DESIGNING AESTHETIC EXPERIENCES FOR YOUNG MATHEMATICIANS: A MODEL FOR MATHEMATICS EDUCATION REFORM

CRIANDO EXPERIÊNCIAS ESTÉTICAS PARA JOVENS MATEMÁTICOS: UM MODELO PARA REFORMA DA EDUCAÇÃO MATEMÁTICA

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ABSTRACT

Although reform is typically associated with change on a grand, pervasive scale, our model is much less intrusive pedagogically. We are not seeking a revolution in mathematics education, but a strategic focus on mathematics worthy of attention, worthy of conversation, worthy of children's incredible minds, which thirst for knowledge and for opportunities to explore, question, flex their imagination, discover, discuss and share their learning. We believe that occasional, well-designed aesthetic mathematics experiences "that are immersive, infused with meaning, and felt as coherent and complete" (Parrish, 2009, p.511), and the associated experience of complex, surprising, emotionally engaging, and viscerally pleasing mathematics, can serve as "a process of enculturation" (Brown, Collins and Duguid, 1989, p. 33) with lasting impact on students' (and teachers') dispositions, living fruitfully in future experiences (Dewey, 1938), by raising expectation and anticipation of what mathematics can offer, and what the intellectual, emotional and visceral rewards might be when quenching a thirst for mathematics.

Keywords: aesthetic experience, elementary mathematics, reform

RESUMO

Embora reforma seja tipicamente associada com uma mudança generalizada em grande escala, o nosso modelo é muito menos intrusivo pedagogicamente. Nós não estamos buscando uma revolução na Educação Matemática, mas um foco estratégico em Matemática que seja digna de atenção, digna de discussão, digna das mentes incríveis das crianças, que têm sede de conhecimento e de oportunidades para explorar, questionar, flexionar sua imaginação, descobrir, discutir e compartilhar sua aprendizagem. Acreditamos que experiências matemáticas estéticas ocasionais e bem desenhadas, "que são envolventes, infundidas com significado, e sentidas como coerentes e completas" (Parrish, 2009, p.511), e a experiência associada de uma Matemática complexa, surpreendente, emocionalmente envolvente e agradável visceralmente, podem servir como "um processo de enculturação" (Brown, Collins e Duguid, 1989, p. 33), com um impacto duradouro sobre o caráter de alunos (e professores), vivendo produtivamente em experiências futuras (Dewey, 1938), elevando a expectativa e antecipando o que a Matemática pode proporcionar, e quais recompensas intelectuais, emocionais e viscerais podem advir de quando uma sede de matemática é saciada.

Palavras-chave. Experiência Estética, Matemática Elementar, Reforma

In *The Little Prince*, Antoine de Saint-Exupéry (1943/1999) tells the story of a merchant selling pills that quench thirst. "With these pills, you save fifty-three minutes in every week," advises the merchant. "As for me," says the little prince, "if I had fifty-three minutes to spend as I liked, I should walk at my leisure toward a spring of fresh water" (p. 50). "Thirst," commonly defined as "a feeling of needing or desiring," may describe a thirst for water, knowledge, justice, and the like. "Thirst," as is the case with any human experience, has both utilitarian and aesthetic qualities.

In this paper, we explore nurturing children's "thirst" for aesthetic mathematics experiences. We begin with a discussion of school mathematics, which commonly lacks an aesthetic quality. We then consider the arts as a possible remedy – as a way of infusing the aesthetic in school mathematics – and propose the answer may actually reside in the quality of mathematics itself. We end by offering a model for designing aesthetic experiences for young mathematicians, with examples from research classrooms in Brazil and Canada.

1. School mathematics

Imagine a person who has never experienced the aesthetic quality of quenching a thirst for water. They have drunk water to stay hydrated, to survive, to be able to work, but have never felt, for example, the pleasure of drinking a glass of cool water on a hot summer day, and of consequently anticipating their next thirst. What has been lost from their human experience?

Now imagine a person who has never experienced the aesthetic quality of quenching a thirst for mathematics. Sadly, this may be all too easy to imagine. Papert (1978) explains that common views of mathematics "exaggerate its logical face and devalue all connection with everything else in human experience," thus missing "mathematical pleasure and beauty" (p. 104). Taylor (1997) describes the mathematics of a grade 12 mathematics examination written by his daughter as "tedious" and "unimportant." He adds, "none of this is mathematics [...] I am a mathematician, and these things are definitely not what I do" (n.p.). He makes similar comments about his daughter's textbook: "No mathematician I know would ever read any part of the grade 12 text book for intellectual or spiritual pleasure" (n.p.). In an interview with

Sriraman & Lesh (2007, p. 67), Dienes comments that school mathematics is "not real," which Whitehead (1967) refers to as "inert" and Hewitt (1999) as "arbitrary." Brown, Collins and Duguid (1989) note that "Many of the activities students undertake are simply not the activities of practitioners and would not make sense or be endorsed by the cultures to which they are attributed" (p. 34).

