HOW AFFORDANCES AND CONSTRAINTS OF PHYSICAL AND VIRTUAL MANIPULATIVES SUPPORT THE DEVELOPMENT OF PROCEDURAL FLUENCY AND ALGORITHMIC THINKING IN MATHEMATICS

COMO POSSIBILIDADES E RESTRIÇÕES DE MANIPULATIVOS FÍSICO E VIRTUAL AUXILIAM NO DESENVOLVIMENTO DA FLUÊNCIA PROCESSUAL E DO PENSAMENTO MATEMÁTICO ALGORÍTIMO

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ABSTRACT

The purpose of this study was to examine how the affordances and constraints of physical and virtual manipulatives influence the development of students' algorithmic thinking when learning algebra and rational number concepts. Thirty-six third-grade students participated in two weeks of instruction using physical and virtual manipulatives as instructional tools. The primary design of the study was a teaching experiment in which quantitative and qualitative data were collected to provide a holistic examination. Pre- and post-test items were used in the quantitative analysis following a within-subjects crossover repeated measures design. Students' written work, a user survey, student interviews, field notes, and classroom videotapes were used in a qualitative analysis by coding the text data for evidence of major themes. Quantitative results indicated a significant difference between the physical and virtual manipulatives teaching episodes on students' pre- and post-test performance that was mediated by mathematics content type (fractions vs. algebra). Qualitative results confirmed that the affordances and constraints of the virtual manipulative fraction applets supported students' development of algorithmic thinking.

Keywords: algorithmic thinking , physical and virtual manipulatives, algebra and rational number

RESUMO

O objetivo deste estudo foi examinar como as possibilidades e restrições de manipulativos físicos e virtuaim influenciam o desenvolvimento do pensamento algorítimo de alunos na aprendizagem de conceitos da algebra e dos números racionais. Trinta e seis alunos da terceira série participaram de duas semanas de instrução usando manipulativos físicos e virtuais como ferramentas de ensino. O primeiro design do estudo foi um experimento de ensino em que foram coletados dados quantitativos e qualitativos para fornecer uma análise holística. Itens dos pré e pós-testes foram utilizados na análise seguindo método *within-subjects crossover repeated measures.* Trabalhos escritos dos alunos, uma pesquisa de usuários, entrevistas estudantis, notas de campo, vídeos em sala de aula foram utilizados em uma análise qualitative, codificando os dados do texto pela evidência de grandes temas. Os resultados quantitativos indicaram uma diferença significativa na performance dos alunos no pré e pós-teste entre os episódios de ensino com manipulativos físicos e virtuais, que foi medida pelo tipo de conteúdo matemático (frações versus algebra). Os resultados qualitativos confirmaram que as possibilidades e restrições dos aplicativos para ensino de fração contribuiram para o desenvolvimento do pensamento algorítmico dos alunos.

Palavras-chave: pensamento algorítimo, manipulativos físicos e virtuais, algebra e números racionais.

1. Introduçao

The purpose of this study was to examine how the affordances and constraints of physical and virtual manipulatives influenced the development of students' algorithmic thinking and procedural fluency through the use of mathematics tools. In particular, the present study investigated the development of procedural fluency as third graders engaged with physical manipulatives and *virtual manipulatives* (Author, 2016) as they explored mathematical tasks to learn addition of fractions with unlike denominators and balancing linear equations in algebra.

Procedural fluency (NRC, 2001) is defined as the ability to carry out procedures flexibly, accurately, efficiently, and appropriately. While procedural fluency is often related with the content of the Common Core State Standards for Mathematics (CCSSM), it can also be related to the standards for mathematical practice (National Governor's Association Center for Best Practices, 2010). For example, being competent in procedural fluency can support and extend a learners ability to *Look for and make use of structure* (Standard 7) and *Look for and express regularity in repeated reasoning* (Standard 8). In this study, we used procedural fluency to include algorithmic thinking and reasoning through the use of multiple representations. Algorithmic thinking or "*algorithmetizing," a term* coined by Gravemeijer and Galen (2001), refers to the guided reinvention of the algorithm and the opportunities to explore and make sense of the algorithm while modeling mathematical processes. The idea of guided reinvention of the algorithm is an appealing alternative to "teaching the algorithm", which often happens in mathematics classrooms characterized by learning the formula and memorizing steps without conceptual understanding. In contrast, encouraging algorithmic thinking and reasoning allows students to re-invent mathematical algorithms through scaffolded and well-designed tasks. By emphasizing the need to create opportunities for students to explore and make sense of algorithms, teachers can ensure that students develop procedural fluency with an emphasis on understanding, flexibility, efficiency and accuracy, which are all necessary for developing mathematics proficiency.

2. Review of the Literature

A brief discussion of physical and virtual manipulatives is necessary to understand the presentation mediums used in the study. Teachers use commonly both physical and virtual manipulatives, particularly in the elementary grades, for learning mathematics concepts. Teachers use manipulatives so that students can appropriate mathematical ideas as the referents for the objects and their manipulations (Chao, Stigler, & Woodward, 2000; Clements, 1999). In 2013, Author (2013) published a meta-analysis on virtual manipulatives and Carbonneau, Marley and Selig (2013) published a meta-analysis on physical manipulatives. These two major recent reviews demonstrate that researchers worldwide still share interest in studying manipulatives (physical and virtual) for mathematics teaching and learning.

