4$. He/she could find the solutions 2 and -2 through attempt or error, in the same way that I know that somebody came home when I hear them whistling or see their jacket in the hall. Nevertheless, to have got to these results does not *prove* that these are the solution of the equation. The proof is not based on such evidences. The proof is given through a method of solution of the equation, for example, by using Bháskara's formula, which, once applied, provides the following results: $x = +\sqrt{4}$ or $x = -\sqrt{4}$, hence $x = 2$ or $x = -2$. According to Wittgenstein, the general method of the solution of an equation is not a stratagem (an artifice) to get to the solutions of the equation, but it is in itself an elucidation of the essence of the equation, that is to say, one establishes *internal* relations and not external ones[12], i.e., the proof attributes a *sense* to the presented solutions.

Going back to the afore mentioned example of the apples which must be equally distributed, we *proved* that only 3 people will be able to have 3 apples each by calculation the division of 11 by 3, independently of the empirical situation. In all these cases, there are no hypothetic relations, but grammatical ones, which tell us what makes sense saying or what does not make sense saying. It does not make sense saying, for example, that each person got 4 apples. This does not mean that Wittgenstein would affirm that mathematics would not be used in scientific[13] or daily activities; on the contrary, the philosopher only draws our attention to the fact that precisely because we very often express our hypotheses about the world by using mathematical calculation, we do not perceive that the nature of its propositions is another one. This is the criticism Wittgenstein will direct at the set theory proposed by the members of Bourbaki's group, which intended to be descriptive of a mathematical reality. Nevertheless, Wittgenstein insists, in mathematics one does not describe, one only *does* (PR, XIII, 159). With Socratic irony, he continues:

> Mathematics is ridden through with the pernicious idioms of set theory. *One* example of this is the way people speak of a line as composed of points. A line is a law and isn't composed of anything at all. A line as a coloured length in visual space can be composed of shorter coloured lengths (but, of course, not of points). And then we are surprised to find, e.g., that 'between the everywhere dense rational points' there is still room for the irrationals! What does a construction like that for √2 show? Does it show how there is yet room for this point in between all the rational points? It merely shows that the point *yielded* by the construction is *not rational*.
> And what corresponds to this construction and to this point in arithmetic? A sort of number which manages *after all* to squeeze in between the rational numbers? A law that is not a law of the nature of a rational number.

According to Wittgenstein, every number is constructed according to different conventional laws. We even learn natural numbers by using different techniques. How does one teach, for example, number 2 to a small child? Perhaps by pointing at two objects and by pronouncing the word "two". But how does one know if the child understood that "two" is a number and not the form or the colour of the object (IF, §§29, 30)? At the beginning of *Philosophical Investigations*, when Wittgenstein systematises his observations on the apprenticeship of language by the child, he incorporates these preliminary reflections on the nature of mathematics made in the early thirties, in which he had already accomplished his therapy of the Platonic conception of mathematics. Understanding that "two" is a word which is used as a number (and not to refer to a form, a colour or to any other aspect of the object one points to) will depend on the role this word plays in language in determined contexts: the child learns the technique of counting by memorising the cardinal numbers in a determined order, he/she learns how to compare objects, to group them in determined ways, finally, the concept of number will gradually be shaped in the *use* one makes of them. Thus, to understand that it is a number does not mean that the child will see an essential meaning going through all the applications of the word "number", referring to some entity in a third reign, but just

[12] An analogous example is found in *Remarks on the Foundations of Mathematics*, Part VII, 46.

[13] Most of the scientific laws are expressed in quantitative terms, such as the law of thermodynamics which expresses the behaviour of the molecules by means of the mathematical relation V = cT/P, where V is the volume, T the temperature P the pressure and c a constant. The confusion stars when one does not distinguish the laws of probability which are being confirmed/refuted (hypothetical assertions about the relationship between these magnitudes) from their mathematical expression (*a priori*).

similarities and differences to a larger or smaller degree, similarities which Wittgenstein would denominate “family resemblances” (PI, §67). In this linguistic perspective, numbers do not describe a reality outside the mathematical system, they are just new constructions invented by mathematicians. Well, if everything can be limited to the domain of techniques, and if understanding a mathematical concept is but being capable of appropriately mastering them, would there remain no place to conjecture in mathematics?

