MATHEMATICS TEACHER'S SPECIALIZED KNOWLEDGE (MTSK) IN THE "DISSECTING AN EQUILATERAL TRIANGLE" PROBLEM

CONHECIMENTO ESPECIALIZADO DO PROFESSOR DE/QUE ENSINA MATEMÁTICA (MTSK) EM UM PROBLEMA DE "DISSECÇÃO DE UM TRIÂNGULO EQUILÁTERO"

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ABSTRACT
This paper focuses on describing a model for the analysis of the mathematics teachers' knowledge from the perspective of what is peculiar to teaching mathematics, and gives an example of its application to the knowledge deployed by a teacher during a lesson. We first present the analytical model, the MTSK (mathematics teacher's specialized knowledge) model, developed at the University of Huelva (Spain), and then demonstrate its potential in the case of a high school teacher working through a problem with her students. We conclude with a summary of the advantages the model offers in detailing the elements of specialized knowledge displayed by the teacher, and illustrate how this knowledge enables her to plan highly productive learning opportunities for her students. A final reflection notes the potential of the MTSK model as a tool for teacher training.

Key words: Teacher's Knowledge, Mathematical Knowledge, Pedagogical Content Knowledge, Specialized Knowledge, Problem Solving.

RESUMO

Neste artigo descrevemos um modelo para a análise do conhecimento do professor de/que ensina matemática, considerando a perspetiva de que esse conhecimento é particular para o ensino da matemática e fornecemos um exemplo da aplicação desse modelo ao conhecimento mobilizado por um professor durante a prática de sala de aula. Primeiramente apresentamos o modelo analítico MTSK (conhecimento especializado do professor de/que ensina matemática), desenvolvido na Universidade de Huelva (Espanha), e posteriormente mostramos o seu potencial no caso de um professor do Ensino Médio na resolução de um problema com os seus alunos. Concluímos com um sumário das vantagens oferecidas pelo modelo em detalhar os elementos do conhecimento especializado mobilizados pelo professor e ilustrando como esse conhecimento possibilita planejar oportunidades de aprendizagem produtivas para os seus alunos. Finalizamos com algumas notas reflexivas relativamente ao potencial do MTSK como uma ferramenta para a formação de professores.

Palavras-chave: Conhecimento do Professor, Conhecimento Matemático, Conhecimento Pedagógico do Conteúdo, Conhecimento Especializado, Resolução de Problemas.

1. Introduction

For many, the raison d'être of research into mathematics education is to participate in bringing about improvements in teaching and learning mathematics. It is no easy task, however, to give a precise picture of what this improvement might look like, as it lies on the confluence of multiple dimensions, and is subject to the numerous conceptions and beliefs of those parties whose right (and obligation) they feel it is to intervene in such an endeavor (teachers, researchers, students, parents, business leaders, among others). But, whatever the ultimate shape of improvement, one thing that all parties can agree on is the importance of teacher training (both initial an in-service) as the incontestable means to the much contested ends. While it is true that teacher training means different things to different educational cultures, each with their own deeply rooted traditions and training programmes, it is no less the case that all training seeks to develop those professionals, who will be responsible for successfully implementing the educational proposals and reforms directed (one supposes) towards bettering the quality of life in our societies through bettering the quality of teaching and learning of their younger members.

When we turn our attention to the teachers themselves, further complexity arises amongst the already complex panorama of (mathematics) education, in the form of an intricate network of notions such as professional development (e.g., Even and Ball, 2009), professional identity (e.g., Brown and McNamara, 2011), good practice (e.g., Givvin, Jacobs, Hollingsworth, and Hiebert, 2009), expert teachers (e.g., Li and Kaiser, 2011), and professional knowledge (e.g., Ponte and Chapman, 2006), among others. The variety and quality of research into these areas have been profuse, and have added depth and complexity to our understanding of the professional needs of teachers. In this paper, we focus on teacher knowledge.

In this respect, a wide range of perspectives have been developed, although most are indebted to the work of Shulman (1986), in particular his three dimensions mapping the knowledge domain – subject matter knowledge (SMK), pedagogical content knowledge (PCK) and curricular knowledge. In the large majority of perspectives, SMK or PCK has been assumed to be the core aspect of (research into) teachers' knowledge, and hence of teachers' initial and continuous training. In this respect, various studies (from large-scale to individual case studies) and research groups focusing on teachers' knowledge (e.g., König, Blömeke, Paine, Schmidt, & Hsieh, 2011; Tatto, et al., 2012) have been developed, grounded on the idea that evaluating and measuring (e.g., Beswick, & Goos, 2012) or understanding such knowledge (e.g., Ball, Thames, & Phelps, 2008; Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013) is the best way to improve the quality of teaching. Thus, from the perspective of evaluation, and with PCK clearly in the foreground, a range of large and medium scale studies have been carried out, such as COACTIV and TEDS-M and its follow up studies (Krauss, Baumert, & Blum, 2008). In a review of research focusing on PCK, Depaepe, Verschaffel, and Kelchtermans (2013) identify several different PCK conceptualizations, leading in turn to a diversity of methods employed to study it.

When the focus switches to SMK, the different perspectives are grounded largely on the MKT conceptualization (Ball et al., 2008), and two main aspects are considered: the specialized nature of teachers' mathematical knowledge and the awareness of its relation to more advanced knowledge (*specialized content knowledge* –SCK – and *horizon content knowledge* – HCK). Such perspectives on mathematics teachers' knowledge (applying the ideas of Shulman to the specific area of mathematics) triggered a change in focus of the work being developed with prospective and in-service mathematics teachers (e.g., Schoenfeld, & Kilpatrick, 2008; Silverman, & Thompson, 2008; Speer, King, & Howell, 2015; Wasserman, & Stockton, 2013).

