CONCEPTUAL NEXUSES OF ALGEBRAIC KNOWLEDGE: A STUDY FROM THE STARTING POINT OF THE HISTORIC AND LOGICAL MOVEMENT

NEXOS CONCEITUAIS DO CONHECIMENTO ALGÉBRICO: UM ESTUDO A PARTIR DO MOVIMENTO HISTÓRICO E LÓGICO

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ABSTRACT

The objective of this article is to present a reflection on what we have called conceptual nexuses of the algebraic knowledge from the historic and logical movement of the concepts. A conceptual nexus is defined as a link between the historic and logical forms of understanding the concept, which do not necessarily coincide with the different languages that express the concept. This study, of a theoretical character, was carried out by taking as singular some episodes highlighted in the history of algebra and acknowledging in their particularities (manifestations of language, thought forms, etc.) the conceptual nexuses that configure algebraic thought: the relationship among variable magnitudes in general. It was assumed that the knowledge of the historic and logical movement of algebraic concepts and its theoretical thought forms generate implications for the constitution of algebra's teaching object and for the organization of teaching, thus contributing to overcome epistemological impasses in the appropriation of knowledge both by those who teach and those who learn.

Key words: historic and logical movement, concepts; theoretical thought; algebraic knowledge; algebraic thought.

RESUMO

Este artigo tem como objetivo apresentar reflexões sobre o que temos denominado de nexos conceituais do conhecimento algébrico a partir do movimento histórico e lógico dos conceitos. Define-se nexo conceitual como elo existente entre as formas de pensar o conceito, históricas e lógicas, as quais não necessariamente, coincidem, com as diferentes linguagens que expressam o conceito. O estudo, de cunho teórico foi realizado tomando-se como singular alguns episódios evidenciados na história da álgebra e reconhecendo em suas particularidades (manifestações da linguagem, de formas de pensamento etc.) os nexos conceituais que configuram o pensamento algébrico: a relação entre as grandezas variáveis de forma geral. Adotou-se como pressuposto que o conhecimento do movimento histórico e lógico dos conceitos algébricos e suas formas de pensamento teóricas geram implicações para a constituição do objeto de ensino da álgebra e para a organização do ensino, contribuindo, assim, para a superação de impasses epistemológicos na apropriação do conhecimento tanto por aqueles que ensinam, quanto por aqueles que aprendem.

Palavras-chave: Movimento histórico e lógico; conceitos; pensamento teórico; conhecimento algébrico; pensamento algébrico.

1. Introduction

Those who teach mathematics in the basic education level often ask whether it is possible to organize the teaching of algebra in a way that it is not presented as a set of technical procedures (resolution of equations, development of expressions, etc.) devoid of meaning.

When thinking about this question with a view to a breaking up that prioritizes a final and symbolic stage of algebra, it is possible to stand up for considering algebra as a field of knowledge stemming from the human historic and cultural experience, constituted on the basis of the historic and logical movement of its concepts, and not only of its contemporary applications and its more formalized aspect.

The understanding of this movement becomes, then, a principle for the constitution of algebra's teaching object, in a way that takes into account the essence and the conceptual nexuses of this form of knowledge, and not only its formal definitions and technical aspects (Panossian, 2014).

It must be stressed that the understanding of the historic and logical movements of algebraic concepts requires the study of the history of Mathematics (more specifically the history of Algebra). Nevertheless, the main objective of this study is not to organize teaching actions that show to the students the chief facts, dates or subjects of history. This is a study that, as it was carried out by teachers, allows them to understand the movement of the constitution of algebraic concepts in a theoretical way, recognizing their movement, cause and effect elements, form and contents of the algebraic knowledge, among other conceptual nexuses.

Thus, this article is developed by explaining what is understood by historic and logical movement of the concepts through dialectic materialism as a theory of knowledge and its relationship with theoretical forms of thought. The conceptual nexuses of algebra recognized from their historic and logical movement are presented as a particular case identifying possible implications of this study for the organization of teaching.

2. Historic and logical movement of concepts

In order to understand the historic and logical movement of concepts it is necessary to resort to the contribution of authors who dealt with the theory of knowledge and with materialistic dialectics (Cheptulin, 1982, Kopnin, 1978; Kosik, 1976). The historic movement of a phenomenon, an object or a concept is reflected by the forms of thought (logical movement). For Kopnin, “what is logical reflects not only the history of the object itself but also the history of its knowledge” (1978, p. 186), and the unity of what is historical and what is logical becomes a premise for the understanding of the essence of an object, of a concept in its structures, its history, its development.

> In this perspective, Kopnin (1978) and Davydov (1982) speak of internal nexuses that present themselves in theoretical thought. Internal nexuses are different from external nexuses. The latter are limited to the perceptible elements of the concept, while the former make up the logical-historical of the concept. External nexuses are dealt with by language. They are formal. (…) External nexuses are also a language of communication of the concept presented in its formal status, but do not necessarily denote its history. They afford little mobility for the subject to elaborate the concept. […] Theoretical thought generalizes the concept. The proof is, for example, to learn algebraic concepts only through the representation of the letter x. Students continually ask: *after all, what is the exact magnitude that I can replace when the letter x comes up in the problem?* (Sousa, 2014, p. 65-66).