Physical manipulatives have been around since the beginning of time. Some examples of historic manipulatives are counting beads, the abacus and counting sticks. Physical manipulatives are objects that can be handled and arranged to stimulate understanding of abstract mathematical ideas and are considered cognitive tools or thinking tools. Research on physical manipulatives has a long history. Historically, much of the research on physical manipulatives began in the 1970s with some of the earliest meta-analyses on physical manipulatives being conducted by Sowell (1989) and Suydam (1985; Suydam & Higgins, 1977). This research has continued until today, examining different types of physical manipulatives and their effects on elementary and secondary students (e.g., Carbonneau, Marley, & Selig, 2013; Gürbüz, 2010; McNeil, Uttal, Jarvin, & Sternberg, 2009). Most researchers who have studied manipulatives over the years are familiar with Sowell's (1989) meta-analysis of 60 studies conducted 25 years ago on the effectiveness of mathematics instruction with physical manipulatives. A major finding of this seminal study was that physical manipulatives were most effective when compared to symbolic-only instruction and when physical manipulatives have been used long-term. In 2012, Charbonneau, Marley, and Selig updated the research on manipulatives in a new meta-analysis that identified 55 studies comparing physical manipulatives-based instruction to a control condition of abstract mathematics symbols-based instruction, and found small to moderate effect sizes in favour of the physical manipulatives-based instruction. Carbonneau et al. (2012) extended the original work by Sowell's (1989) by identifying moderators of physical manipulatives' effectiveness (e.g., an object's perceptual richness, level of guidance during learning, and students' age). Ball (1992) cautioned that students do not automatically make the connection between their actions with physical manipulatives and their actions with symbols. Kaput's (1989) explanation for this disconnect was that the cognitive load imposed during the activities with physical manipulatives was too great for students. He stated that the problem with physical manipulatives is that learners cannot keep record of their actions with the materials and fail to see the connection between these actions and the manipulation of symbols.

More recently, the mathematics education community has seen more research on the use of *cognitive technology tools*. Zbiek, Heid, Blume, and Dick (2007) highlighted the importance of cognitive technology tools for mathematics because of their externalized representations, dynamic actions, and multiple linked representations that promote representational fluency. One of these cognitive technology tools is a *virtual manipulative*, first defined by Author (2002) and recently updated as "*an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated,* *that presents opportunities for constructing mathematical knowledge*” (Author, 2016, p. 1). According to Dorward and Heal (1999) there are several benefits to virtual manipulatives with linked representations:

> While appropriate use of good physical manipulatives has been shown to increase conceptual understanding, these ‘virtual manipulatives’ directly link iconic and symbolic notation, highlight important instructional aspects or features of individual manipulatives, provide links to related web-based resources, and have the potential to record user movements through stored procedures within each application. In addition, virtual manipulatives are very cost effective, versatile, and provide at least as much engagement as physical manipulatives (p. 1511).

Kaput (1995, p.523) states that different media have different “carrying dimensions” that affect the encoding of information. Clement and McMillan (1996) support Kaput’s statement by explaining how actions, like breaking apart computer base ten blocks then gluing them together to form tens, can help students build mental actions of composing and decomposing numbers. Because the numbers represented by the base ten blocks are dynamically linked to symbolic notations that automatically change based on the users’ action, this helps students make sense of their activities and the numbers. Meta-analyses on virtual manipulatives demonstrate that, across multiple studies, virtual manipulatives have an overall moderate effect on student learning when compared with other instructional treatments (Author, 2013, 2016). Classroom studies and dissertations have confirmed these results (Bolyard, 2005; Author, 2005; Takahashi, 2002). Five categories of affordances have been identified as an explanation for the results reported across numerous studies: focused constraint, creative variation, simultaneous linking, efficient precision, and motivation.

In contrast, one of the largest random-assignment studies comparing virtual manipulatives with physical manipulatives in third- and fourth-grade classrooms demonstrated no significant differences in achievement between the treatments. However, a follow-up study on the same data indicated important hidden predictors that influenced students’ achievement when manipulatives are used for mathematics instruction (Author, 2014). These studies on manipulatives demonstrate that physical and virtual manipulatives can have positive effects on student achievement and learning, but that there is much more to learn about how different manipulatives mediate achievement and learning and what those effects are when students learn different mathematics content.

3. Methods

In this study, we examine the generative process that is involved when students develop algorithmic thinking using different media tools. The following research question guided our inquiry: What affordances and constraints of physical and virtual manipulatives influence the development of students’ algorithmic thinking and procedural fluency (i.e., adding fractions with unlike denominators and balancing linear equations)? Our overarching research hypothesis to test involved an assumption that both manipulative types would result in student learning gains, but that those learning gains would be related to the affordances and constraints of the different manipulative types. Based on the large body of research that has been conducted on the effectiveness of physical manipulatives and virtual manipulatives, one of our primary assumptions was that we would see growth in student learning during teaching episodes with physical manipulatives and virtual manipulatives. However, to confirm this assumption, we first pre-tested and post-tested students to document learning gains before conducting a more in-depth analysis of the affordances and constraints of the manipulatives, and how those affordances and constraints influenced the development of students' algorithmic thinking.