In the Brazilian context, a recent review reveals a different picture of research into mathematics teachers' knowledge. This study (e.g., Fiorentini and Crecci, 2017) reviewed all Master's and PhD theses presented between 2001 and 2012 (859 items) focusing on mathematics teachers (from kindergarten to university), and revealed that the large majority (over 95%) used Shulman (1986) or Tardif (2002) for their theoretical groundwork, and left aside the specificities recognized by the international community as being relevant to mathematics teachers' knowledge (SMK and PCK), focusing instead on the generalities of teaching. Although as yet undocumented, there has been a slight shift in focus in Brazilian mathematics journals over the last two years, chiefly with regard to the MKT conceptualization (e.g., Trivilin & Ribeiro. 2015) or on Davis and Renner's (2009) work on mathematics for teaching (e.g., Coutinho & Barbosa, 2016) to analyze teachers' classroom practice.

In Portugal, chiefly since 2005, with the implementation of the Bologna process and due, in large part, to the work of mathematics educators such as Canavarro, Ponte, Serrazina and Santos, amongst others, it has been compulsory for teacher training programs to focus on the development of teacher knowledge. Both initial and in-service programs – financed by the government – have followed this mandate. Educators have largely implemented their own research recommendations, based chiefly on a focus on the three Shulman subdomains, but also with a tendency to focus on PCK and curricular knowledge (e.g., Dias and Santos, 2016; Oliveira, Menezes, and Canavarro, 2012; Ponte, 2012).

2. The Mathematics Teacher's Specialized Knowledge (MTSK)

In recent decades, teachers' knowledge has been the object of numerous studies. Since Shulman's (1986) seminal work, researchers have tackled the issue of the nature and composition of this knowledge. Among them, the group based at the University of Michigan (Ball et al., 2008) stands out for this work in developing the MKT model (mathematical knowledge for teaching), which represented an application of Shulman's model to the area of mathematics teaching. The MKT model foregrounds the fact that the knowledge required by mathematics teachers differs from that required by other professions (hence the distinction it makes between teachers' specialized content knowledge – SCK – and others' common content knowledge – CCK). At the same time, it retains many of Shulman's original components, with various subdomains within MK and PCK.

In our experience in using MKT to do research (e.g. Climent & Carrillo, 2003; Climent, Romero, Carrillo, Muñoz-Catalán, & Contreras, 2013), we came up against various limitations of the model as a framework for analysis. Chief amongst these was the lack of clarity in the demarcation between sub-domains, notably between SCK and CCK, and between SCK and KCS (knowledge of content and students). To take a simple example, according to the conventional algorithm for multiplying natural numbers, intermediate results are aligned to the left of the result immediately above; but how can we determine whether knowing why this is so pertains to CCK or SCK? Is this knowledge something required by other professions? Is it something the person-in-the-street might know? Does it depend on what we understand school knowledge to include? This problem of demarcation is compounded by the way in which some of the definitions of the subdomains are worded, focusing on what the knowledge in question enables the knower to do, rather than on the content itself. This is the case of SCK, which is described as the kind of knowledge teachers draw on when "looking for patterns in student errors or in sizing up whether a nonstandard approach would work in general" (Ball, Thames & Phelps, 2008, p. 400). In response to these limitations, we suggest that the element of specialization pertaining to teachers' knowledge should not be considered as confined to a single subdomain of mathematical knowledge, but to all aspects of the teacher's knowledge that interest us, whether MK or PCK. In other words, specialization is intrinsic to the deployment of mathematical knowledge in the context of teaching and learning. By the same token, rather than define MK in contradistinction to other mathematical practitioners, we felt it would be beneficial to do so purely in terms of the mathematics involved. From this, it follows that we do not differentiate between common and specialized knowledge. Instead, we draw on Ma's (1999) notion of "profound understanding of fundamental mathematics", a kind of knowledge that makes connections, is structured around the curriculum and is longitudinally consistent with the core mathematical notions. The foundation of this profound knowledge is what we denominate *knowledge of topics* (KoT). Alongside this, we consider syntactic knowledge of mathematics (Schwab, 1978) fundamental for the teacher to generate new knowledge and to teach their students to do mathematics (the subdomain *knowledge of practices in mathematics*). Finally, still in the domain of MK, we recognize knowledge of interconceptual connections between mathematical content (Figueiras, Ribeiro, Carrillo, Fernández, & Deulofeu, 2011), comprising not only connections to more advanced content, but to more elementary content, too, as well as other types of connections we shall describe below (*knowledge of the structure of mathematics*, KSM). With respect to PCK, we define the subdomains regarding mathematics teaching and learning in such a way as to prevent any seepage with between this domain and that of mathematical knowledge. To this end, we center the learning of mathematical content on how the content itself determines the learning, that is to say, on the transformation of knowledge of the subject into forms which facilitate its learning (a focus denominated by Marks, 1991, the *interpretation* of content). From this perspective, we arrive at the subdomains of *knowledge of mathematics teaching* (KMT) and *knowledge of features of learning mathematics* (KFLM), which are comprised respectively of the teacher's theories of learning and teaching (both personal and academic). Finally, the organization of PCK is completed with Shulman's subdomain of *curriculum knowledge*, enlarged to include what students might be expected to learn at a particular level according to curricular and other guidelines. In summary, the MTSK model seeks to offer first a change of perspective in how we classify certain significant elements of mathematics teachers' knowledge, and secondly an enhanced degree of specificity in describing the subdomains, through, as shall be seen below, an inventory of categories.

Another important feature of the mathematics teacher's specialized knowledge model (MTSK) is that it includes the domain of beliefs about mathematics and mathematics teaching and learning (drawing on Bromme's, 1994, identification of the philosophy of mathematics as a component within the teacher's knowledge set).