5. Conjectures *versus* laws of concept formation

> How can there be conjectures in mathematics? Or better: what sort of thing is it that looks like a conjecture in mathematics? Such as making a conjecture about the distribution of primes.
> I might, e.g., imagine that someone is writing the primes in series in front of me without my knowing they are the primes – I might for instance believe he is writing down numbers just as they occur to him – and I now try to detect a law in them. I might now actually form an hypothesis about this number sequence, just as I could about any other sequence yielded by an experiment in physics.
> Now in what sense have I, by so doing, made a hypothesis about the distribution of primes?
> You might say that a hypothesis in mathematics has the value that it trains your thoughts on a particular object – I mean a particular region – and we might say ‘we shall surely discover something interesting about these things’.
> The trouble is that our language uses each of the words ‘question’, ‘problem’, ‘investigation’, ‘discovery’ to refer to such basically different things. It’s the same with ‘inference’, ‘proposition’, ‘proof’.
> The question again arises, what kind of verification do I count as valid for my hypothesis? Or, can I – *faute de mieux* – allow an empirical one to hold for the time being, until I have a ‘strict proof’? No. Until there is such a proof, there’s no connection at all between hypothesis and the ‘concept’ of a prime number.
> The concept of a prime number is the general law by means of which I test whether a number is a prime number or not.
> Only the so-called proof establishes any connection between my hypothesis and the primes *as such*. And that is shown by the fact that – as I’ve said – until then, the hypothesis can be construed as one belonging purely to physics. – On the other hand, when we have supplied a proof, it doesn’t prove what was conjectured at all, since I can’t conjecture to infinity. I can only conjecture what can be confirmed, but experience can only confirm a finite number of conjectures and you can’t conjecture the proof until you’ve got it – and not then either.
> The concept ‘prime number’ is the general form of investigation of a number for the relevant property; the concept ‘composite’ the general form of investigation for divisibility etc. (PR, XIII, 161)

In these passages, Wittgenstein synthesises various fundamental ideas: the first of them is that our concepts are used in a much different way in distinct language games, even if they maintain a certain family resemblance between them. Particularly the concepts of proof, inference, investigation, discovery, problem and proposition. By means of the example of the sequence of prime numbers, the philosopher distinguishes experiments from proofs, and conjectures from laws of concept formation. Mathematical propositions are not conjunctures which will be confirmed by experience. To be prime, a number *must* follow the rule that it can only be divided by 1 or by it itself. Wittgenstein’s arguments are simple in order to reject the idea that it would be possible to make conjectures regarding a law on prime numbers in the sense of empirical sciences, that presupposes the verifiability of their hypothesis and which does not work in the case of prime numbers, since one cannot simply do it *ad infinitum*. The connection between a conjuncture and the concept of prime number can only be established internally by means of proof, it is that which establishes essential connections (Wright, 1980). Thus, mathematical proof is not an instrument of discovery, like the experimentations of empirical sciences, but, according to Wittgenstein, it is an instrument of *conceptual change*: "The mathematical proof draws our attention to the essential relations between the mathematical concepts. For example, that a rectangle can be obtained from two equal parallelograms and two triangles which have half of its area"[14].

From this linguistic perspective, the function of the proof is to demonstrate an essential conceptual connection. Thus, a theorem is not a generalisation such as the hypotheses and the laws of physics. Whereas the hypothesis is corroborated by various verifications, in the case of a theorem, a unique proof is enough to elevate it to the level of a legitimate proposition of the system in which it is inserted. Wittgenstein further suggests that other proofs would each be different instructions for the use of the theorem, in the same way in which one learns the meaning of the words of the ordinary language by means of the use one makes of them (Wittgenstein, 1989a, part VI). Thus, one establishes different meaning conditions for the application of a theorem, different aspects of this proposition, or yet different "ways of seeing" (Gottschalk, 2006, p.81). In this sense, mathematical investigation radically differs from empirical sciences, since there is no region to be discovered by means of proceedings of empirical verification, but only rules one *invents*. To make affirmations about the "discovery" of irrational numbers or of a "law" which foresees the sequence of prime numbers leads us to confusions in the sense that one believes in the existence of an objective reality outside language, which would gradually be discovered.

Also, in the school context, to expect a pupil to *discover* the rule of formation of their sequence by showing them only a finite amount of the first prime numbers, such as 2, 3, 5, 7, 11... and by asking them to continue the sequence, only reveals once more the equivocation of considering mathematical "laws" as analogous to empirical sciences. One could imagine that the pupil continues the sequence in different ways, according to distinct rules, which are all different from the rule the teacher expects. Why would this pupil act in an erroneous way? To have them continue the sequence *how* the teacher expects it, the teacher must train them in the rule of formation of prime numbers. By writing the next number of this sequence, one is simply following a practice where its rules were collectively established, in which one masters determined techniques (counting, the division algorithm...), with public criteria (instead of private ones) in order to evaluate if the continuation of the sequence is being done in the right way. In other words, to know how to continue the sequence presupposes a training (*Abrichtung*[15]) on following these different techniques; more than eventual intuitions or conjectures to be verified in situations of the pupil's daily life. In which sense could one then speak of *comprehension*? What does it mean to say, for example, that a student understood a mathematical problem?