Thus, for the understanding of the relationship between the historic and logical movement of concepts on the one hand and teaching on the other, it is necessary to see that, when we mention the historic and logical movement of concepts, we mean initially the process of objectification in the form of instruments (material or psychic) of the historic experience of mankind. And when we deal with single individuals is made reference to the process of appropriation of concepts, taking into consideration that:

> Concepts, historically shaped in society, exist objectively in the forms of activity of man and his results, that is, in the objects created in a rational way. Single individuals (and specially children) capture and assimilate them before they learn to act with its peculiar empirical manifestations. The individual must act and produce things according to the concepts that already exist in society as norm; he did not create them, but captures, appropriates them. Only then he deals with things in a human way. (Davidov, 1988, p.128).

The principles of historic-cultural theory and theory of activity contribute with the understanding of the process of appropriation of knowledge and the psychic development of subjects. When in activity, characterized by Leontiev (1983) as a psychological process that satisfies a need of man in his relationship with the world, the subject is able to appropriate the meaning of concepts and instruments as a product of human construction, generating his own scientific development. To appropriate a concept means, therefore, to understand the meaning that individuals have historically given to it, which can be done in activity, over the objects, instruments and phenomena in themselves or carried out by the action and communication developed through them.

In order to express the historic and logical movement of concepts, one assumes the study of the categories of dialectic materialism considering that “[…] matter is an objective reality, existing outside and independently from consciousness […]” (Cheptulin, 1982, p. 68), given to man through his senses.
Two other essential characteristics of reality are also considered: interdependence and fluency (Caraça, 1952). Interdependence means that all things, objects and phenomena of objective reality are linked to each other; fluency means that the world is in permanent evolution and therefore all things, objects and phenomena of reality are in movement and in a permanent process of change.

According to Davydov (1982) when the nexuses, or rather the relationship of interdependence among objects and phenomena is captured, in this constant movement of transformations and changes of the objects within an integrated system, one finds oneself before a theoretical thought. The latter seeks the “mastery of objectively inter-related phenomena that constitute an integrated system” (p. 306), as its main function is to “[…] clarify the essence of the object as a general law of its development” (p. 310).

In the following section an attempt is made to highlight singular moments and particularities identified in the historic and logical movement of algebraic concepts, describing the phenomenic aspects by means of records of the history of algebra and the recognized conceptual nexuses that reveal the essence of algebraic knowledge”: the relationship between variable magnitudes in a general way.

3. The historic and logical movement of algebraic concepts

In order to understand the historic and logical movement of algebraic concepts one must take as the starting point of the analysis some singular episodes of the history of algebraic knowledge. These episodes were investigated from secondary sources, authors who write about the history of mathematics, following its facts and beginning an analysis about them, for instance Baumgart (1992), Caraça (1952), Eves (1995), Klein (1992), Radford (2011), among others.

Considering that the historic and logical movement cannot be confused with the history of the object chronologically oriented, the intent here is to understand the movement of formation of algebraic concepts and changes that have occurred and that reveal themselves historically in their internal nexuses and their theoretical relationship.

The adopted methodological movement assumes as singular the records of the history of algebra and as particular the forms of thought, their manifestations in language, and the processes of formation of concepts analyzed by means of categories of dialectic materialism, in search of showing some changes that occurred in algebraic knowledge.

The starting point is the understanding that algebraic knowledge, just as other forms of knowledge, is a product of human activity (Leontiev, 1983). Thus, one seeks the understanding of the human needs that triggered the emergence of algebra, as well as the actions, operations and conditions that marked each stage of its development. In this connection, it is necessary to understand how algebra constitutes itself as scientific knowledge.

It must be stressed that in relation to algebraic knowledge, as well as with other forms of knowledge, there is not “one” exact moment of objectification and elaboration of a given concept, but rather a movement that starts with human activity and progresses in ever more complex levels of generalization. Thus, for Dantzig (1970), algebra deals with operations under symbolic forms, and considers that “[…] algebra is as ancient as human ability to deal with general propositions: as old as the ability to discriminate between some and any. (p. 77).

In his research, Sousa (2004) analyzed the relationship between the knowledge of teachers and algebraic concepts. He developed among teachers teaching activities that considered the conceptual nexuses present in symbolic and non-symbolic algebras and recognized that “there is a lack of knowledge of the conceptual nexuses that make up symbolic algebra; the concepts of fluency, of variable, of field of variation and of non-symbolic algebra” (Sousa, 2004, p. 28).

Next, for explanatory purposes, we separated some conceptual nexuses revealed through the analysis of the historic and logical movement of the concepts, but it is important to stress that it is understood that they are all interlinked.

3.1 The recognition of magnitudes

Since ancient times men try to understand the physical, social, economic and political phenomena around them in order to attempt to master Nature. It is the search for the understanding of the causes and effects of reality.