The study used a mixed methods design with a qualitative teaching experiment as the focus of the design and quantitative data used to investigate researchers' initial assumptions. These qualitative and quantitative data together were collected to provide a holistic examination. A classroom teaching experiment involves the researcher as the teacher in an interaction with the students over a period of time in a sequence of teaching episodes (Kelly & Lesh, 2000; Steffe, 1983). This study included the four basic elements of a teaching experiment, which include: the teaching agent, the students, a witness and a method of recording the teaching episodes. A within-subjects crossover design was used to allow all of the students to participate in instructional episodes using physical manipulatives and virtual manipulatives during the study (Campbell & Stanley, 1963). Because all students used both the physical manipulatives and the virtual manipulatives, this allowed each student to serve as his or her own comparison during the quantitative portion of the analysis. To avoid any residual effects, instructional episodes focused on two different mathematics topics: addition of fractions with unlike denominators and balancing linear equations in algebra.

4. Participants

The participants in this study were 36 third grade students in two classes at the same elementary school. The student demographics included 83% White, 11% Asian, 3% African American, and 3% Hispanic. There were 22 boys and 14 girls. Teachers of the third grade placed the students at this school in mathematics achievement groups using results from standardized tests. The students selected for this study were in the middle achievement group working on the third grade level in mathematics. The two classes were randomly assigned to the virtual manipulative treatment group or the physical manipulative treatment group for the first week of instruction on fractions, and then groups switched treatments for the second week of instruction on algebra.

5. Mathematics Concepts and Materials

During the treatment, the third grade students learned addition of fractions with unlike denominators and balancing equations in linear algebra. These two concepts were chosen because, traditionally, they are taught in the middle grades heavily relying on the algorithmic approach with a sequence of steps and rules, sometimes lacking concretization of the concepts.

The two virtual manipulative applets used in this study came from the National Library of Virtual Manipulatives (*Adding Fractions* and *Algebra Balance Scales*). The two physical manipulatives used were fraction circles and Hands-On Equations®. Examples of both manipulatives types are shown in Figure 3.

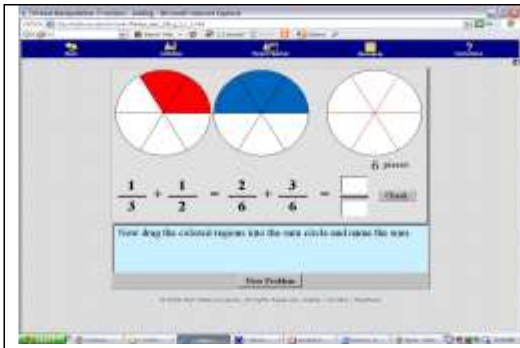
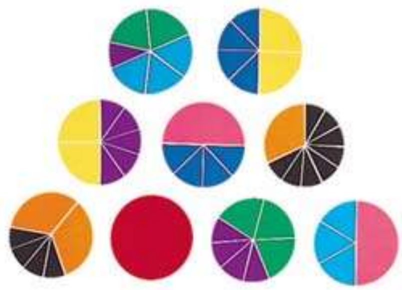
 Virtual Manipulative Fraction applet	 Physical Manipulatives: Fraction Circles
 Virtual Manipulative Algebra Balance Scales	 Physical Manipulative: Hands-On Equations®

Figure 3. Instructional materials for fraction and algebra unit

6. Data Sources

The data sources used in this study were both qualitative and quantitative. The quantitative data included pre- and post-test scores of students' mathematics content knowledge. The purpose for collecting pre- and post-test scores of students' mathematics content knowledge was to confirm or refute our assumptions that the teaching episodes with physical manipulatives and virtual manipulatives would result in growth in student learning. The researcher-designed pre- and post-tests contained three sections, with a total of 20 items. The first section included multi-representational items (i.e., test items with pictorial and numerical representations); the second section included items with only numerical representations; and, the third section contained two word problems which required students to draw a picture, represent the problem with a number sentence, and explain their solution strategies in writing (See figure 4).

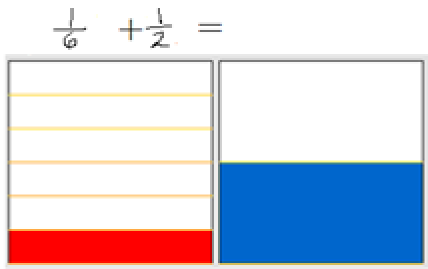
Fraction Examples	**Algebra Examples**
Section 1	
	
Section 2	
$\frac{2}{4}+\frac{3}{8}=$	$2x+2=10$ x=______

Figure 4. Examples of pre- and post-test items.

The qualitative data included classroom videotapes (as a method of recording the teaching episodes), students' written work, a user survey, student interviews, and field notes. All class sessions were video-taped to capture student activity during the lessons. Students' written work contained drawings, solution procedures, and numeric notations. The User Survey gathered information on students' preferences in each manipulative type. Student interviews were mini-interactions with students while they were working during class sessions and these were also video-taped. Researchers took field notes during all classroom sessions. These qualitative data sources were used to construct a holistic picture of how affordances and constraints of the physical manipulatives and virtual manipulatives effected the development of students' algorithmic thinking processes during the class sessions.