> A schoolboy, equipped with the armoury of elementary trigonometry and asked to test the equation sin x = x – etc.[16], simply wouldn't find what he needs to tackle the problem. If the teacher nevertheless expects a solution from him, he's assuming that the multiplicity of the

[14] *Apud* Waismann, F. *Wittgenstein y el círculo de Viena* [*Wittgenstein and the Vienna Circle*]. México: Fondo de Cultura Económica [Foundation for Economic Culture], 1973.
[15] This term is translated as *drill*, and it is used by Wittgenstein in the sense of training animals. According to him, we are inserted in language games through this type of training. Cf. Stickney (2008), Gottschalk (2013).
[16] Wittgenstein refers to the development of the series x – (x^3/ 3!) + ...

> syntax which such a solution presupposes is in some way or other present in a different form in the schoolboy's head – present in such a way that the schoolboy sees the symbolism and now translates the rest from an unwritten symbolism and now translates the rest from an unwritten into a written form.
> The system of rules determining a calculus thereby determines the 'meaning' of its signs too. Put more strictly: The form and the rules of syntax are equivalent. So if I change the rules – seemingly supplement them, say – then I change the form, the meaning.
> I cannot draw the limits of my world, but I can draw limits within my world. I cannot ask whether the proposition *p* belongs to the system *S*, but I can ask whether it belongs to the part *s* of *S*.
> (…)
> The schoolboy that lacked the equipment for answering the second question, couldn't merely not answer it, he couldn't even understand it. (PR, XIII, 152)

In the previous example given by Wittgenstein, if the child does not yet have at his/her disposal the rules which regulate the second side of the equation, they simply cannot operate with the signs presented to him/her as equivalent to the first side of the equation. Thus, in this Wittgensteinian perspective, *to understand* is not something mental (in which the child's mathematical potential would manifest itself in some way), but it comes closer to the idea of mastering techniques, which are of *conventional* nature, therefore a equipment that must be acquired through *training*. Besides, a problem such as this would actually be a problem only if the question had a sense (when there is a way in the system in which the question is put). In other words, there is comprehension if the child is capable of using determined mathematical concepts to solve it, in the sense of being able to follow the 'laws of formation" of mathematical concepts used on both parts of the extended system.

6. Some therapeutic results in mathematics teaching

On the basis of these remarks made by Wittgenstein on the nature of the mathematical knowledge, one can extract some "therapeutic" results, with the hope that they will help to avoid confusion in pedagogical practices, such as the belief that there would be a natural evolution of the child's daily knowledge towards mathematical concepts, or that mathematics can be extracted from empirical experimentations. From the perspective of language as constitutive of meanings, in spite of its descriptive uses, one concludes that, in the same way that there is not one method (unique and natural) which is based on premises and gets to a determined mathematical proposition, neither there is not either a natural way based on the child's previous knowledge (cultural knowledge) and which leads to institutionalised knowledge. Its propositions only make sense within an *a priori* system. To apply them to other contexts is to know how to relate them to empirical facts, it is to know how *to use them,* which does not mean that they are abstracted from experience.

The Platonic conception of mathematics criticised by Wittgenstein is not restricted to mathematicians represented by Hardy's voice, who emphatically postulated a mathematical reality independently of human rules or use. It has deeply penetrated pedagogical practices and the confusion has multiplied because no attention was paid to the distinct uses of the mathematical concepts in regards to those of other domains of knowledge. For example, by teaching a pupil how to "divide" two numbers or how to "draw a straight line through two points", the teacher is simply teaching new rules and is not leading the pupil to discover new objects in an ideal pre-existent world. There is no way to have the pupil discover on his/her own rules in "situations of apprenticeship",