According to the foundations of dialectic materialism, the category cause is associated with the interaction of objects and phenomena that provoke changes in it. The concept of effect, for its part, is linked to the changes made in objects and phenomena as a function of their interaction. "What engenders the other and conditions its emergence is reflected in the concept of cause; what is engendered and conditioned is reflected in the concept of effect" (Cheptulin, 1982, p. 126). It is understood that the causes of the changes in objects and phenomena should be looked for not only in external actions, but in their internal nature.

An initial question that is brought before men is the existence of a single principle that governs the diversity and plurality of the universal objects and phenomena. The first thinkers endeavored then to understand whether this single principle of nature existed (Caraça, 1952). The first Ionian answers led to an explanation based on a main substance. For Thales it was water, for Anaximenes it was air. "[…] for Heraclitus the essential aspect of reality is the transformation constantly undergone by things through the action of fire" (Caraça, 1952, p. 67).

Heraclitus' explanation was not based on the stability of one element, but rather in the principle of tension of contrary ones, which triggers movement, breaking up one balance and building another.

Pythagoras' answer was different from previous ones. He believed that all the understanding of the universe was based on the relation among numbers. Everything is number. Particular stress is given here to the identification of sequences of numbers which, arranged in points, formed geometrical figures originating triangular and pentagonal numbers. Not to mention Pythagoras' theorem, but it is precisely through it that the Pythagorean mathematical organization of the universe is attacked when it faces the problem of immeasurability and even more with Zenon's arguments. The difficulties raised by the problem of immeasurability might then be resolved through the study of the infinite and movement, and that the straight line cannot be thought about through the juxtaposition of monads, but in its continuity. It was thus impossible for the Pythagoreans, through the knowledge they possessed, to control the movement of quantities of some magnitudes, for example, the diagonal of the square in relation to the side.

In this constant movement of the phenomena of objective reality and with the human objectives and action, there arises the need to recognize what we today call magnitudes.

The quality of an object is understood through magnitude. And one can understand that the changes that occur in the mathematical knowledge make possible to assign quality to a quantity.

The categories of quality and quantity stand out. For Cheptulin (1982), the changes from a qualitative state to another, called "leaps, can be of two different types: the leaps that produce rupture and change the fundamental quality of the object or phenomenon, modifying its essence; and the leaps that develop by gradually accumulating elements of a new quality or modifying qualities that are not the fundamental ones of the object or phenomenon. The first kind of leap represents the revolutionary form of qualitative changes and the second kind represents the evolutionary form" (p. 218).

The dialectic pair quantity/quality is also found in Caraça (1952). This author considers that the study of the laws and phenomena of the objective reality, whose main features are fluency and interdependence, are only possible if they are defined in isolation, that is, cuttings from totality which possess components that relate to each other independently and allow for the study of the object or phenomenon from which the cutting was made. Thus, intrinsic qualities of an object or phenomenon do not exist, but these are considered in relation to another object or phenomenon. If different degrees of intensity can be assigned to these qualities (more than, less than, greater than, and others) then they admit variation according to the quality.

Therefore, in this article magnitude is understood as the quality of an object that can be quantified, in the sense given by Caraça (1952). For example, the concept of number attained at the time of the Pythagorean philosophers did not permit the control of a magnitude (in this case the dimension of the diagonal of the square) since it was not possible to establish the relationships between this diagonal and the side of the square.

Already in the 14th and 15th centuries movement was one of the key subjects of natural philosophy (Roque, 2013). However, movement was determined by one quality and this was understood as an essential property of a body; thus, for instance, for medieval thinkers, speed was not dissociated from movement and could not be treated as a magnitude, but rather as the attribute of a body. Nicolas Oresme, a French thinker of the 16th century, opposed this notion and pointed out the intensity of a quality, for instance, the notion that a body is not cold in itself but may be more or less cold, and this stresses the quality assigned to the quality. In the case of speed, he considered it as a quality related to space and time and is thus close to the concept of quality as presented by Caraça (1952).

In the Renaissance, when trade expanded and sciences developed rapidly, "[…] mathematics was instrumental to create new forms of understanding the world" (Radford, 2011, p. 234), a world that was completely dominated by numbers, proportions and measuring processes for everything. It is the time of Galileo and Leonardo da Vinci who are part of the development of a capitalistic society. Control over Nature is a necessity and becomes ever more complex.

Newton's assertion taking account of change in the conceptualization of mathematical magnitudes stands out:

> I do not consider mathematical magnitudes as formed by parts, however small, but as described by a continuous movement. Lines are not described and engendered by the juxtaposition of their parts, but rather by the continuous movement of points; surfaces by the movement of lines; solids by the movement of surfaces; angles by the rotation of sides; time by a continuous flow (Newton[1] apud Lacasta; Pascual, 1998, p. 28.29).

[1] NEWTON. Sir Isaac Newton's Two treatises of the quadrature of curves, p. 1

What Newton attempts to apprehend by means of magnitudes is movement, and he affirms: “I looked for a method to determine magnitudes starting from the velocities of movement or increase that engenders them” (p. 29).