Sinclair and Watson (2001) observe that although many mathematics educators make references to the aesthetic appeal of mathematics, "Very few of them mention the possibility of students' aesthetic experience in mathematics learning" (p. 39). Root-Bernstein (1996) states, "Students rarely, if ever, are given any notion whatever of the aesthetic dimension or multiplicity of imagining possibilities of the sciences" (p. 62). What do we do about this? Where do we turn for ideas to design classroom experiences that offer the aesthetic dimension of mathematics?

2. Mathematics as an aesthetic experience

Greene (1978, 1995) and Eisner (2002) encourage educators to learn from the arts and from artists. As will become evident in the next section, we have in fact used the arts in the development of our pedagogical lens. However, we need to distinguish between mathematics and school mathematics, and to be cautious about suggesting that mathematics or the work of mathematicians might be devoid of "art."

Consider the not-uncommon scenario where school art is taught instrumentally, where young children copy "art" created by their teacher, by cutting, colouring and pasting ready-made models of petals, stems and leaves to create "artistic" flowers. Seeking guidance for enhancing the aesthetic quality of school art, we might encourage art educators to learn from mathematics and from mathematicians, such as Peter Taylor (2009):

> When one is doing mathematical biology there's a lot of things to pay attention to, and there's a lot of papers to read, and a lot of ideas to think about, but the things I choose to work on, the things I give to my graduate students, are things where the structure fills me with a sense of beauty, where the aesthetics kind of speak to me and lead me on. (n.p.)

Looking at mathematics for ideas to improve school art experiences, we might focus on the common generalization that mathematicians discover things while artists do not, and design art experiences where students do not simply copy teachers' work, but have opportunities to discover personal ways of seeing and representing their world artistically. However, this would diminish our view of art, as doing art does in fact involve discovery. Boyd (2001a, 2001b, 2009) talks about "artistic discovery" and Root-Bernstein (1996) notes that "there is something to be discovered in the arts, just as there is in the sciences" (p. 54).

Sixty years ago, Snow (1956) bemoaned the divide between the arts and the sciences. Root-Bernstein (1996) notes that although it is common in recent times to generalize that the sciences are "objective, analytical, and rational" and the arts are "subjective, emotional, and based on intuition," if we look to the past the sciences and the arts were "considered to be very similar, certainly complimentary, and sometimes even overlapping ways of understanding the world" (p. 49). The problem with school mathematics is not that it lacks the arts, but rather that it lacks the aesthetic that is common to mathematics, the arts, and other disciplines: the aesthetic that makes the experience of these disciplines human.

Below we present an overview of our model for designing aesthetic mathematics experiences, then discuss its seven elements, organized in three categories – aesthetic, mathematical, and implementation (see Figure 1) – with examples from classrooms in Brazil and Canada.

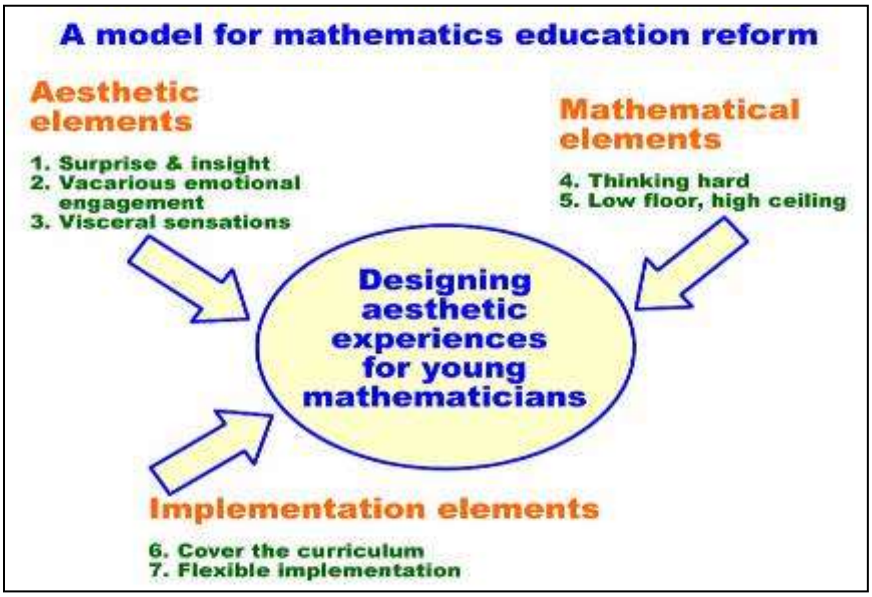

Fig. 1. A model for mathematics education reform.

3. Aesthetic elements

Our model designs aesthetic mathematics experiences that prepare children to answer "What did you do in math today?" by discussing school math with family and friends just like one would a favourite book or movie. To this end, we draw on important connections among elements of narrative (what makes for a good story) and mathematics (what makes for a good math experience).