to the extent that they themselves are the ultimate basis which organise the empirical space and not vice versa (*grammatical conventions cannot be justified by describing what is represented*). Besides, the word "to divide" may have various meanings; it can also be used in empirical situations when one divides objects among people, as in the example of the division of the 11 apples. Nevertheless, in an empirical situation, one considers it right to say that Mr. So and So got 2 apples and the other one, who was hungrier, accepted 4 apples and there were still 5 apples left. Nonetheless, it is not this division the teacher expects the pupil to do, since in mathematics, to divide presupposes specific proceedings, algorithms which say *how* to do it: "Tell me *how* you're searching and I'll tell you *what* you're searching for" (PR, III, 32). The proposition $11 \div 3 = 3$ is empty of meaning; it begins to make sense with the *use* one makes of it in the context of mathematics (Wittgenstein, 1998). And this use must be learnt, as it is a *conventional* use.
This intended natural rationality which would be the basis for the construction of mathematical concepts, as some educators defend it, such as, for example, in cognitivist constructivism of Piagetian inspiration (or even in the approach of competence pedagogy, in which one presupposes mental operations, which must still be unveiled by cognitive sciences) is questionable from Wittgenstein's linguistic point of view. There is no *natural* way to get to the concept of irrational number, prime number or even to continue a mathematical sequence. According to Wittgenstein, there is no *a priori* existence of what mathematicians understand by irrational number, or by any other kind of number, but a theory of numbers (a "ruler") which allowed the creation of this new concept, or better yet, which *authorised* the establishment of a new internal connection[17]. In order to have the pupil use this concept, he/she must *learn* how to move in this new space, how to apply the new "ruler". It does not make sense to expect the pupil to *discover* new mathematical objects, as if there were a natural way leading to them.

Thus, the reflections and the examples Wittgenstein presents in the works considered above point out to a conception of mathematics which ruptures the conception in vigour that, to some extent, has guided the teacher's work in the classroom. By means of his philosophical therapy, the philosopher makes another view of the mathematical activity possible: as constructions of concepts and relations which are not descriptive of something *a priori* existent, be it in an ideal world in a third reign (Platonic) or the product of a development supposedly innate of cognitive structures. It is a *linguistic* activity, a work of language that results in *rules*, which are part of a larger system of rules, internally related to each other. It is there that one finds the mathematician's creative aspect: they invent new internal connections, which enable us to attribute meaning to the facts of the world in unusual ways. The different numbers they create form the syntax of the assertions about the empirical world, in the same way in which "Euclidian geometry is the syntax of assertions about objects in the Euclidian space. And these objects are not lines, planes and points, but *bodies*". Numbers, lines, planes and points are *rules* one follows in order to describe bodies in the space and to establish relationships between them; but if one considers these concepts as if they were descriptions of ideal objects, it only leads to confusion, including in the pedagogical field. When the teacher states that "Paul crossed the football field in a straight line from a certain point to another point" and asks the child to calculate the distance Paul covered, the concepts used as "straight line" and "point" are part of the Euclidean

[17] Wittgenstein uses the metaphor of the ruler in order to show the paradigmatic character of the propositions of mathematics, which play a role analogous to our measurement patterns.

geometry (syntax), rules that one follows in order to give the correct answer in *this* language game; but if someone were describing the path of Paul in another geometry in which space were curved, this same statement would be simply meaningless. Thus, these concepts are meaningful only within a given grammar; they do not describe ideal objects or what is "really" going on, but only allow the description of empirical facts from a certain perspective. One could still imagine that the child could follow some other rules to solve the problem, for example, that Paul went in straight line south and then east, measuring both lines. Why should it be wrong?

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In conclusion, the question is not what has priority, "to know what" or "to know how": the symbols themselves are empty of meaning. It in is the *use* we make of them (normative or descriptive), intrinsically linked to different activities in specific contexts, that they acquire life. "Experience decides if a proposition is true or false, but not its meaning" (PR, III, 23). This is a Wittgensteinian maxim which could also be one of Rousseau's maxims, if the Genevan philosopher had taken his own reflections on language more seriously by proposing his negative education, where human reason should be developed in a natural way, derived from observations and experimentations. Actually, as Rousseau pointed out in *Emile*, the human reason is not innate, but acquired in our forms of life and expressed in different tongues. However, if we *look* and not think (Wittgenstein, PI, §66) we will see that there is not a unique rationality guiding our actions, common to all the tongues, but a multiplicity of them, effective and possible. Part of them is constituted inside the occidental mathematics systems, when one *uses* mathematical propositions normatively to infer from empirical propositions to others or applies mathematical concepts to describe facts of the world. This is *one* of the possible *conventional* ways to meaningfully organize experience, in which learning occurs primarily through a workout that involves the mastery of different techniques, regardless of what happens inside the brain of the child or in the empirical world. As Wittgenstein (1967, §419) emphasizes: "Any explanation has its foundations in training. (Educators ought to remember this.)".

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