7. Procedures

The study employed a within-subjects crossover repeated measures design to examine the research questions (Campbell & Stanley, 1963). All subjects participated in both treatments (virtual and physical manipulatives), which allowed each student to serve as his or her own comparison during the analysis. To avoid any residual effects, researchers introduced two different mathematics units, fractions and algebra, as the topics of study. The researchers chose concepts that are taught traditionally using an algorithm, namely adding fractions with unlike denominators and balancing equations, to examine ways manipulative representations might serve as conceptual supports in helping students understand how and why those procedures work. In the first phase, Group One participated in fraction lessons using the physical manipulatives while Group Two participated in fraction lessons using the virtual manipulatives. In the second phase, each group received the opposite condition. That is, Group One participated in algebra instruction using virtual manipulatives, and Group Two participated in algebra instruction using physical manipulatives. A pretest on fraction and algebra concepts was administered at the beginning of the study. Students learned fraction content using virtual or physical manipulatives during the first unit. During the second unit on algebra, students switched treatment conditions and learned algebra content. Fractions and algebra content tests were administered at the end of each unit.

8. Data Analysis

Data analysis of the videotaped instructional episodes, students' written work, a user survey, student interviews, and field notes was guided by a framework of affordances and constraints developed from the categories reported by Sarama and Clements (2009). Author proposed five interrelated affordance categories: *focused constraint* (i.e., VMs focus and constrain student attention on mathematical objects and processes), *creative variation* (i.e., VMs encourage creativity and increase the variety of students' solutions), *simultaneous linking* (i.e., VMs simultaneously link representations with each other and with students' actions), *efficient precision* (i.e., VMs contain precise representations allowing accurate and efficient use), and *motivation* (i.e., VMs motivate students to persist at mathematical tasks). Traditional methods of open and axial coding were used to frame categories and make sense of the students' experiences (Strauss & Corbin, 1998; Merriam, 2009; Moghaddam, 2006).

Students' written work contained drawings, solution procedures, and numeric notations. This work was examined and categorized along dimensions of students' solution strategies. The User Survey gathered information on students' preferences in each manipulative type. Student interviews, field notes, and classroom videotapes were used to examine the representations that students used to solve problems during the teaching episodes in both manipulative environments. The solution strategies and the representations that students used to perform those strategies were compared to examine their unique features.

The quantitative portion of the analysis consisted of the use of SPSS to run basic descriptive statistics on students' pre- and post-test scores. A 2 x 2 ANOVA was run to compare students' scores and examine learning gains for each of the groups.

Results: Students' Procedural Fluency and Algorithmic Thinking

9. Analysis by Manipulative Type and Mathematics Content

To test our assumption that both manipulative types would result in learning gains, we analyzed student achievement by (a) manipulative treatments (physical versus virtual) (b) mathematics concepts (physical fraction versus virtual fraction conditions, and physical algebra versus virtual algebra conditions), and (c) test items categorized by representational modes (pictorial/numeric versus numeric only modes).

The pretest administered at the beginning of the study showed that students in both groups had very little prior knowledge on either topic, fractions or algebra (see Table 1). There were no significant differences between the groups prior to the treatments. Table 1 shows descriptive information for all measures across treatment groups and mathematics content.

Table 1.

Mean for the Pretest and Posttest by Treatment Type and Mathematics Content (N=36)

	Group 1: Pretest	**Group 1: Posttest**	**Group 2: Pretest**	**Group 2: Posttest**
	Physical Manipulatives		Virtual Manipulatives	
Fraction	12.50	45.55 (SD=17.05)	13.00	75.55 (*SD* = 19.91)
	Virtual Manipulatives		Physical Manipulatives	
Algebra	30.00	83.33 (*SD* = 14.34)	22.00	80.00 (*SD* = 20.16)

The mean algebra posttest score for Group One, the virtual manipulative treatment, was 83.33 (*SD* = 14.34) and for Group Two, the physical manipulatives treatment, Hands-On Equations®, was 80.00 (*SD* = 20.16). For fraction content, the mean posttest score for Group One, the physical manipulative treatment, was 45.55 (*SD* = 17.05) and for Group Two, the virtual manipulative treatment, was 75.55 (*SD* = 19.91). A 2 x 2 factorial design using ANOVA for two independent variables (i.e., manipulative types and mathematics content) produced a significant main effect for manipulative types, $F(3,68) = 15.03$, $p < .001$. A statistically significant difference existed between the manipulative treatments, virtual versus physical, on students' overall performance on the mathematics posttests, showing that students' scores depended on the manipulative treatment they used. In this case, the students who used the virtual manipulative treatment outperformed their peers using the physical manipulative treatment. This significance is most revealing between the physical and virtual fraction treatment showing that students who used the virtual fraction applet significantly outperformed students in the physical fraction treatment.

10. Analysis of Students' Solution Strategies on the Fraction Test

An analysis of students' solution strategies on the fraction posttest items indicated marked differences between the responses from students in the virtual treatment versus students in the physical treatment. Students' responses provide some insight into their scores on the final posttests. A frequency count showed differences in the way students solved the numeric only test items (see Table 5).

Table 5.

Analysis of Students' Solution Strategies for Numeric Only Items on the Fraction Posttest

Solution strategies	Group One: Physical Fraction Circles		Group Two: Virtual Fraction Applet	
	Number of Students	Percent	Number of Students	Percent
Primarily used pictorial representations	8	44.5 %	2	11.0%
Primarily used fraction algorithms	2	11.0 %	14	78.0%
No strategy shown	8	44.5 %	2	11.0%

On the fraction posttest, eight students from Group One (who worked with the physical fraction manipulatives) used pictures, two used a fraction number sentence, which indicated some understanding of the algorithmic process, while eight others did not show any strategy. In Group Two (the virtual fraction treatment), 14 students used an algorithm that showed an understanding of the process of renaming then combining fractions, two students drew pictures, and two others did not show any strategy. As this result shows, students in the virtual fraction treatment were better able to use an algorithm for their solutions.