In addition to the beliefs domain, the domains of mathematical knowledge (MK) and pedagogical content knowledge (PCK) are also each divided into three subdomains (figure 1). The subdomains of MK register the teacher's knowledge of *definitions, properties and fundamentals*, *phenomenology and applications*, *representations*, and *procedures*, all of which categories related to mathematical content (*knowledge of topics,* KoT), connections between mathematical content (connections involving an increase in complexity, connections involving simplification, auxiliary connections and cross-curricular) (categories of *knowledge of the structure of mathematics,* KSM), and ways of doing mathematics, such as defining, making conjectures, providing proof, and solving problems among others (*knowledge of practices in mathematics*, KPM). For its part, the subdomains of PCK comprise knowledge of strategies, techniques, tasks and examples; resources; and *theories of teaching* (categories of *knowledge of mathematics teaching*, KMT). The ways students interact with content, their expectations when faced with a particular content area, their difficulties and errors, and learning theories (categories of *knowledge of features of learning mathematics,* KFLM), and expected learning outcomes according to educational and cognitive level of the students (*knowledge of mathematics learning standards*, KMLS, in which we distinguish the categories learning expectations, expected conceptual or procedural level of development, and the sequencing of previous and subsequent topics). A more detailed account of the subdomains and categories can be found in Carrillo, Montes, Contreras & Climent (2017).

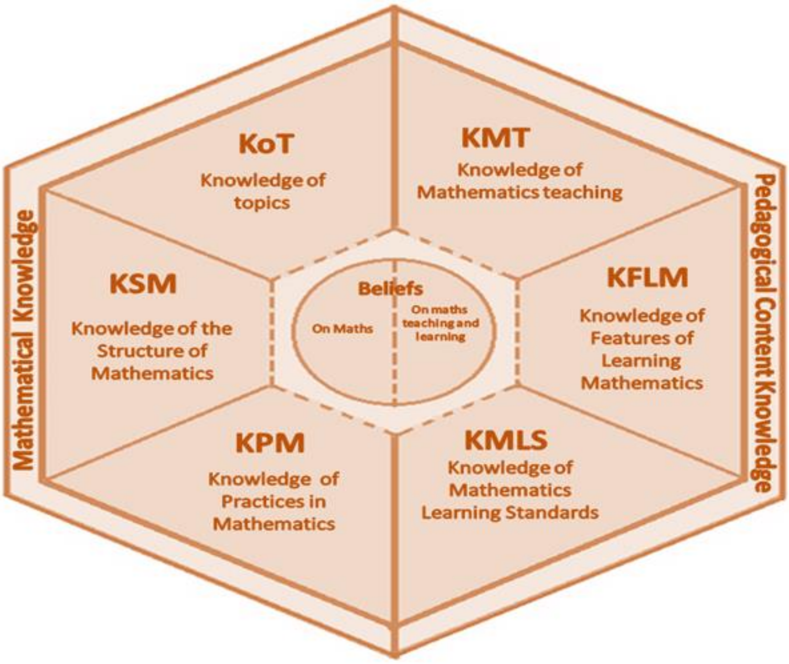

Figure 1: The MTSK model showing domains and subdomains

The MTSK model provides an analytical tool that enables details of teachers' knowledge to be registered (for instances of its use, see for example Montes & Carrillo, 2017). The model considers knowledge as complex not only for the number of elements that participate in it, but, above all, for the connections between these elements and subdomains. In this paper, we use the model to analyze the knowledge deployed by a teacher in the course of a lesson.

3. Paula's knowledge in the management of a solution to a problem

Paula is a teacher of Spanish nationality with, at the time of this study, some 18 years' experience in the profession. After completing a degree in mathematics, she sat the teaching entrance exam and earned a position as a high-school level and Spanish Baccalaureate teacher (7th to 13th grade) in the State system. She was an active participant in numerous courses, workshops and other in-service training, and formed part of a collaborative research project with other teachers at varying levels in the education system. One of the objectives of the group was to respond systematically to teaching and learning situations arising in the course of the members' work. This involved making recordings of lessons (as observers rather than participants) and subsequently analyzing them for the knowledge brought into play by the teacher. The recordings were complemented by an interview carried out by one of the members in the role of researcher. The choice of Paula as an informant in one such study arose from the group's interest in exploring how she teaches problem solving from the perspective of the MTSK model.

She had come across the problem, specifically about "Dissecting a square", in the book *Thinking Mathematically* (Mason, Burton and Stacey, 1988), which she had been recommended on one of the courses she attended, and this had provided the starting point for the lesson under analysis below. Encouraged by the course in problem-solving she had attended, Paula took on the challenge of "Dissecting a square", convinced of the learning benefits it might offer. At the same time, she had realized during the course that she had not pitted her mathematical abilities against this kind of problem for some time, which led her to dip into the recommended book and give the problem a go. We will not go into the solution to the problem here, but rather focus on her subsequent decision to present the same problem to her 11th grade students, albeit substituting the original squares for equilateral triangles.

Her handling of the adapted problem was analyzed according to the categories in the MTSK model, which was applied to both the lesson transcriptions and the interview. This system of categories was directed throughout at providing a fine-grained analysis of Paula's knowledge and at no point sought to subject it to any kind of value judgement, solely to understand it.

This section offers short commentaries on key excerpts of the ensuing lesson, which was two hours in length, in conjunction with corresponding extracts from a follow-up interview with Paula, in which the researchers asked her to amplify some of the pivotal episodes that occurred. The first of these is from the interview, and concerns her decision to switch squares for triangles.

R (Researcher): Why did you decide to do this problem with your 11th graders, and what made you change it to equilateral triangles?
P (Paula): There isn't any specific area in 11th grade related to the topic of the problem, but I thought it could encourage the kind of habit of thought which we really want to awaken in students of this age.
R: And what would they be?
P: Problem-solving strategies, I'd like to see them develop, realizing that there might be more than one solution to a problem, that there are various ways to arrive at the same result.
R: And the equilateral triangles, why use them?
P: Well, to be honest, I could have left it with the squares, but I thought that one or two of the students might think I was talking about square numbers and get confused. When I completed the problem with squares, I tried it with triangles and it struck me that it was very similar, it had the same solution. So that was why I thought it would be good to try it out with equilateral triangles.