To recognize the fluency of objects and phenomena of objective reality makes it possible to understand the infinite relationships and constant transformations of that reality. A consequence of this finding is to understand that the attributes of an object or phenomenon are always related to other objects and phenomena. Thus, what may be called magnitude of an object includes necessarily its relationship with other objects. For this reason, magnitude is understood here as the quality of an object (that may be quantified) in its relationship with others.

Thus, considering the fluency of objective reality, the need to recognize the magnitude of objects and phenomena stands out for the constitution of the conceptual nexuses of algebraic thought as well as the need to list and control them.

3.2 The movement of numeric fields to the control of quantities

The fluency of phenomena, which philosophers in different moments tried to understand and explain, are revealed in the resolution of everyday problems linked to the control of quantities for different peoples in different historic moments, for example Babylonians, Egyptian, Hindus, Chinese, Arab and others.

The control of quantities is seen as a fundamental element of Mathematics. Indo-Arab numerals that are used today are singular examples among many other symbols produced by a large part of mankind in different spaces and times, expressing the idea of number to carry out this control of quantities.

Through the record of the history of mathematics qualitative changes in knowledge are recognized to identify rupture leaps (that change the fundamental quality of objects or phenomena) and gradual leaps (that bring about changes in non-essential qualities of objects of phenomena) (Cheptulin, 1982).

The use of numeric symbols and the possibility of operation with them resolved a large part of the everyday problems of different peoples. New numeric fields were created in order to deal with this movement of control of quantities. This is what happens, for instance, with the need for the creation of numbers that can be represented in the form of a ratio, which progress gradually modifying the quality of the number, or else with the organization of a field of whole numbers in which the negative quantity acquires a meaning.

Regarding numeric fields, a qualitative leap of rupture can be considered, overcoming the impasse of the Pythagoreans about the qualitative aspects of things and their acknowledgement of the number as a universal essence (Cheptulin, 1982).

The Pythagoreans, then, considered that the numeric quantity (which they only knew in the natural and rational fields) defined all objects and phenomena. In this way they reduced things to countable properties and managed to compare them by means of the ration between these numbers. This ratio “expressed a relationship between numbers that was hidden in something and through that relationship such a thing could be described” (Roque, 2013, p. 47).

A qualitative rupture leap happens in the acknowledgement that not all objects can be described in numbers by means of discreet quantities and that, therefore, it is necessary to understand movement in some other way. According to Roque (2013), an important consequence of the discovery of the incommensurables is the separation of the universe of magnitudes from the universe of numbers. “If we do not know how to calculate all we can do is to show” (Roque, 2013, p. 47). The straight line, as a geometric element, was for a long time considered as a model of continuity.

This Pythagorean impasse brings about the need of a new understanding about numbers. However, the leap of rupture and the creation of another numeric field representing the opposition to the concept of rational number were only accepted by the scientific community much later, already in the 19th century, with the publications of Dedekind which looked for a formal definition of numeric continuity that is not linked to geometry.

> It so happened that when comparing the ensemble of rational numbers with the straight line, the ideal model of continuity, Dedekind created the concept of cut and through it he defined the rational and irrational numbers within the unity of a system, an ensemble continuous by its equivalence with the straight line, not formed by points, but by numbers. After the construction of the arithmetic continuity, the real straight line, geometry would be negated, considering the dialectic thought. Tus Dedekind could teach differential calculation with the formal coherence of the numeric field (Dias, 2007, p. 194).

By considering numbers as concepts directly related to the magnitudes of objects of phenomena (quantities of the sensitive concrete) the development of new numeric fields keeps its form limited to the numeral that may express the quantity of objects, its length, its volume., the quantity of time and others. For example, one can know a square whose side measures 5 (five) and determine its diagonal represented by $5\sqrt{2}$; however, by means of this representation the relationship between the magnitudes (diagonal and side of the square) is not expressed in a general form, but merely in a particular form for each situation.

In this way, one agrees with Sousa (2004, p. 66) when he says that “in considering the most general concept of number, algebraic thought cannot be related only to the physical and formal presence of the number: the numeral”; in algebraic thought it is necessary to think of the number without the numeral. For this reason, to understand the movement of numeric fields for the control of quantities is an element for the recognition of the conceptual nexuses of algebraic thought.

3.3 Form and content of algebraic knowledge

For a long time it was enough to resort to numbers in order to control quantities; however, everyday problems became ever more complex. The numeric symbol associated to rhetorical language becomes insufficient to control the movement of quantities and does not offer potentially possibilities of elaboration of new methods of solution for everyday problems and those within the realm of mathematical knowledge.

In this way, and in direct relationship to the algebraic knowledge, one can recognize leaps that, in an evolutionary and gradual manner, modify mathematic language in different moments of rhetorical, syncopated, geometric or symbolic algebra.

For instance, the rhetorical language of the Babylonians allowed them to solve their problems with peculiar methods that were developed with numeric symbols and with words of the natural language

(Baumgart, 1992). The most widely used method was what could today be called parametric, since it established two unknown terms from the relationship with a third term (the parameter).