Students in Group One, who worked with the physical fraction circle manipulatives, relied more on pictures to help them solve the numeric items (See Figure 6).

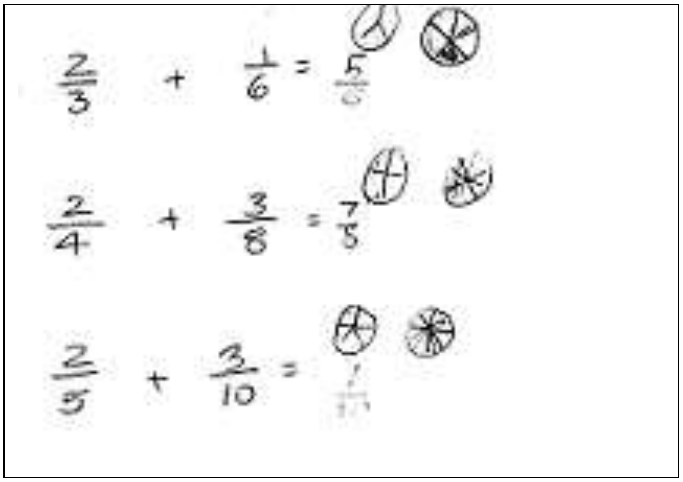


Figure 6. Group One student's work on the fraction posttest showing a pictorial solution strategy

Students from Group One who used the physical manipulatives were able to solve simpler problems by using drawings, but they had difficulty illustrating more complex fraction problems. For example, problems like $\frac{3}{4}+\frac{1}{8}$ were simpler "friendlier" problems, because 8 is the common multiple of 4 and can be illustrated easily once each quarter is split into eighths by drawing a line through each quarter to change $\frac{3}{4}$ pictorially into $\frac{6}{8}$. Complex fraction problems, like $\frac{1}{4}$ + $\frac{1}{5}$, where both fractions need to be renamed before being added, became complex for students and it was harder for them to illustrate their answers. This finding suggests that although physical models and pictorial representations can be helpful when initially learning fraction concepts and visualizing fractions, over-reliance on pictures becomes limiting when students need to solve more complex problems.

Further examination of the fraction posttests for Group One (physical fraction treatment), revealed that 11 of 18 students did not correctly solve any of the eight problems in the single representation (numeric only) test items. Out of the eleven, five students left the section blank and six students attempted the problems, but solved all of them incorrectly. In analyzing the written work, researchers recognized that several students exhibited common fraction error patterns as described in Ashlock's (2001) *Error Patterns in Computation*. Some students found common denominators, but failed to change the numerators (i.e. $\frac{2}{3}+\frac{1}{4}=\frac{3}{12}$, they found the common denominator of 12, but failed to change the numerators and simply added 2+1); others exhibited the "adding across" error pattern (i.e., $\frac{2}{5}+\frac{3}{10}=\frac{1}{3}+\frac{1}{5}=\frac{2}{8}$, where they added the numerators, 1 + 1 and the denominators, 3 + 5).

Group Two (virtual fraction treatment) relied more on algorithms that were modeled on the applet by the linked representation feature. Further analysis of their posttests revealed that most students who successfully answered the numeric only items changed the unlike fractions into fractions with common denominators, as was modeled by the virtual fraction applet. (e.g. $\frac{3}{4}+\frac{1}{8}=\frac{6}{8}+\frac{1}{8}=\frac{7}{8}$). Figure 7 shows an example of how one student solved the numeric only items by using the algorithmic process.

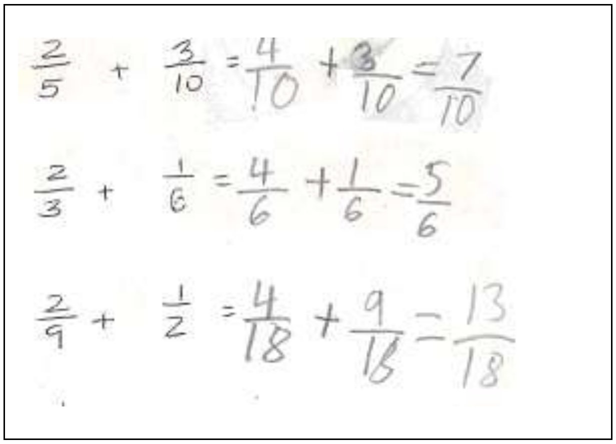


Figure 7. Group Two student's work on the fraction posttest showing an algorithmic process.

11. Analysis of Solution Strategies for Word Problems on the Fraction Test

In solving the word problem test items, most students in Group One (physical fraction circles) explained their process using a picture to illustrate the problem. One student explained: "I drew a picture and took the half and I put it in the third." (See Figure 8). Although, the student had the correct answer, there was no evidence of the renaming process.