There are various indicators of Paula's knowledge that can be identified in this excerpt. Her first turn demonstrates her knowledge of the curriculum (KMLS – *learning expectations*), which goes beyond the list of topics to also include habits of thought which are not specific to any particular topic, but are generally applicable to the subject. In explaining what she means by them, she gives her reasons for choosing this particular problem and so illustrates her knowledge of tasks for mathematics teaching (KMT – *strategies, techniques, tasks and examples*). In her last turn in this extract, she shows her knowledge of how her students typically interact with mathematical content (KFLM – *ways of interacting with content*), and likewise displays a manner of regarding mathematics education in which it is important to avoid situations which might cause confusion (beliefs). There are also indications of KoT, in that she reports having solved the problem, but we cannot be fully sure of this from the extract.

We now shift our attention to the classroom, to the moment when she presented the problem to her students. The class was mixed-ability, with 12 boys and 15 girls. There was high degree of participation during the lesson, as the extracts below illustrate.

P: The problem I've got for you today is different to the ones you usually do in class and I think you're going to like it [She projects it onto the screen]: "A number *n* is called 'nice' if an equilateral triangle can dissected into *n* non-overlapping equilateral triangles. What numbers are 'nice'?" You need to work on your own for this problem. Please, put up your hands when you think you get the solution.

After some four minutes, Marcelo raised his hand and Paula went over to his desk.

M (Marcelo): Teacher, the first nice number is 4, the second is 16, then 64 and so on.
P: Why is that?
M: Look, here is 4 [*points to figure 2*]. Then I can divide each of these triangles in four more and so on.
P: Do you know how you can express those numbers?
M: Umm…ah, yes, they're multiples of 4.
P: Like 8, right?
M: No, not 8.

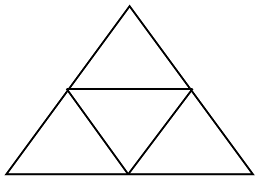

Figure 2: The number 4 as a nice number.

P: But 8 is a multiple of 4, isn't it?
M: The one after 4 is 16, which is 4^2, and after that 64, 16x4. I said it wrong, they're powers of 4, 4 raised to *n*.
P: So, would you say that [*writes on Marcelo's sheet*] $n= 4^n$?
M: No, no, I need to use another letter. I should say that the 'nice' numbers, which we'll call *n*, are those in the form of 4 raised to the power of *k*.
P: Very good, but there's more. What you've discovered is that all those in the form of 4^k are 'nice', not that all the 'nice' numbers are 4^k. And, by the way, you'll also need to demonstrate that all those triangles you've drawn are actually equilateral. But don't do that just yet; concentrate on finding more ways to dissect the triangle, then you can work on the proof.
M: That's impossible, teacher.
P: Don't be in such a rush, you'll find more.

This lesson extract illustrates how Paula regards her role as teacher. Rather than supply answers, she responds with an interrogative; instead of correcting errors directly, she sets the pupils to question their work so that they find it for themselves; and she asks for reasons, her students must explain their results (beliefs). Likewise, Paula's requirement for the students to provide an explanation or some form of argument in favor of their results derives from her knowledge of how mathematics is done, specifically how one goes about validating or proving in mathematics (she is not satisfied with a drawing as proof, she requires proof that the triangles the student has drawn are equilateral), and in particular in geometry (the fact that a shape or certain elements seem to fulfil a specific property does not prove it to be the case) (KPM). Also in relation to her KPM, she shows that she understands how to deploy the counterexample (in this instance "8") as part of mathematical reasoning. In the interview following the lesson observation, we asked Paula why she had asked Marcelo whether his solution was $n=4^n$. She replied that she wanted Marcelo to be precise when he gave his results, as otherwise he would have difficulties in expressing his solutions later as a result of inadequate notation. She went on to add that the expression he used ($n = 4^n$) represented a frequent error among the students, the tendency to use the same letter to refer to different quantities (an instance of Paula's KFLM, specifically awareness of her students' most common areas of weakness - *strengths and weaknesses*). The importance of mathematical precision also forms part of her knowledge of the ways of doing mathematics (KPM). Her next intervention was also related to KPM, dealing with the difference between the concepts of necessary and sufficient in a condition.

As can be seen in the previous extract of her interaction with Marcelo, he had been somewhat surprised when she had informed him that there were more 'nice' numbers. Paula had encouraged him to continue. When she told him not to spend his time proving that the triangles must be equilateral, she is illustrating her awareness that interrupting Marcelo's train of thought to set him to proving intermediate results might well break that highly productive train of thought (KFLM – *strengths and weaknesses*). Meanwhile, other students had put their hands up, and Paula circulated and had conversations very similar to that with Marcelo. However, her conversation with Daniela followed a different pattern. When Paula asked her for her solution, the following exchange ensued:

D (Daniela): I got 4. Then I thought I'd extend the edges of the triangle. So, for 4, I had 1 triangle on top and three underneath; now I've got 1 on top, three in the middle and 5 along the bottom, which gives me 9. If I carry on like that I get 16 and then 25. Each time I need to add an odd number: 1+3 (4) +5 (9) +7 (16) +9 (25) (figure 3).
P: So what numbers are those?
D: I don't know, they're all the addition of odd numbers.
P: Like the number 8, which is 3+5, or 26, which is 11+15, for example?
D: No, those aren't 'nice' numbers. They have to start with 1 and then go in sequence.
P: They need to be the addition of consecutive odd numbers starting with 1. OK, but what are the numbers 4, 9, 16 and 25?
D: I don't know.
P: They're the perfect squares. Do you see?
D: Oh, of course. Silly me!