But the restrictions in regard to language and known numeric fields require the creation of different modes of action of the solution of problems; for instance, the use of geometric resources to solve problems involving irrational numbers, which were not known at the time. Baumgart (1992) states that Greek algebra, which was essentially geometric, followed the same method of solution, translating problems in terms of segments of straight lines and areas illustrated by geometric figures. “For even if $\sqrt{2}$ cannot be expressed in terms of whole numbers or their ratios, it may be represented by a segment of a straight line that is precisely the diagonal of the unitary square” (p. 8).

These moments reveal modes of action of mankind to solve the problems that had arisen, that is, particular modes of action to solve specific situations within a given model.

In this movement of algebraic knowledge, the magnitudes involved in everyday problems are identified and listed. However, such relationships are particular, represented by means of numeric or geometric symbols and not aimed to reach a general expression of general modes of action to solve all problems, but rather to solve a group of problems that have common features.

Thus, starting from contemporary algebra, dealing with objects of different natures (for instance, matrixes and vectors) and permitting the enunciation of general situations, one may consider that Euclidean geometry did not contemplate generality in its enunciations and utilized particular geometric properties. In this way, evidences of an algebraic thought were not considered (Roque, 2013).

But for dialectic materialism the content of an object or phenomenon must be considered as a process of interaction between the elements that constitute it, and must consider the actions that this object or phenomenon provokes in others. Content cannot be confused with the interior, just as form should not be confused with the exterior. The category form reflects the structure of the content and penetrated the interior and exterior domain.

> The determining role in the content-form relationship is performed by the content. It determines form and its changes result in corresponding changes in form. By its turn, form reacts on content, contributes to its development or slows it down (Cheptulin, 1982, p. 268).

This is what happens to the movement of algebraic language, as form of the content of algebraic thought. The different forms of language attained in human experience (rhetorical, syncopated, geometric, symbolic) made possible limitations or progress in relation to algebraic content. In this context, language as a phenomenon constitutes a determining particularity for the constitution of algebra.

According to Cheptulin (1982) this dialectic relationship (form and content) makes possible qualitative leaps to the extent that form no longer corresponds to content and begins to repress it.

> The non-correspondence of form with the new content, as the latter develops, becomes ever more acute and finally a conflict between form and content erupts: the new content rejects the old form, destroys the relatively stable system of movement and based on a new relatively stable system of movement (the form) changes, reaching a new qualitative level (Cheptulin, 1982, p. 268).

In this connection, the existing interaction between content and form in the moment of rhetoric algebra is recognized, for instance. Algebraic thought and the way to reflect it in that historic moment provide impetus to the development of algebraic knowledge up to a certain stage. This interaction between content and form is different in the moments of syncopated or geometric algebra.

In the moment of rhetoric algebra, numbers and words were representations sufficient to express modes of action to solve specific problems, such as problems or area or perimeter (Baumgart, 1992). As everyday problems and those of mathematical science became more complex, the need to solve situations of quantity control in an ever more generalized way arose.

Diofanto, who also resorts to the modes of action of rhetorical algebra, brings about changes in the form of algebra, more that in its content. His studies generate conceptual changes by using abbreviations of words as symbolism; abbreviations are used as representations or indication of objects. In his book Arithmetics, Diofanto produces solutions for 130 problems but does not resort to general methods for their solution, but rather to ingenious stratagems that cater to needs of specific problems (Eves, 1995).

He used abbreviations of words as signals, for example arithmos, meaning a given set of given things (Klein, 1992, p. 131). On the other hand, says Roque, arithmos meant an indeterminate quantity, different from numbers that are formed by a given quantity of units, but on which the same properties of numbers were applied. Thus, one can understand that it dealt essentially with numeric magnitudes and not with other forms of magnitude.

With his syncopation, Diofanto succeeded in changing the representation of numeric magnitudes by means of abbreviations, but he could not find general methods of resolution of problems and his objective was not to recognize more than one solution for the problems he solved.

It is possible to say that at that historic moment there was a change in the form, but not in the content of knowledge. It should be stressed that in Diofanto's time it was possible to establish relationships between what would be today whole and rational numbers. Negative and irrational numbers were not yet object of consideration, and thus his arithmo meant at most the number of monads, or fractions. Therefore, Diofantos' records cannot encompass the possibility of a symbolic technique of calculation.

> From the standpoint of modern algebra, only one additional step was needed to make Diofanto's logistics a perfect one: the replacement of the "determined number" by "general" numeric expressions, from the symbolic to numeric values – a step that meant later a significant progress in the treatment of equations in general, finally taken by Viète (Klein, 1992, p. 139.)

Both Diofanto and Viète are known as "Father of Algebra", but this entails the adoption of a certain conception of algebra. Diofanto may be recognized under that nickname if we consider the progress in relation to the methods of solution and the systematization of equations through syncopated recording. However, the generality of such methods had not yet been reached. Viète, for his part, was given that nickname because of the introduction of symbolic records and the possibilities for generalization of his analytical art. More important than assigning that title to one of the other mathematician is the acknowledgement of the advancement and constant changes that are being made possible in algebraic knowledge, modifying the very conceptions of algebra in accordance with the development of human activity.