As discussed in Gadanidis and Borba (2008), the aesthetic elements of our model are based on Boorstin's (1990) criteria for analyzing movies, which correspond to Norman's (2004) three categories – namely, reflective, behavioral, and visceral – for analyzing the emotional design of everyday things. Paraphrasing Boorstin's (1990) criteria of what makes movies work, good math experiences afford us three distinct pleasures: (1) the pleasure of experiencing the new, the wonderful and the surprising in mathematics; (2) the pleasure of experiencing emotional mathematical moments (either our own, or vicariously those of others); and (3) the visceral pleasure of sensing mathematical beauty.

3.1. Surprise and insight

Boorstin (1990) describes the "joy of seeing the new and the wonderful" (p. 12), which corresponds to Rodd's (2003) call for experiencing "awe and wonder" in mathematics education. Boorstin says that audience "demands surprise – so long as surprise comes with a rational explanation" (p. 13). "For the writer, this means constantly creating expectations that (for the right kind of reasons) aren't quite fulfilled" (p. 50).

Movshovitz-Hadar (1994) and Watson and Mason (2007, p. 4) see mathematics as full of surprises. Kleiner and Movshovitz-Hadar (1994) suggest that mathematical paradoxes, which offer one source of surprise, "have had a very substantial impact on the development of mathematics" (p. 973). Paradoxes and the mathematical surprises and insights they afford help students experience important moments and processes in the evolution of mathematics. In the design of mathematics tasks, Zaslavsky highlights the important role of "cognitive conflict" and the associated feelings of "perplexity, confusion and doubt" (p. 299), which parallels our emphasis on mathematical surprise. Burton (1999), analyzing the practices of mathematicians, notes that their "world of knowing" is "a world of uncertainties and explorations, and the feelings of excitement, frustration and satisfaction, associated with these journeys" (p. 138). Sinclair & Watson (2001), in their review of Fisher's (1998) book, *Wonder, the rainbow and the aesthetics of rare experiences*, highlight the importance of mathematical surprise and the moments of insight that "reveal our *logics of feeling*, as Gattegno (1974) put it, those intuitive and aesthetic modes of thinking that allow us to formulate conjectures and ideas" (p. 40).

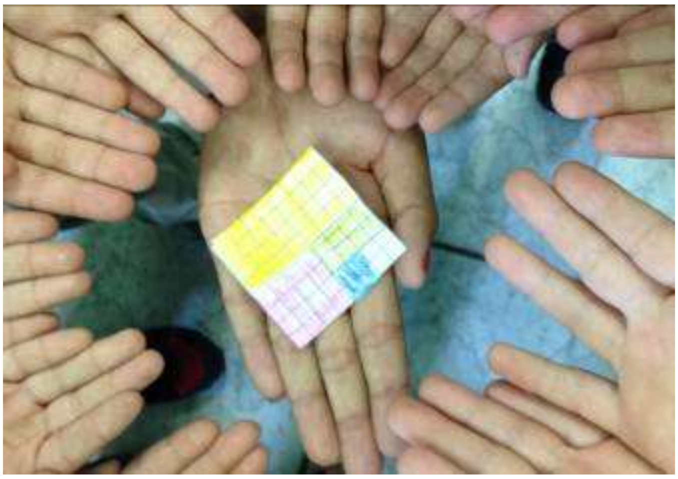

Fig. 2. Infinity in my hand.

Figure 2 shows grade 2 students in Brazil sharing the surprise "You can hold infinity in your hand!" Below we discuss how students investigated the following pair of ideas: (1) Can you walk past the classroom doorway? and (2) How big is infinity?

3.1.1. "Can you walk past the classroom doorway?" asked the teacher. "Yes," replied students. The teacher gave a volunteer these instructions: "Walk half way to the doorway. Then half the remaining distance, then half the remaining distance, and so forth." As the student modeled the walk, the teacher asked: "How many times would we do this on your way to the door?" "It would go on forever," students suggested. "That's an interesting puzzle," the teacher commented. "If we don't think about it, we can walk past the doorway. But once we imagine all of the parts you have to travel to get there, we feel stuck in our classroom." As we shall discuss in the next section on vicarious emotional engagement, the teacher also acted out a story on this theme. A common teaching strategy we use is "turn and talk," where students discuss the paradox with a partner, and then share it with the whole class. Typically, there is no consensus and students share a variety of ideas: a) it's not possible as the steps are infinite; b) we'll get there because eventually what's left will be too small to walk half; and c) imagine walking twice as far as the doorway, then you'll get to the doorway in the first step. The last idea intrigues students, and when a grade 3/4 teacher in Canada asked students to "turn and talk," they made interesting connections and extensions: a) it doesn't have to be twice as far - set the target past the doorway and you'll get to the doorway; b) every step has an infinite number of steps in it; and c) "are we going to be able to go out to recess, if half the time passes, then half of the rest of the time passes, and so on?"

3.1.2. How big is infinity? In the grade 2 classroom in Brazil, discussing in pairs and as a whole class, the consensus emerged that the stumbling block was infinity. If we have to walk five, ten or even one thousand fractional distances on the way to the doorway, it would be possible. But, how do you do this forever and still get to the doorway? "How big is infinity?" asked the teacher." Infinity is as big as the entire universe," said one student. "It is everything, everything, everything," added another.