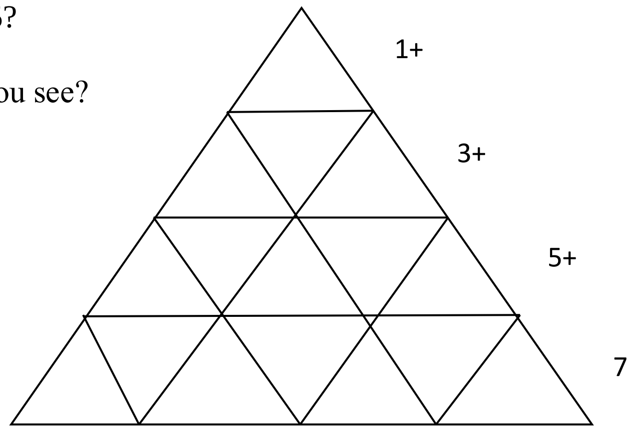

Figure 3: Obtaining square numbers through the sum of consecutive odd numbers

Like Marcelo, Daniela obtains the expression $n=k^2$, and Paula suggests that there are more solutions, which leaves Daniela a little puzzled.

As before, it is possible to point of evidence of beliefs and KPM, but the thing that most surprised us in this instance was that Paula was inclined to tell Daniela directly that her number sequence was the perfect squares. We asked her to explain her reasons for doing so.

P: I didn't give any importance to the fact that she didn't know the name for that particular sequence of solutions.
R: But perhaps the pupil hadn't realized that all the numbers were perfect squares.
P: True, but at the time I thought it was just the name that she didn't know, I didn't think that she might not have been aware that 4 is the square of 2, 9 of 3 and so on.

Paula explains her interpretation of the sequence to Daniela. Daniela had obtained her numbers through successive additions, which guaranteed her a way of constructing the sequence, that is, a means of generating 'nice' numbers. However, the pattern of the sequence as the squares of whole numbers was not necessarily obvious to the pupil. Paula's intervention in this respect illustrates her beliefs about the role of terminology in the generation and application of mathematical knowledge (beliefs): for her, not knowing the denomination of a term should not prevent anyone from doing mathematics. But, at the same time, as she herself recognized, the supposition that this interpretation of the sequence was transparent to Daniela (KFLM – *ways of interacting with content*) led her to (unintentionally) reveal it without giving the pupil the opportunity to discover it for herself.

Nevertheless, Paula's exchange with Daniela did not finish with the revelation of the perfect square sequence.

P: Look, Daniela, what you've done is to take a triangle and extend it to obtain more solutions, but what I actually asked you to do was tell me what the 'nice' numbers are if you take any triangle. Think about that for a few minutes.

After a while, Paula returned to Daniela's desk.

P: OK, have you thought about it?
D: What I did before is right, because I can start with a triangle, it doesn't matter which, and if I divide the sides into 2, I get 4 triangles, and if I divide it into 3 I get 9, and so on.

Paula was anxious for Daniela to recognize the equivalence of the two approaches (KoT – *procedures* – distinguishes the strategies of dissection and reconstitution to solve the problem and recognizes that in this case they are equivalent), and to be aware of the need for precision in her argumentation (KPM).

While the exchanges above were taking place, the rest of the students continued working on the problem, and the majority arrived at the sequence of perfect squares. At this point, Paula got the attention of the whole class and summarized what they had found so far, that the numbers expressed in the form k^2 are 'nice' numbers. One of the students then asserted that those expressed in the form 4^k were also 'nice', to which another responded that the form k^2 was inclusive of the form 4^k. With that, the pupils considered the problem had been brought to a successful conclusion, but then Paula informed them that there were still more solutions to be found, which caused some consternation among the class. After a few moments of reflection, Claudia called the teacher over.

C (Claudia): Can the triangles be different?
P: I didn't say they had to be the same.
C: OK, then I've got 7.
P: How?
C: I split it into four and then split one of those four into another 4 (figure 4).

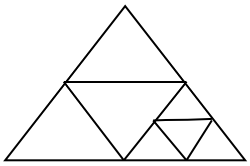

Figure 4: 7 as a 'nice' number.

P: Very good, but there are more.
C: Yes, of course, now I can get 10, 13.
P: How can you express this sequence?
C: As n=4+3k. I start with 4 and keep dividing one of the triangles into 4.
P: That's very good, Claudia. Now tell your classmates that you've got 7.

After various attempts, everyone realizes that the triangles can be different sizes. After a few minutes, many students reach the sequence 4+3k. Others arrive at the sequence k^2+3. In general, the class is surprised by how many (infinity) 'nice' numbers there are, which leads some to suspect that all numbers ("except a few at the beginning") will turn out to be 'nice'.

This excerpt confirms Paula's knowledge of how to encourage her students to stay focused on solving the problem (KFLM – *interests and expectations*). It also underlines the importance of making sure the students understand the formulation of the problem and do not make unwarranted assumptions. In the follow-up interview, Paula told us that it was her intention that the students, after initially taking for granted that the triangles should be the same size (thus illustrating her awareness of how her students tended to respond to content items – KFLM – *ways of interacting with content*), would realize that such a condition was not given in the statement ("This is very important when they solve problems. Understanding the statement properly is fundamental") (KPM).

Once the students reach this realization, they begin to obtain new numbers in no particular order. For example, one student obtains the number 17 starting from 25; another obtains 8 starting from 16. Then another student, Mario, informs that he has identified all the 'nice' numbers.

P: Are you sure?

Ma (Mario): Yes. Between us, we've obtained 7, 8 and 9, and some more, but I'm going to focus on these three. So, if we do to 7 what we did with 4, we get 10, if we do the same to 8 we get 11, and if we do it to 9 we get 12, and so we can carry on indefinitely.
P: Very good. So, what are the 'nice' numbers?
Ma: Well, the numbers 4 and 7, and after that all of them.
P: And what about the numbers 2, 3, 5 and 6?
Ma: They're not 'nice'.
P: Why not?
Ma: Because you can't get them by dividing a triangle into another 4 triangles.
P: That would seem to be true, but it really only means that we can't dissect a triangle into, say, 6 using this way. It doesn't mean that we can't dissect it another way. I'll leave you to think about it for a while to see if you can find a way to get 6 or if you can find an argument in support of the idea that 6 is not a 'nice' number.