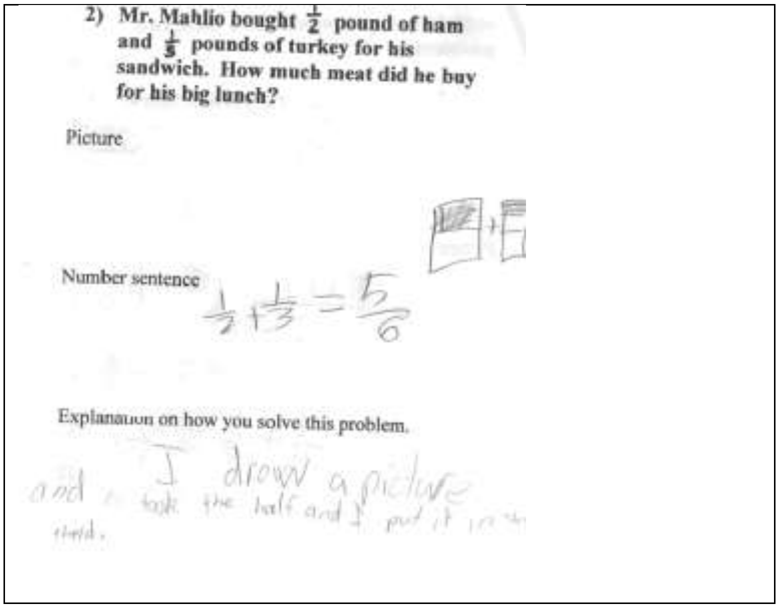


Figure 8. Example of Group One student's explanation using a number sentence and pictorial representation to solve the problem.

In contrast, when most students in Group Two (virtual fraction applet) solved the word problems, they drew pictures, wrote a correct number sentence, and used the formal algorithmic approach to solve the problem by renaming each fraction with common denominators. Some examples of their explanations are shown in figure 9.

- "I said to myself 2, 4, 6 and 3, 6, 9 and got my common denominator."
- "I found a multiple of 2 and 3."
- "I multiplied the [number of divided parts] by 2 for $\frac{1}{3}$ which equals $\frac{2}{6}$ and I divided 6 in half which is $\frac{3}{6}$ and then I added $\frac{2}{6}$ and $\frac{3}{6}$ which equals $\frac{5}{6}$."

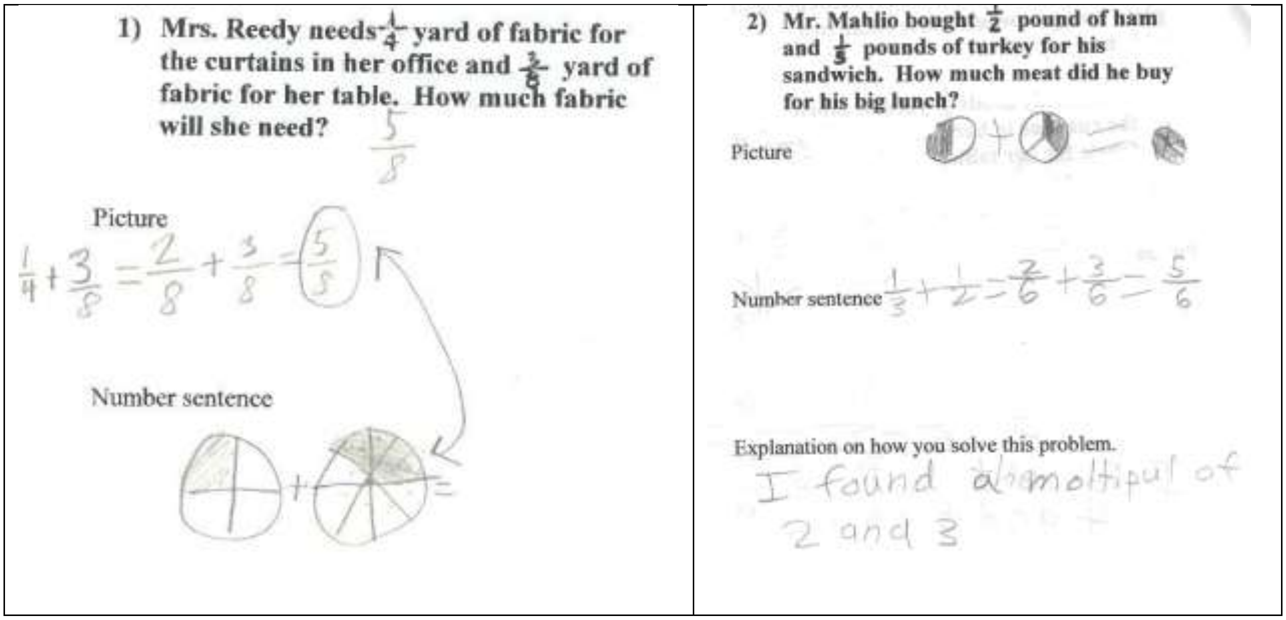


Figure 9. Examples of Group Two student's solutions on a fraction word problem with pictorial and numeric representations.

12. Unique Affordances in the Virtual Environment

To address our research question, focused on the affordances and constraints that may explain student learning, we analyzed classroom videotapes, field notes, student interviews and a user survey to identify unique affordances in each environment (virtual and physical). The unique affordances identified in the virtual environment included: 1) Explicit link between visual and symbolic modes; 2) Dynamic features; 3) Guided step by step support with formal algorithms; and 4) Immediate feedback and self-checking system.

Explicit link between visual and symbolic modes. The virtual manipulative applets linked representations of visual and symbolic modes. These links existed in both the fraction and the algebra applets. An excerpt from an in-class interview highlights this linkage:

> After I press the check button, I got to the second screen that lets me see the new number sentence next to the first sentence like this: $\frac{1}{3}+\frac{1}{4}=\frac{4}{12}+\frac{3}{12}=\square$. On top I saw a picture of the fractions with the new number of pieces both divided into 12 parts.