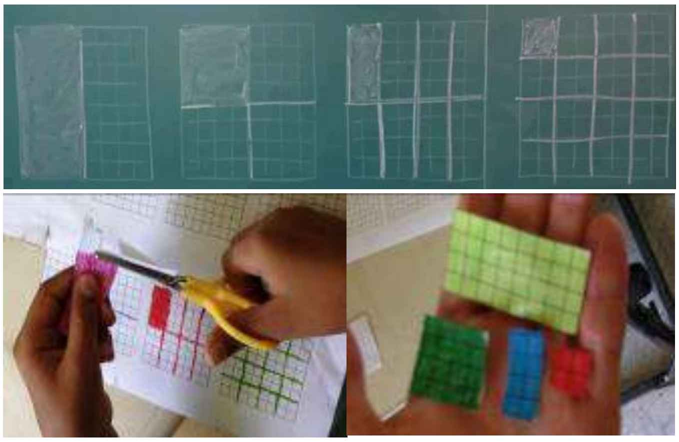

Fig. 3. Shading fractions.

"Let's try it once more," suggested the teacher. "This time, we'll draw pictures of the different parts that we travel." Each student was given a sheet of identical squares, each containing an 8x8 grid. As a volunteer modelled walking to the doorway, the teacher asked how they might represent this by shading the first square. With some scaffolding the first square was shaded on the board by a student volunteer, as shown in Figure 3. Students also shaded their first square the same way. The teacher took the time, with student input, to name this part as one half, to represent it as 1/2, and to discuss what the "1" and the "2" mean in this context. This process was repeated for the next part (1/4). For the grade 2 students in Brazil, this was their first school experience with fractions. Then students worked in pairs to shade in squares to represent the next two distances travelled (1/8 and 1/16), and these were modelled on the board. "Imagine doing this forever, cutting out all the shaded parts, and joining them to form a new shape. How big would that shape be? Would it fit on your desk? In our classroom? In Brazil?" Some students suggested that all the parts fit in a single square. "That can't be," said the teacher. "You told me infinity is as big as the universe. How could it fit in a little square?" The discussion continued and a class consensus emerged that the fractional parts will either completely fill the square or come very close to doing so, without spilling beyond the sides of the square. In a grade 3 classroom in Canada, the teacher posed this problem: "1/2 + 1/4 + 1/8 + 1/16 + ... = ?" explaining that the 3 dots mean that the pattern of fractions continues forever. "It's a whole," said one student. "It's like the fractions in the square. They fit to make one square," added another.

Extensions are available for students, depending on interest, time allotted by the teacher, and grade level. Students explore different representations of the shaded fractions, such as using triangles or by shading every other square in the grid to represent one half. The painting shown in Figure 4 depicts various representations created by grade 3 students in Canada. With research dissemination funding, we have created such paintings for many of the math surprises we have explored in classrooms. Posters have been printed for distribution to schools and the original paintings have been donated to project schools. Another extension is to investigate whether 0.999... equals one, by considering the fractions 9/10, 9/100, 9/1000 and so forth. Students use calculators to model sums, and more recently they use coding languages like Scratch or Python. Coding and mathematics are a natural fit, helping make complex concepts tangible, math relationships dynamic, and exploration student-controlled (Gadanidis, 2014; 2015).

Fig. 4. Infinity in my hand representations.

There are other ways of engaging students with the surprise associated with infinity and limit (Gadanidis, 2012). Working in a grade 4/5 classroom in Canada, the teacher asked for an interesting way to engage students with linear measurement. We collaboratively designed the following sequence of experiences: Students taped a metre stick against a wall. They dropped a rubber ball from the metre height and recorded the height of its first bounce. They then dropped the ball from the first bounce height and recorded the new (second) bounce height. They did this five times and constructed a bar graph of the five bounce heights. They drew a smooth curve through the tops of the bars, and extended it to predict the next five bounce heights. Then students were given this puzzle: Imagine a ball that always bounces half its drop height, and it bounces like this forever. How long is the path it travels? Is it infinite? How long would the ball bounce? Forever? We sometimes get asked: "Is it possible to use the same surprise in different grades?" The sequence of experiences shared in this paragraph show how different contexts can be used to investigate the "infinity in your hand" surprise. Many more contexts can be found in the book *Images of Infinity* by Hemmings and Tahta (1984). Also, many curricula use a spiral approach, where core ideas are revisited in subsequent grades, with added complexity. Extensions shared in the previous paragraph may be used in a similar manner.

The mathematics experiences discussed above satisfy Boorstin's first "pleasure". Infinity is "new and wonderful" for young children. Holding infinity in the palm of your hand offers mathematical surprise and insight, and in the context shown in Figures 2-4 it "makes sense."