Once again, Paula highlights the need to provide reasons for any conclusions reached. Likewise, she is aware of the difference between a condition being sufficient and it being necessary when she explains to Mario that a particular way of dissecting a triangle is not a necessary condition for obtaining 'nice' numbers (KPM). She offers Mario a heuristic tip to keep him on track to solving the problem (focus on a specific instance, the number 6) (KPM, regarding knowledge of strategies for problem-solving).

Meanwhile, the other pupils have continued their efforts and one reports obtaining the number 6, starting from 9: they dissect the original triangle into 9 small triangles and then group together 4 contiguous ones at the apex, leaving 5 at the base (figure 5).

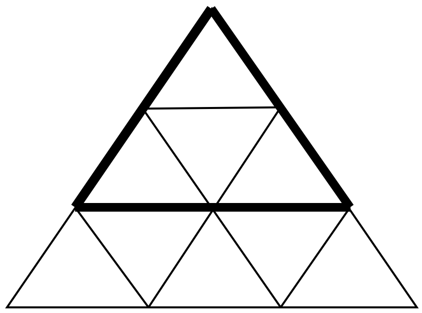

Figure 5: Dissection of 6 as a 'nice' number

Paula addresses the whole class while various pupils draw their dissections on the board. She informs them that, effectively, all numbers are 'nice', with the exception of 2, 3 and 5 (adding that 1 can be considered 'nice' in the trivial sense that it requires no dissection to meet the conditions). She goes on to say that, for the moment, the class will not enter into why these particular numbers cannot be considered 'nice', but that in any event, the argument could be that these numbers were not obtained via any of the methods employed – "that can provide some reinforcement for our suspicions that they are not 'nice', but no more". In this way, Paula demonstrates her knowledge of how a result can be validated mathematically, as well as knowledge of the role of conjectures and how these are constructed (KPM). She adds that the students now need to prove that all the triangles they have drawn are equilateral. In fact, she had already challenged one student to show that the dissection met the condition of being equilateral triangles.

Paula's intervention with Juan, a pupil experiencing difficulties in generalizing his results, is also noteworthy. Juan had not managed to obtain many 'nice' numbers by himself, but seeing that he had obtained 4, 7, 10 and 13, he called Paula over to his desk and told her the sequence was 4+3k.

P: Why do you think that numbers with the form 4+3k are 'nice'?
J (Juan): Because we've got 4, 7, 10 and 13. So, the rule is 4+ 3k.
P: And how can you be sure that 16 and 19 are 'nice'?
J: Because they're next in the sequence.
P: Juan, when we studied sequences, we looked at what the terms in the sequence had in common and we inferred the rule, but normally there are more possibilities. I mean, if I give you the sequence 2, 4, 6, 8 and ask you what comes next you'll say 10. But there could be other possibilities; I might write 11 and we would still be able to find a rule. What I mean is that if the problem imposes the condition of obtaining a dissection, an arithmetic argument is of no use to us, we have to use a geometric reason. Try and find that reason. If you find it, go and work with Claudia.

The extract below reproduces Paula's answers to our question about her conversation with Juan in the follow-up interview.

R: Do you think Juan understood the significance of what you told him?
P: No, I don't think that Juan understood what I said to him at any deep level, but I think that at least he recognized and acknowledged that working out an arithmetic rule wasn't enough. The students tend not to see the need to check things, they're happy to assume that everything works in a particular way. And they also tend to generalize far too readily. In this particular instance they were beguiled by how easy the sequence was for them, so it was more unlikely that they would see the need for geometric reasoning.

In our view, this excerpt illustrates Paula's knowledge of the different types of proof in mathematics (KPM), and through her questions she attempts to encourage Juan to recognize the need to find the appropriate reasoning (KMT – *strategies, techniques, tasks and examples*). At the same time, she demonstrates her awareness of how her students typically react in these situations, and, in particular, what they find difficult (KFLM –*strengths and weaknesses*). Her use of examples for highlighting the need to back up the pattern with geometric reasoning is also worthy of mention. We cannot be sure from this extract alone that she knows the procedures for identifying the pattern from n terms, but there is evidence that she knows this is possible, as it is this which allows her to give Juan an example so as to encourage him to try and find a geometric explanation. Here we can see knowledge of the topic of sequences (arithmetic progressions, KoT, in connection with the topic of the problem, KSM) and knowledge of examples for learning (KMT).

We will now go back to the moment in which she reveals that all numbers are 'nice' except for 2, 3 and 5, and then elicits the argument explaining why all the triangles are equilateral. Paula then asks her students to give a dissection of the number 1253. Various students say that it takes the form 6+3k, 7+3k or even 8+3k; others seem not to understand, and Paula asks Pedro to explain.