Dynamic features. On the virtual fraction applet, one of the dynamic features is an arrow key, which allows the user to break fraction pieces into multiple parts to find the common multiples. This feature allowed students to experiment and test their ideas on how to rename fractions. When the researcher asked one student, "How does the arrow key help you?" The student replied, "I can click on it to see that the lines even up whenever I hit a multiple of that number.

For example for this fraction $\frac{1}{11}$, I noticed that the lines evenly divided at 22, 33, 44, 55, and so on. There is a pattern in the list that I see." From the dialogue, it was clear that the student had a strong conceptual understanding of how the tool could be used to identify multiples of a number.

Guided step by step support with formal algorithms. Both of the virtual applets chosen for this project included an emphasis on learning the formal algorithm. These applets were concept tutorial tools rather than exploratory tools because they had built in constraint-support systems that guided students through the algorithm. This was useful to students' learning as demonstrated in the following student's comment: "I like how you go step by step through the fraction problem. It helps me think through the steps." The fraction applet modeled the formal algorithm for renaming fractions with unlike denominators, thereby supporting students' development of procedural fluency. In the previous example, the student understood the reason behind renaming a fraction before adding fractions with unlike denominators. She also noticed the pattern of finding equivalent fractions, that is, that fractions can be renamed into multiples of the denominator.

Immediate feedback and self-checking system. Both the fraction and the algebra applets contained a check answer button to verify students' final answers. In addition to the final check button, the applets included several prompts during the procedure. If students entered the wrong numeric response, the computer would provide the prompt: "The two sides don't match the equation" or "You can't subtract 4x from both sides unless there are at least 4xs on each side." This immediate and specific feedback and the self-checking button kept students from practicing the problems in an erroneous way. It also provided immediate positive reinforcement.

13. Unique Constraints of the Physical Manipulatives

There were two unique constraints of the physical manipulatives: physical constraints that did not allow modeling and disconnections between manipulatives and symbols.

Opportunities for open-ended exploration but limited by representations. Unlike the virtual fraction applets, the physical fraction circles and equivalence mat was more exploratory in nature and did not scaffold students learning. The fraction circles lacked the constraints and scaffolded tutorial features of the virtual tools. Some students were able to use the manipulatives for carefully selected problems to obtain the answer, but later found some problems were not possible to model due to limited pieces of the fraction circles. Without the fraction circle pieces to model the problem, students could not always connect the fraction circles with the symbolic notation.

Over reliance on the manipulatives and disconnect between physical manipulatives and symbolic. One drawback identified in the observations was the over reliance on the fraction manipulatives and the equivalence mat. Here is an excerpt from the field notes:

> Some students combined the two fraction addends like $\frac{2}{6}$ and $\frac{3}{12}$ and randomly placed them on the fraction mat until they saw the fraction lining up to a common denominator

without much thought about how each addend must change in order to be combined using a common denominator.

From these notes, researchers observed that students over-relied on the fraction mat instead of looking at the relationships between renaming and combining fractions with common denominators.

14. Discussion

The significant results for the fraction applet treatment show the influences of the applet's affordances for learning the fraction addition procedures. The non-significant result from the algebra unit were just as revealing, in that it indicated that the physical and virtual manipulatives used with the learners for this content were equally effective for teaching students to solve linear equations. In the case of the algebra content, affordances in both media promoted student learning. The examination of the test items shows that *algorithmetizing* was reinforced by the virtual fraction applet. Students were able to rename and combine fractions because the virtual manipulative applet assisted students in developing this procedural fluency, a feature that was not present in the physical manipulative fraction environment. In the fraction posttests, students in the virtual fraction applet treatment transferred procedural fluency to help them solve fraction problems.

The virtual fraction applet used visual images and numeric sentences that appeared contiguously on the screen, allowing students to observe this relationship between the two representational forms. The dynamic visual images and the symbolic notations provided more opportunities for students to translate between the two representational forms and, in doing so, reinforced the relationship among procedures and concepts of renaming and combining like denominators. The virtual fraction two-step applet closely resembled the mental action that takes place when one needs to add fractions with unlike denominators: first renaming the fractions with common denominators then combining the two fractions.

Results from the test items also suggest that students' performance varies with different types of items: dual versus single representations. Building mental images for symbolic and numeric representations is an important skill to develop. This study suggests that dual coded representations better facilitated students' understanding during instruction and during assessment.

Equally important to note is not only what supported learning in the virtual environment but what hindered learning with physical fraction circles. In the physical fraction environment, there was an over reliance on the fraction circle pieces and difficulty translating and transferring the physical actions performed on the manipulatives to the test items. Although flexible strategies were encouraged in the physical environment, they became limiting when the mathematical task became more complex. Students said, "It was hard to find the right fraction pieces because they were not labeled and challenging to find equivalent fractions with so many pieces on the mat" and "it was 'tricky' to solve the problems like $\frac{1}{3}+\frac{2}{5}$ because there was not a common denominator of fifteenths in the deluxe fraction kit for thirds and fifths." The multiple pieces with the different colors and sizes of fraction pieces became an extraneous cognitive load in the learning process.