3.2. Vicarious emotional engagement

Boorstin (1990) describes the second pleasure that we get from movies as the vicarious pleasure of putting "our heart in the actor’s body," which can be "profoundly moving" (p. 114). Teachers and parents experience this when witnessing a child grasp a mathematical idea. Students experience the same pleasure when they share their learning and re-experience surprises through the eyes of family and friends. Malmivuori (2006), Op 'T Eynde, De Corte, and Vershaffel (2006), Maturana (1988), and Zan, Brown, Evans and Hannula (2006) stress the co-constructive interactions of affect and cognition.

Fig. 5. Infinity story.

Vicarious emotional engagement may be facilitated through children's literature. In the grade 2 classroom in Brazil, the teacher performed a retelling of the story of Rapunzel (see Figure 5), where the prince realizes that Rapunzel is unable to leave her cell despite her cell door being open, as she can't imagine travelling the infinite fractional distances she has to pass to get to the door. The vicarious connections students make through the mathematical predicaments of characters create an emotionally intense learning experience (Whittin and Gary, 1994). Sinclair and Watson (2001) write about the vicarious experience of mathematical wonder:

> Moreover, learners may need to be inducted into the wonder of mathematics, to experience wonder vicariously through the teacher (including the stages of pleasure and frustration that sense-making requires) and, more urgently, to set aside the illusion of mathematics as systematic knowledge so complete that there is nothing more to expect. (p. 41)

In the grade 2 classroom in Brazil, students scripted and performed dialogues they might have at home with parents, with the teacher scaffolding their attention to eliciting mathematical surprise. Ideas and comments were used to create lyrics to a song (Figure 6), which students performed for their grade 4 peers and also at a parent evening organized by the school. In older classrooms, the skits and songs become more elaborate. For example, one group of grade 3 students in Canada scripted the dialogue/lyrics shown in Figure 7.

Infinity

Infinity, little infinity, let's all learn
Let's think about what it is and what size it has
I can see it beginning, but the end I can't

If I try to reach the end
I will see that it never ends
But I can hold infinity in my hands
1/2, 1/4, 1/8 and so it goes
It's all cool, it never gets out of a square

Fig. 6. Grade 2 lyrics.

Infinity in my hand

Hey Serena, do your chore
Take the garbage out the door
I can't Daddy, I learned in math
That I can't do it any more

It can't be true Serena
Your teacher is crazy
You can walk out the door
You've done it many times before

Halfway there Dad
And half of what's left
And half of what's left
It never ends
Infinity gets in the way
I heard my teacher say

Fig. 7. Grade 3 lyrics.

In some cases, the teacher records student comments, organizes them by theme, and returns this collective knowledge to students as a handout (Gadanidis, 2012). Students use these ideas to create skits or comics (see bottom image in Figure 8) to be shared with parents. This merging of what individual students know and understand creates and shares a collective knowledge that is more than any one person possesses, thus affording an opportunity to raise students' level of understanding. Davis and Simmt (2006) have noticed the emergence of such collective knowledge in their work with teachers.

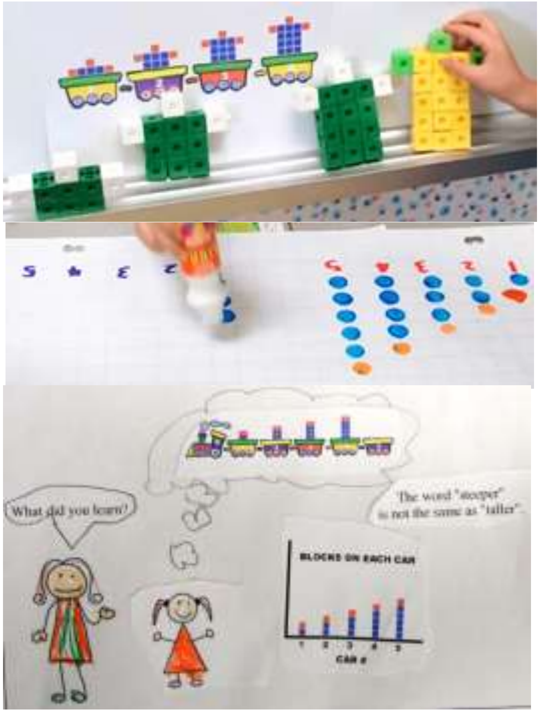

Fig. 8. Linear functions in grades 1/2.

Similarly, in some cases the teacher organizes parent comments by theme and shares them with both students and parents. Sometimes the comments form a song that students sing for their parents. Figure 8 shows images from a "linear functions" investigation in grades 1 and 2 in Canada. Below are some of the parent comments that were included in the song performed by their children.

> "I found school math hard, so I loved to watch her excitement, her complete understanding and explanation of it all."
> "It's an interesting way to teach young children basic patterns that lead to more complex algebra."
> "I was surprised how advanced the math exercises were and how so very easily my son grasped it all."

When students are given opportunities to share their “identity texts” with peers, family, teachers and the general public, they are likely to make gains in self-confidence, self-esteem and a sense of community belonging through positive feedback (Cummins, Brown & Sayers, 2007). Hull & Katz (2006) note “the power of public performance in generating especially intense moments of self-enactment” (p.47). Burton (1999) suggests that "learning mathematics as a narrative process where the learners have agentic control over authorship makes a substantial difference" in terms of achievement and attitude towards mathematics (p. 31).