Pe (Pedro): If we've got a 'nice' number, then we've seen that we can obtain a number which is three units higher and so on. I think that they have chosen 6, 7 and 8 because they are the first three consecutive numbers which are 'nice', and from there we can get to all the other numbers, can't we, teacher?
P: That's absolutely right, you've explained it very well. Adding three at a time from 6, then 7, then 8, we can get to all the numbers we want [writes on the board 6+3+3+3+3..., 7+3+3+3..., 8+3+3+3...]. For example, how can we get to 9 this way? And 10? And 11? (Various students indicate 6+3, 7+3, and 8+3, respectively). And what about 12? 13? 14? [They associate these with 6+3x2, 7+3x2 and 8+3x2]. Do you think we can get to any number starting with these expressions? [It seems that the students understand the expressions and are aware that they generate all the numbers].
P: But I asked you to dissect the number 1253. How can we obtain it?
R (Roberto): You would need to start with 6 and carry on dissecting till you got as close as you can to 1253. If you can't do it with 6, then you'll have to try with 7, and failing that with 8.
A (Antonia): Do we have to do all that? That's very long.
P: Let's imagine for a moment that we have to do it. We would want to start with the right number from the beginning, I mean, it'd help to know which of the three numbers to start from. How can we work it out?
J: I'd divide by 6 and if I got remainder 3, then I'd start with 6.
P: OK, Juan, do it.
[Juan divides 1253 by 6 and gets a quotient of 208 and a remainder of 5; he then repeats this for 7, getting a quotient of 179 and a remainder of 0; finally, he divides by 8 and gets a quotient of 156 and a remainder of 5. He concludes that they should start with 7.]
M: There's something I'm not sure about. We're supposed to be dividing by 7 to find out how close 1253 is to being a multiple of 7. But that would be 7k, and here we've got 3k. So, shouldn't we be dividing by 3 to see if the remainder is 6, 7 or 8?
D: But the remainder can't be more than 3.
M: That's true. The remainder can only be 0, 1 or 2.
P: Think about it, if you've got a multiple of 3, that is a number in the form 3k, what happens if you add 6 to it?
Ss (various students): It's still a multiple of 3.
P: And if you add 7?
M: You get a number which, if you subtract 1, is a multiple of 3.
P: Do you see?
Ss: Yes.
P: And if we then add 8?
Ss: We take away 2 and get a multiple of 3.
P: So, divide 1253 by 3.
[The students perform the calculation.]
Ss: You get a quotient of 417 and a remainder of 2.
P: And what does this tell us?
M: That we need to start from 8.
Pe: Why?
R: Because 8 is 6 plus 2.
J: I don't get it.
P: Let's see. Pedro and Juan, and anyone else who doesn't get it, how many times do you have to successively dissect a triangle into four smaller ones?
J: 417 times.

P: Correct. And you have 2 left over, which means that if you start with 2 you'll get to 1253. Do you get it now?
J: But you can't start with 2.
P: No, you can't. But 8 is 2 plus two times 3. So, 2 plus two times 3 plus 415 times 3 will give you 1253, won't it?
D: Oh, yes, so we can say that we can use the sequences 3k, 1+3k and 2+3k.
P: That's right. But you need to bear in mind that we can't start dissecting from any number, because some are not 'nice'.
A: Humph, this is really hard for me to understand, but anyway, it's a drag if you have to do the dissection so many times.
P: Absolutely. Which is why we need to go one step further. We need to find a quicker way to get the dissections. Now you need to think hard about the kinds of calculations we can do with the natural numbers (0, 1, 2, 3, 4, and so on). It's a new area for you, but you'll see that you can do it. How do mathematicians work things out with the natural numbers? Well, if they can think of a system which works for all the numbers they say what it is and that's it finished. But if they can't come up with this kind of system, they think of one system for the even numbers and another for the odd numbers. And if that doesn't work either, then they think of a system for the multiples of 3, another for the multiples of 3 plus 1, and another for the multiples of 3 plus 2, and so on. This is what we are going to do now. First, we'll think of a way to dissect the triangle which works for all the even numbers (except 2, of course), and then we'll find another way that works for all the odd numbers (except for 3 and 5).

This longer excerpt begins with Paula revoicing (Chapin, O'Connor and Anderson, 2009) Pedro's contribution with the aim of making sure the other students have understood what he said. As confirmed in the post-lesson interview, she was aware that when her students were called on to explain their reasoning, they did not always make all aspects explicit and often left out important elements (knowledge of how the students interact with content, KFLM), which tended to make it difficult for their classmates to understand (knowledge of difficulties in learning mathematics, KFLM). Her strategy here, whereby she writes 6+3+3+3+3..., 7+3+3+3..., 8+3+3+3... on the board, seems to be effective. Straight afterwards, Paula allows Juan to apply his reasoning, even though it is wrong.

Paula demonstrates her knowledge of aspects of divisibility among the set of natural numbers, specifically the equivalence of expressions in which the constant term can be decomposed in order to facilitate the calculation, taking account of the remainder of the whole division (for example, the sequence 7+3k, $k\geq0$, has the same elements that the sequence 1+3k, $k\geq2$) (KoT). Indeed, the association Paula makes between the division and the result (underlining the number of times a triangle need to be dissected – the quotient – and the 'nice' number from which the dissection should begin – the remainder) is non-trivial. When asked about this in the follow-up interview, she stated that "in sequences we can see divisibility, and in divisibility you've got division, multiplication and the remainder, which they don't give much importance to because in the majority of the problems they do what they're interested in is the quotient. What I wanted was for them to see these connections, although it was difficult for them." We can see evidence here of Paula's knowledge of the connections between different content items (KSM – *cross-curricular connections*), as well as her knowledge of the difficulties encountered by her students (KFLM). The lesson excerpt above ends with a challenge to find a more elegant solution (KPM), in which Paula explains a typical procedure for working in mathematics, especially with regard to the set of natural numbers (KPM). This excerpt also marks the end of the lesson.

4. Final reflections

When one does research into complex phenomena, it is impossible to include all the variables and dimensions in one study alone. All one can hope to do is, as plausibly as possible, approach the phenomenon with respect to one or two of these dimensions. Many such dimensions intervene in the teaching and learning of mathematics; in this paper, we have focused on the specialized knowledge of a mathematics teacher, and with the aid of the MTSK model we have attempted to foreground some of the more interesting details which make up this knowledge.