It is important to observe that at Diofanto's time the object of algebra was related to the resolution of problems and the production of solutions for them, and not necessarily to the definition of a general method of resolution. But the need to establish ever more general methods of resolution of problems whether of everyday or of internal interest to the mathematical science was increasingly characterized as one of the objects of algebraic knowledge. In the resolution of problems, it was no longer enough to recognize the magnitudes involved and to establish the particular numeric relationships, but rather to generate a general method that could solve the larger part of the problems.

Around the 12th and the 13th centuries, the time of the Italian Renaissance and of the creation of the institutions called Scuole d'Abaco, turned toward the formation of an elite to work in trading (Radford, 2011), the chief objective of algebraic knowledge was to develop techniques for solving a great number of problems, and the paths that would culminate in the Viète's specious logistics were beginning to be trod (according to the next section).

It is important to point out that, according to Puig and Rojano (2004), the division of algebra in stages of evolution called rhetorical, syncopated and symbolic, to which this article makes reference, was carried out in the middle of the 19th century by Nesselman[2], starting from its language. The algebra of the Sumerians, Babylonians, and Greeks are examples of the moment of rhetorical algebra, in which words are used as a resort to express the details of the calculation. The stage of syncopated algebra keeps the same nature of the moment of rhetorical algebra, but abbreviations of words are used to represent the calculations carried out. The third stage, known as symbolic, envisages the possibility that a system of signs represent all forms and operations; what is essential, more than the use of symbols, is the possibility to operate with them without making reference to the concrete objects or explanations in form of words.
This division can be questioned after the historic studies, not only because it does not take geometric algebra into consideration but also because it does not highlight the thought processes of algebraic knowledge by emphasizing its forms of representation, as Roque (2013) summarized. "In order to characterize algebraic thought it is not enough to associate it to the use of symbols, and even less to the use of abbreviations" (p.112).

Puig and Rojano (2004) also show that it is not enough to follow the development of the history of algebra only through its language and symbolism; it is necessary to know its methods and thought forms. For this, they present processes of problem resolution with different languages (rhetoric and syncopated), which with regard to the form of thought run into the same obstacle: the difficulty of carrying out operations with unknowns.

On the other hand, Radford (2011) shows that in the book Liber Abaci, by Pisano, the use of rules of restoration, combination and transposition of terms allow for the carrying out of operations with unknowns, but the difficulties generated by the use of rhetorical language are kept. However, it is important to point out that the algebra that was being developed in the Italian Middle Ages permits the solution of different problems using the same technique.

Puig and Rojano (2004) stress the passage from the syncopated to the symbolic in Viète, through relationships and calculations established between what is called species or form of things:

[2] NESSELMAN, G.H.F., **Versuch einer kritischen geschichte der algebra, 1 teil, Die Algebra der Griechen,** [Essay on a critical history of algebra. 1st part, The algebra of the Greeks], Berlin: G. Reimer, 1842.

> The transition from syncopated to symbolic algebra starts with Viète, for whom the specious logistics, the analytical art which he wished to call by this name more rather than by algebra was the calculus with the species, or forma rerum (form of things). But to represent this calculus with species, Viète developed symbolic expressions in which what is represented by letter is not the species, but the known and unknown quantities (Puig and Rojano, 2004, p. 207).

By presenting the form and content of algebraic knowledge through these historic records, it is understood that another conceptual nexus that is part of the algebraic thought can be revealed.

3.4 The acknowledgement of variable magnitudes

Another fundamental conceptual nexus for algebraic thought is the understanding of variation. Between the unknown element and the one that varies, the acknowledgement of variable magnitudes was undergoing transformations as we follow the historic and logical movement of algebraic concepts.

Considering the numeric and the geometric fields that stand out in ancient civilizations, Radford demonstrates in his historical investigations that in both fields proportional reasoning was used as a means for the resolution of problems. This form of reasoning was very developed in mathematical thought and the methods of false positioning were used in a more sophisticated way.

In the method of false positioning, one takes a number as a false solution in order to solve the problem. The differences found between the result attained through the false solution and the result that should have been attained is treated proportionally in order to find the exact solution.

> This peculiar numerical choice of an unknown seems to have permitted scribes to systematize a numerical method for the solution of problems and for this reason to reach an important step with regard to the conceptual development of the ancient proportional reasoning (Radford, 2011, p.123).

However, one can find an algebraic general mode of resolution of problems that fits the context of the problem only when those unknowns (unknown values) are represented by a name or symbol that fits the context of the problem (for example, length, width and others) and identifies the proper magnitude, that is, the unknown that is exactly searched (and not a false number).

Many of the problems presented in Diofanto's Arithmetic are related to the methods of false positioning for the resolution of problems. However, the old numeric problems that dealt with palpable and concrete quantities are transformed into problems about abstract magnitudes. The term arithmos encompassed the concept of unknown and replaces particular terms such as length and width used by the ancient scribes. It is a more general concept. "In function of this generality, this concept can be applied to a great variety of situations. Arithmos thus became a genuine algebraic symbol" (Radford, 2011, p. 146) but it possesses a generality that is more related to mathematical objects than to methods pf resolution.