3.3. Visceral sensations

Boorstin (1990) describes the third pleasure as visceral: "the gut reactions of the lizard brain – thrill of motion, joy of destruction, lust, blood lust, terror, disgust” (p. 110). In our work we focus on visceral sensations of mathematical beauty. Taylor (2009), reflecting on his work as a mathematician, comments that "The more beautiful something is, the more true it’s likely to be" (n.p.). Sinclair (2001) adds that an aesthetic math experience often involves a sense of pattern or a sense of fit. Grade 2 students in Brazil experienced the pleasure – the beauty – of mathematical pattern and fit as they noticed that the infinite set of fractions they represented as area diagrams all fit in a single square (see Figures 2 and 3). The grade 4 students from Brazil shown in Figure 9 experienced a similar pleasure when they explored concrete representations of odd numbers and noticed they "hide" in squares. Zwicky (2003) states that such "geometrical representations" attract our attention, and "say: Look at things like this" (p. 38).

Fig. 9. Odd numbers "hiding" in squares.

4. Mathematical elements

Artistic mathematical expression and complex, challenging mathematics concepts go hand-in-hand (Gadanidis, Hughes & Cordy, 2011). Greene (1978) relates an account by Kierkegaard (1947) of deciding, as an author, to work against the common societal trend of "making life easier for people" and "with the same humanitarian enthusiasm of others," to make things harder, "to create difficulties everywhere" (p. 194). There are issues with expecting young children to engage with complex, challenging mathematics: (1) Are they developmentally ready? and (2) Are there representations that make such mathematics accessible to them? These issues are addressed by the next two elements of the model.

4.1. Thinking hard

As discussed in Gadanidis (2015), there is a historical pattern of adults underestimating their children. Comments about the new generation lacking math or writing skills, not paying attention, and the like, are not new. Daniels (1983) documents such concerns as far back as ancient Sumeria. Consequently, it is perhaps not surprising that we seem to be attracted to educational theories of what children *cannot do*, such as Piaget's stages of cognitive development.

Egan (2002) notes that "Piaget's ideas and overall approach absolutely dominate in education" (p. 105). Papert (1980), who studied with Piaget, disagrees with the linear progression of his developmental stages, suggesting it does not exist in children's minds but in the poverty of school culture. Dienes, in an interview with Sriraman and Lesh (2007) comments that "Children do not need to reach a certain developmental stage to experience the joy, or the thrill of thinking mathematically and experiencing the process of doing mathematics" (p. 61). Piaget (1972/2008) himself cautioned about how generally his stages of development might apply. Egan (1997) challenges Piaget's notion that young children are not capable of abstract thinking, as it is an integral component of language development. Schmittau (2005) notes,

> Although historically in mathematics and traditionally in education, algebra followed arithmetic, Vygotskian theory with its emphasis on scientific concepts and theoretical understanding, supports the reversal of this sequence in the service of orienting children to the most abstract and general level of understanding from the beginning of their formal schooling. (p. 17)

Fernandez-Armesto (1997) laments that "Generations of school children, deprived of challenging tasks because Piaget said they were incapable of them, bear the evidence of his impact" (p. 18).

Underestimating young children goes hand-in-hand with the pervasive pedagogical belief that good teachers make mathematics easy-to-learn. This belief leads to taking complex ideas apart, to teaching their separate, simpler components, one at a time. Rancière (1991) describes this process for the case of teaching a child to read: "But now it is time to read, and he will not understand words if he doesn't understand syllables, and he won't understand syllables if he does not understand letters" (p. 7). Rancière calls this "the principle of enforced stultification" (p. 7), where the focus on refining teacher explanation has the effect of the child "understanding that he doesn't understand unless he is explained to" (p. 8). Sullivan (2000) adds that schooling strives to get students to pay attention whether or not the curriculum has interest or relevance" (p. 211). Similar comments have been made by others, such as Holt (1964), Nunn (1920), and Whitehead (1967).

Boorstin (1990) would say that easy-to-learn "math movies" do not work. As we try to guess what might happen next in the easy-to-learn movie, all of our guesses are correct, and the movie becomes uninteresting. We take pleasure from movies, from math classroom experiences, and from life generally, when we are surprised by a turn of events, and think hard to make sense of an unexpected development. Movshovitz-Hadar (1994), note the need for non-trivial mathematical relationships in eliciting mathematical surprise.

In designing aesthetic mathematics experiences we purposely seek to engage young children with complex mathematical ideas, such as infinity and limit, representations of the sums of odd and even numbers, and growing patterns that model linear functions. Our pedagogical goal is for children to experience attending deeply as they try to make sense of complex, puzzling and consequently interesting mathematical ideas.