In the virtual fraction environment, the explicit link between the visual mode and the numeric mode allowed for dual coding and facilitated learning. The guided step by step features supported algorithmic thinking and the immediate feedback and self-checking system reinforced learning. The dynamic features in the virtual environment allowed the fraction pieces to break up into unlimited multiple parts, thereby helping students to understand the embedding and disembedding nature of the fraction part-whole relationship.

In the virtual fraction treatment, students were able to use both representational forms in their responses by drawing pictures and writing a fraction sentence with the renamed fractions, as was modeled in the virtual environment. The dual representations in the virtual fraction applet that linked the virtual fraction pieces that could be changed into multiple parts with the numeric algorithmic process aided students' understanding and supported their cognitive load. This action was also evident in students' work on the posttests where students rewrote the problem $\frac{1}{3}+\frac{1}{4}$ as $\frac{4}{12}+\frac{3}{12}=$. The virtual fraction applet gave students more opportunities than the physical manipulatives to concretize their understanding of the renaming procedure with the step-by step procedure. In addition to the support with the algorithmic process, students received immediate feedback that reinforced their learning of the algorithm.

The design of the virtual applet allowed students to maximize their cognitive capacity and resources to focus on the mathematical behaviors of the objects thereby enhancing the learning. After having exposure to both fraction and algebra content, students reported that the algebra problems were much easier than the fraction problems. One student commented "I thought algebra was supposed to be really hard but it is like finding the mystery number. Using the pictures of the balance helped me comparing the amounts on both sides and figure out the x". When asked why adding fractions was harder, one student said, "Sometimes, I could not easily find a common denominator between the two fractions using the tools and did not have friendly numbers". Many of the students had difficulty processing all the steps that were involved in finding a common denominator using the physical manipulatives, especially because they had to figure out which pieces represented the fractional parts and then, once they found the pieces, they had to find fractional parts that were equivalent for both fraction addends with unlike denominators.

Students who used the fraction circles to add fractions with unlike denominators had difficulty keeping track of the procedures in their head, and failed to see the connection between their manipulation with the fraction pieces and the numeric notations. The manipulation of multiple fraction pieces and the task of adding fractions with unlike denominators may have presented an extraneous cognitive overload. The use of the virtual fraction applet provided students with the conceptual knowledge and the procedural knowledge of adding fractions with unlike denominators. It promoted algorithmic thinking because students learned the procedure while building a conceptual foundation for fraction addition with unlike denominators using the dynamic visual representation.

15. Conclusion

The results make some important connections to cognitive theories and multimedia learning principles. The first connection is to the cognitive load theory. For fraction addition with unlike denominators, which had high intrinsic cognitive load, presenting the mathematics concept through the virtual fraction applet showed multiple representations in a linked format and the least extraneous cognitive load and was more effective than the physical manipulatives. However, with the concept of balancing linear equations, which had a low intrinsic cognitive load, both virtual and physical manipulatives presentation mediums showed no significant differences in achievement. Since the limited cognitive processing capacity forced learners to make decisions about which pieces of incoming information to pay attention to and how to build connections among the selected pieces of information, learners needed to use metacognitive strategies to allocate, monitor, coordinate and adjust these limited cognitive resources (Mayer, 2001). This could explain the way the fraction applet's constraint-support system assisted students to allocate more of the learner's cognitive capacity to the mathematical processes that were most important. The virtual fraction applet promoted algorithmic thinking, because students learned the procedure while building a conceptual foundation for fraction addition with unlike denominators using the interactive dynamic visual and numeric representations. Kaput (1992) stated that constraint-support structures built in to computer based learning environments "frees the student to focus on the connections between the actions on the two systems [notation and visuals], actions which otherwise have a tendency to consume all of the student's cognitive resources even before translation can be carried out" (p.529). The dual representation virtual fraction environment offered many meta-cognitive opportunities, such as keeping record of users' actions and of the transformation of numeric notation, which allowed learners to use their cognitive capacity to observe and reflect on the connections and the relationships among the representations, thereby promoting algorithmic thinking. The physical fraction circles proved to be less effective because the learner's cognitive resources were expanded on keeping track of fraction pieces, finding equivalent fractions using an equivalence mat, and recording notations on paper. The task may have induced cognitive overload because demands on the learners did not leave any cognitive resources to observe relationships between actions on the physical manipulatives and the symbolic manipulation.

The second connection relates to how multiple representations contiguously on the screen help build the algorithmic thinking processes. Building mental images for symbolic and numeric representations is an important skill for students to improve in their mathematical understanding. It encourages the sense making process when learning the algorithmic process so that the efficient process of solving complex mathematics problems is understood. Encouraging algorithmic thinking by observing the relationships between facts, procedures and concepts is critical to building procedural fluency. Developing algorithmic thinking enables students to understand their methods and carry them out proficiently so that they can think about more important things, such as why they are doing what they are doing and what their results mean.

This study also shows that certain features of virtual manipulatives may be very effective for enhancing student achievement for some mathematical concepts, while other virtual manipulatives may have the same influence on student achievement as physical manipulatives for mathematical concepts. As the development of virtual manipulatives advances, instructional designers must work with educators and researchers to carefully design the affordances and constraints in the virtual applets for teaching specific concepts by taking into account how design principles influence learning. In particular, cognitive processing theories, such as dual coding theory and multimedia principles, must be considered when designing learning tasks. Further study is needed to learn how and why specific tools, both virtual manipulatives and physical manipulatives, are more or less effective for teaching specific mathematics concepts.

16. References

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