Through Diofanto's book, On Polygonal Numbers, Radford (1996) considers that there is evidence that the concept of variable emerges in this text associated to the formula and not to the function. "His concept of formula is not based on the continuous flow of quantities but in: (a) an explicit relationship between numbers that are seen as monads (that is, unities), or fractional parts of monads, and (b) an explicit sequence of calculations permitting the determination of a given number to the identity of another number". (Radford 1996, p. 50).

Radford (1996) indicates that one of the chief differences between unknowns and variables may be in relation to context, the objective and the proposed intention. Thus, the situation can be related to the resolution of a problem, and in this case it is necessary to find an unknown value; therefore, one has an unknown. Or else, in another situation of establishing the relationship between magnitudes in a general way, it is necessary consider that they vary and this situation refers to the variable.

Still through the study of Diofanto's texts, Radford (1996) indicates another difference between unknowns and variables related to their representation. In On Polygonal Numbers he considers that the concept is an abstract one and can be represented geometrically and by letters, and in Arithmetica the concept is unknown (Arithmos) and cannot be represented geometrically.

It is understood that Viète, who, with his specious logistics, permits the passage from syncopated to symbolic algebra, performed the quality leap in relation to the manifestation of language and form of thought. The fact of assigning letters to unknown and also to known values of the equation, which today can be understood as parameters, was very helpful to the development of algebra.

Viète's intention (2006) with his The Analytic Art was to solve all problems, which he does through modes of analysis (zethetic, porisitc and exegetic).

> It is in fact through zethetics that an equation or proportion between a term that is found and the given terms; poristic, through which the truth of a declared theorem is tested by means of an equation of proportion, and exegetic, through which the value of the unknown term in a given equation or proportion is determined. Thus, all the analytical art, assuming these three functions by themselves, can be called science of the correct discovery in mathematics (Viète 2006, p. 12).

For Viète, algebra was a means of symbolic calculation involving magnitudes in an abstract form, and he manipulated magnitudes independently from their nature by creating symbolic calculation procedures that could be applied both to geometric and numeric magnitudes. A single symbol could represent all types of magnitudes.

Viète's logic is called specious, the logic of magnitudes "in species" in which he used letters to represent symbolically abstract magnitudes. The symbols used by Viète may express the relationship between quantities in a syntactic way, without necessarily resorting to the interpretation of a particular problem.

However, it is not only the use of symbolism what guarantees the establishment of relationships between magnitudes. It is necessary to understand the meanings assigned to the symbols, not the particular meanings linked to specific problems, but rather the universal meanings that these symbols take.

The device used by Viète was to assign vowels to known magnitudes and consonants to the unknown ones. But this vowel-consonant notation did not live long and was replaced by the notation proposed by Descartes, who used the first letters of the alphabet for the given quantities and the last letters for the unknown ones, which is predominant still today (Dantzig, 1970).

As currently used, the symbols exist independently from the concrete object they represent. The role of symbols in algebra is to permit progress in the form of elaboration of new objects of thought. Symbols in algebra make possible the representation of scale magnitudes (distance, time, length, mass and others) associated to a numeric intensity, and also the representation of vector magnitudes (force, acceleration and others) which require, besides numeric intensity, also the representation of direction and guidance. They are also related to the composition of new mathematical structures, matrixes, rings, bodies and others, on which axioms and properties are established.

Symbols permit working with the relationship between magnitudes, without their association to numeric, geometric entities or entities of another kind, which stands out as essential for algebraic knowledge by showing that that general method allows the resolution of particular case.

Viète's thinking can be understood as a theoretical form of thinking that realizes abstractions (that were already being realized in the process of human development) and makes the synthesis. That is, it identifies a fundamental relationship that is taken as universal and applies it to what is particular, in this case to numeric and geometric records, showing that such general method permits the resolution of particular cases.

The assignment of a symbol and the recognition of the variable as an element that permits the establishment of the relationship between two magnitudes may be considered as a qualitative change and a leap of rupture that modifies the essence of algebraic knowledge and makes possible another form of movement and development.

3.5 The need for the generalization of mathematical objects and methods

Just like other human thought processes, the process of algebraic generalization does not unfold in a way unlinked from human practices; it is connected to the conditions of the time in which it takes place. Thus, for instance, the generalization made possible in Euclid's time did not develop in the same way as in Viète's time, since it could count upon the experience historically accumulated by Euclid's time. Each one of them developed knowledge in his own time starting from what potentially could be found in their own objective reality.

The processes of generalization present in the moments of rhetoric, syncopated and symbolic algebra can also be recognized.

At the moment of rhetoric algebra, there were processes of generalization of specific modes of action to solve problems of everyday life. At the time of syncopated algebra, the use of abbreviations permitted advancements in processes of generalization of those procedures, still linked to numbers or geometric elements. The introduction of abbreviations and the use of the term arithmos by Diofanto allowed for the generalization of mathematical objects, but not its methods. In the development of geometric algebra, of which Euclid is the main representative, segments are used to express the relationships between quantities.