4.2. Low floor, high ceiling

Following Papert's (1980) lead, we engage young students with complex mathematical ideas using a) a low mathematical floor, requiring minimal prerequisite knowledge, and b) a high mathematical ceiling, offering opportunities to explore more complex concepts and relationships and more varied representations. As school mathematics is typically organized and differentiated by grade level, there is not a tradition of representing and communicating complex mathematical ideas in ways that they can be accessed by non-experts. This predicament is a weak link in mathematics education. If we cannot identify and communicate big ideas, we are lost in the forest of mathematical detail, and reinforce Rancière's "principle of enforced stultification" (p. 7).

Boyd (2001b) notes that good storytelling involves solving *artistic* puzzles of how to create situations where the audience experiences the pleasure of surprise and insight. Solving such artistic puzzles in mathematics pedagogy in ways that afford young children opportunities to engage with, to be surprised, and to gain insights into complex ideas of mathematics is not yet common in mathematics education, and it is not easy to do. Based on our experience solving such artistic puzzles, we have found that the process leads to deeper mathematical understanding and enhanced mathematical pleasure for students, teachers and researchers (Gadanidis & Borba, 2008; Gadanidis, Hughes & Borba, 2008; Gadanidis, 2012).

One of our strategies in creating a low floor, high ceiling context is to engage mathematicians with the experiences that young children were involved in, and make the video interviews available to the students and their teachers. For example, mathematician Graham Denham was interviewed on "infinity in my hand." An extension he discussed involved finding different ways of arranging the fraction pieces inside the square (see Figure 10). Here are some of his think-aloud comments (available at www.researchideas.ca/wmt/c1b3.html).

> I could try putting all the square guys all in a row. Where does that leave me? Can I put all the rectangular guys together too? I kind of like that one too. You can imagine how it would continue. The rectangles are getting smaller and smaller, marching off into the corner. The squares are jumping off into the corner as well. [...] It looks like there's something we can look for in this picture as well. Notice that each of these terms, the squares, are only half as big as the rectangles that went before. [...] So if you worked on the details a little bit you could argue that this sum is twice as big as this sum. [...] the contribution from the rectangles [...] is two-thirds, and the squares add up to one-third. That's kind of fun.

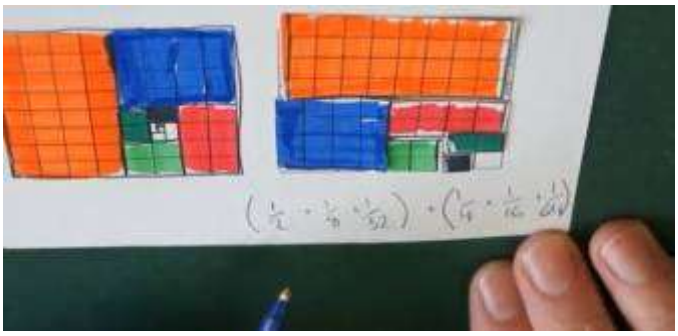

Fig. 10. Mathematician interview.

Having both young students and mathematicians working on similar problems, with similar representations, and making videos of both of their work publicly available, helps publicly "perform" the low floor, high ceiling component of our model. It also helps dispel the adult bias of who is a mathematician. To paraphrase Dissanayake's (1992) view of art, we want to see mathematics as a "normal, natural and necessary" part of human experience, for both young mathematicians and professional mathematicians.

5. Implementation elements

Although reform is typically associated with change on a grand, pervasive scale, and our model is much less intrusive pedagogically, we purposely take up the label "reform" to disrupt its common meaning. We are not seeking a revolution in mathematics education, but a strategic focus on mathematics worthy of attention, worthy of conversation, worthy of children's incredible minds, which thirst for knowledge and for opportunities to explore, question, flex their imagination, discover, discuss and share their learning. Large scale reform creates the danger of asking teachers to abandon their experience and knowledge, which is not a good starting point for teacher buy-in and teacher professional growth.

The history of mathematics educational reform is replete with innovations taken up enthusiastically by early adopters without significant transfer to other classrooms. Scholars such as Fullan (2007), Hess (1998), and Elmore (1990), note that most innovations have little impact on teaching and learning, the costs outweigh benefits, and the situation worsens rather than improves. Our model for designing aesthetic mathematics experiences incorporates two elements to help with its implementation.

5.1. Cover the curriculum

Our work in classrooms starts with teacher interests and concerns. The infinity learning experience discussed in this paper was designed when three grade 3 teachers in one of our project schools in Canada asked for help in teaching area representations of fractions. When designing such experiences, we distinguish between the mathematics content (what teachers need to teach) and the mathematics context (the more complex mathematics ideas we incorporate to enhance the experience).

> Framing interventions as altering the context but not the content of what teachers are required to teach is purposeful, as the vast majority of teachers will not accept an intervention that asks them to not teach the mandated curriculum. (Gadanidis, 2015, p. 162)

This does not mean that we agree with the mandated curriculum or that we don't work towards curriculum reform. However, curriculum reform is not a prerequisite for using our model.

5.2. Flexible implementation

Our aim in project classrooms is to use our model approximately once in every unit of study, or 8-12 times in the school year. One reason for this limited intervention is that good movies, to make an analogy, do not need to have a continuous sequence of surprises. A second reason is that we are not suggesting that ours is the only model for effective mathematics teaching – teachers can continue to use effective components of their existing practice or experiment with other models. Not being expected to change their whole practice all at once makes reform manageable. Teachers need time to try ideas, to see their value for their students, to understand how they work, to reflect on their practice, and to grow professionally and mathematically. Wubbels, Korthagen, and Boekman (1997, p. 23), reflecting on the less successful aspects of a reform program aimed at mathematics teacher candidates, identify being thrown in the "deep end" and not having opportunities to grow "more gradually" as obstacles. Our experience in project schools is that after initially collaborating with teachers to design one or two aesthetic mathematics experiences using our model, based on their needs and interests, teachers typically ask: "Do you have any more like these?"

We acknowledge that "reforms" are like inkblots, where different teachers see and implement different pedagogical directions, and not necessarily what the reform designers intended. Our focus on sharing beyond the classroom, on creating an audience for school mathematics, and on creating public documentaries of our work offers some ways to ameliorate this problem by making visible, and by modelling alternatives to, what has traditionally been hidden from public view. Burton (1999) suggests that teachers need personal experiences of this "world" on which to build their practice. The classroom experiences we propose through our model are one source of alternative experiences for teachers that helps shift their pedagogical attention towards more interesting mathematics. With support from the Fields Institute and the Teaching Support Centre at Western University, we are developing freely accessible online courses for teachers based on our work in classrooms (see www.researchideas.ca/wmt).

6. Experience and attention

Sullivan (2000, p. 211) asks: "What exactly are teachers asking for when they say, 'Pay attention'?" Typically, few of our students are actively attending to the mathematics ideas at play. Most students have learned to be passive observers, waiting to be "explained-to". This is not their natural state, as "Children begin their lives as eager and competent learners. They have to *learn* to have trouble with learning in general and mathematics in particular" (Papert, 1980, p. 40). We cannot directly equate "experience" and "education", as some experiences can be "mis-educative" by limiting growth from further experience (Dewey, 1938, p. 25). Viewing learning as experience in instructional design raises the status of student engagement, as only when students consider the experience worthy of their attention "will the transaction of experience have its full impact" (Parish, 2009, p. 512).

The challenge in mathematics education "is to select the kind of present experiences that live fruitfully and creatively in subsequent experiences" (Dewey, 1938, pp. 27-28). We believe the model presented in this paper for designing aesthetic mathematics experiences is a step in this direction. However, as we noted earlier, we do not propose an either-or choice. In the Preface to *Experience & Education*, Dewey (1938) cautioned that "any reform movement that thinks and acts in terms of an 'ism [...] forms its principles by reaction against (other 'isms) instead of by a comprehensive, constructive survey of actual needs, problems, and possibilities" (p. 6). As an example, let's consider Eisner's (2002) comments on the implementation of an arts-informed model for education reform:

> I want therefore to emphasise here that I am not talking about the implementation of isolated curriculum activities, but rather, the creation of a new culture of schooling that has as much to do with the cultivation of dispositions as with the acquisition of skills. (n.p.)

Although we agree with Eisner's call for "the cultivation of dispositions" (n.p.), for both students and teachers, especially the disposition of attending to aesthetic aspects of the mathematics experience, and although we do notice classroom, school, and school-community cultural shifts in our project schools, our view is that the starting point of mathematics education reform is closer to "the implementation of isolated curriculum activities" - or, more accurately, aesthetic mathematics experiences - rather than "a new culture of schooling" (n.p.). We believe our model can live fruitfully alongside, and in dialogical relationship with, other models used by teachers. We do not see mathematics education reform as picking a place on the pedagogical spectrum, but rather as broadening the pedagogical spectrum to include aesthetic mathematics experiences.

We believe that occasional, well-designed aesthetic mathematics experiences "that are immersive, infused with meaning, and felt as coherent and complete" (Parrish, 2009, p.511), and the associated experience of complex, surprising, emotionally engaging, and viscerally pleasing mathematics, can serve as "a process of enculturation" (Brown, Collins and Duguid, 1989, p. 33) with lasting impact on students' (and teachers') dispositions, living fruitfully in future experiences by raising expectation and anticipation of what mathematics can offer, and what the intellectual, emotional and visceral rewards might be when quenching a thirst for mathematics. Borrowing from work on thinking-with-new-media, when "attending deeply" enters the "cognitive ecology" (Levy, 1993) of mathematics experience, it introduces an attraction or "pull" (Hughes, 2008; Gadanidis & Borba, 2013) that potentially facilitates a "reorganization" (Borba & Villarreal, 2005) of student and teacher thinking about what constitutes mathematics and mathematics learning. Students who have experienced attending deeply to mathematics – what Greene (1978) refers to as "wide-awakeness," Gatagno (1987) labels as "awareness" and Bruner (1962) calls a "searching mind" – will be more likely to seek and even self-create such experiences under other models of instruction.

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