In turn, the meaning assigned to the symbolic recourse by Viète created theoretical conditions for generalization by associating symbols to known and unknown quantities, regardless of their species, and permitted the establishment of the relationships between quantities in an abstract way. Only starting from that understanding was it possible to open the field for the understanding of the variable.

Taking into consideration this cultural and historic movement, one is faced with the need to understand that the development of the process of generalization depends on each time and on each social context.

The notion of generalization thus acquires a new quality, shifting between the generalization of objects and that of methods. For example, the generalization that could be attained by the abacists at the time of the Renaissance deals with concrete objects stemming from trade as a social practice and permitted the establishment of a general rule for a number of particular cases.

> It is a generality "horizontally" organized, expanding as the increasing technical difficulties of the axis corresponding to methodology are overcome. The general and the particular are connected by links that emerged where there is an intersection of the axis of the formulation of problems and the methods used to solve them (Radford, 2011, p.170)

The generalizations of algebra made over mathematical objects and starting from different methods for the solution of problems generated progress in concept of number (from natural to rational, negative, complex and irrational). Beyond the concept of number as a mathematic object, the algebraic concepts permit generalizations about other objects such as matrixes and vectors and about created algebraic structures such as rings and bodies, establishing relationships between these objects, general properties, theorems and others.

The importance of the process of generalization stands out as an element that establishes the internal conceptual nexuses of algebraic knowledge. One agrees with Radford (1999, p. 7) in that "Generalization is not a mere act of abstraction stemming from the concrete; in fact, generalization keeps a genetic link with the concrete in accordance with the mediated system of the activity of individuals and their symbolic and epistemic structure".

4. The essence of algebraic knowledge revealed by the historic and logical movement of concepts: implications for teaching

As mathematical knowledge, algebra constitutes itself and acquires meaning in what it offers regarding possibilities for the interpretation of the fluency of objective reality. Just as in other scientific fields, as Caraça (1952) states, its role is to create pictures that interpret reality. As scientific knowledge, its objective is to create a picture that interprets a moving, flowing reality and in this sense it is characterized by the search of the quantitative relationship between variable magnitudes in a general way; this is its essential theoretical relationship, or cell. The forms through which this essence, or cell, develops change according to the historically accumulated experience.

In order to understand such theoretical relationship (essence) the study of the historic and logical movement of the concepts was carried out from the starting point of the categories of dialectic materialism, considering that human consciousness is linked to "[…] some structural formations of the brain and to some forms of interaction among men, between themselves and Nature, and some forms of their activity" (Cheptulin, 1982, p. 92). It also considered that practical activity as a basis for knowledge reflects specifically the stage of human development, the forms of relationship of men among themselves and of men with Natura in a given period of the historical development of society.

In this way, it is understood that both the process of objectification, that is, of assignment of meaning to objects and phenomena, and the process of appropriation by each individual as he learns to be a man are conditioned by the historical processes of human formation, which justifies the need to understand the historic and logical movement of concepts.

We endeavored in particular to recognize the conceptual nexuses of algebraic knowledge through historic episodes of algebra, by highlighting: the recognition of magnitudes; the movement of numeric fields for the control of quantities; form and content of algebraic knowledge; the recognition of variable magnitudes; the generalization of mathematical objects and methods. Starting from the understanding of these conceptual nexuses, the establishment of a quantitative relationship between the variable magnitudes in general was considered as a stable relationship (essence, general entity or cell) of algebraic knowledge.

It should also be stressed that the understanding of the historic and logical movement of algebraic knowledge, or mathematical knowledge in general, is not restricted to presenting historic records, facts and dates that mark a given event, neither does it intend to establish a parallel between the movement of elaboration of the concept by the subject and the development of the concept in human historical experience. Rather, it intends to identify what is objectified in relation to knowledge and is characterized as an essential theoretical relationship to be appropriated by future generations with reference to knowledge, and to algebraic knowledge in the particular case of this article.

The understanding of this movement of algebraic knowledge, of its conceptual nexuses as well as the awareness of what can be considered as the essence of algebraic knowledge generates implications for teaching. The appropriation of this knowledge permits that structuring of the elaboration of curricular programs, the formation of teachers and the organization of teaching activities from the starting point of a conception of algebra and of the teaching of algebra that takes into account its logic and historic development. Some results of studies and research already carried out by the Group of Study and Research on Pedagogic Activity (GEPAPe) reveal the deepening and the implications for teaching with regard to the historic and logical movement of algebraic concepts (Panossian, Piovezan &Moura, 2013; Sousa, Panossian & Cedro, 2014), as well as implications for the formation of teachers (Moura, 2010; Moretti, 2014; Sousa, 2015).

The review of the constitution of algebra's teaching object by taking into consideration the historical and logical movement of algebraic concepts makes it possible to organize teaching actions that allow the students to recognize the human actions and activities that generated this form of knowledge and thus to recognize the motives and need that led to the conceptual synthesis, for example, of variable, function, series, etc.

5. References

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