Over the course of our analysis of Paula's lesson, we have laid bare the specialized knowledge that she brought into play. We accumulated evidence of knowledge pertaining to each of the subdomains, and we have also demonstrated that, at times, one also finds indications and opportunities to delve further, as in fact we were able to do in the follow-up interview. The MTSK model enabled us to gain access to this information through careful observation, but in addition, and more significantly for us, it allowed us to focus on each individual element of her knowledge and so understand the connections between them, and, in particular, how items from one subdomain receive support from items from another.

Like all analytical models, MTSK enables items of knowledge to be brought to the surface, which, in reality, co-exist in synthetic relation with other items. In narrating our analysis of Paula's lesson, we have attempted to highlight these connections. The lesson excerpts have shown us the confluence of items from various subdomains and how they become coherent when they are considered together.

Paula's teaching has provided evidence of her knowledge of the learning expected of her students at this level (11^{th} grade), the specific topic of the problem, and even the habits of thought encouraged by the subject (KMLS). She displayed sound criteria for selecting learning tasks appropriate to these expectations, as well as suitable questions and examples for getting her students to keep moving forwards towards the ultimate solution to the problem (KMT). She understands how her students interact with mathematical content, where their main areas of difficulty lie, and how to encourage them to keep on looking for solutions (KFLM). Paula demonstrates all this knowledge in the course of her lesson, backed up by her knowledge of the mathematical topic related to the problem, which brings to the fore the solution to the problem itself, mathematical sequences, divisibility and the equivalence of different approaches (KoT). Consistent with this, her performance is mediated by a mode of viewing mathematics teaching and learning founded on perceiving error as natural and a valuable means of learning, and considering meaning above terminological knowledge, and process above result (beliefs). Paula also shows she knows key features of proceeding in mathematics, such as proof, accuracy in communicating one's work, the importance of understanding the statement of a problem, the process of constructing a conjecture, the difference between a necessary and a sufficient condition, the significance of mathematical elegance, and the usual way of going about mathematics with the natural numbers (KPM). It should also be noted that she deploys her knowledge of these features at the moment she considers most propitious so as not to break the creative chain leading to the solution (KFLM). In addition, she provides evidence of knowledge of topics that can be associated with finding a solution to the problem, such as mathematical sequences (KSM).

This specialized knowledge (Paula's MTSK) enabled highly productive learning opportunities to be established, where the students were able to make significant progress towards solving the problem. The problem itself allowed for different levels of execution, from obtaining isolated solutions to obtaining a system that worked for all 'nice' numbers, and even the creation of ways of dissecting the triangle quickly for any 'nice' number. The students were free to try making random attempts or to be more systematic in their search for patterns underpinning the different sequences; they experienced a situation in which knowledge from different topics came together to account for their solutions, and they realized that some 'nice' numbers could be obtained by varying ways of dissecting the original triangle. Likewise, the students worked together to find a collaborative solution, treating all the solutions as their own and drawing on the knowledge of a classmate's solution to motivate their search for a dissection which met this solution.

The MTSK model, in addition to being an instrument for analyzing the mathematics teacher's specialized knowledge, is a formative tool. This aspect has been highlighted in the heart of the collaborative research project team (mentioned above), in which the MTSK model has been useful for reflecting on the specialized knowledge deployed in this experience and on the specialized knowledge of each one of the project components. Further, from the perspective of teacher training, the foregoing analysis presents us with opportunities for training at both initial and in-service levels. Paula's specialized knowledge is what enables her to generate fertile learning situations for her students, and it is precisely the analysis of the situations that Paula instigates that makes the MTSK model such a powerful tool for teacher training. The organization and categorization of the domains of MTSK (mathematical knowledge, pedagogical content knowledge and beliefs) and awareness of the connections between these domains and their subdomains can all be seen to be deployed by Paula in an authentic classroom context, and this provides trainee teachers with the opportunity to approach expertly managed situations of mathematics teaching and learning linked to those subdomains or categories susceptible to being developed. What's more, MTSK could serve to reflect on the knowledge of mathematics teacher trainers. Indeed, Kilpatrick and Spangler (2016) indicate that the subdomains MTSK KoT, KSM, KPM "should be included in the mathematical knowledge, skills, and abilities expected of future mathematics education professors" (p. 301).

References

Ball, D. L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Beswick, K., & Goos, M. (2012). Measuring Pre-Service Primary Teachers' Knowledge for Teaching Mathematics. *Mathematics Teacher Education and Development*, *14*(2), 70-90.

Bromme, R. (1994). Beyond subject matter: A psychological topology of teachers' profesional knowledge. In R. Biehler, R. Sholz, R. Sträer, & B. Winkelman

(Eds.), *Didactics of Mathematics as a Scientific Discipline* (pp. 73-88). Dordrecht: Kluwer Academic Publisher.

Brown, T., & McNamara, O. (2011). *Becoming a mathematics teacher: identity and identifications*. New York: Springer.

Carrillo, J., Climent, N., Contreras, L.C., & Muñoz-Catalán, M.C. (2013). Determining Specialised Knowledge for Mathematics Teaching. In B. Ubuz, C. Haser, & M.A. Mariotti (Eds.), *Proceedings of the VIII Congress of the European Society for Research in Mathematics Education (CERME 8)* (pp. 2985-2994). Antalya, Turkey: Middle East Technical University, Ankara.

Carrillo, J., Montes, M., Contreras, L.C., & Climent, N. (2017). Les connaissances du professeur dans une perspective basée sur leur spécialisation : MTSK. *Annales de Didactique et de Sciences Cognitives*, *22*, 185 - 205.

Chapin, S.H., O'Connor, C., & Anderson, N.C. (2009). *Classroom Discussions: Using Math Talk to Help Students Learn, Grades K-6*. Sausalito, CA: Math Solutions.

Coutinho, J.L.E., & Barbosa, J.C. (2016). Uma matemática para o ensino de combinação simples a partir de um estudo do conceito com professores. *Educação Matemática Pesquisa (Online)*, *18*, 783-808.

Climent, N., & Carrillo, J. (2003). El dominio compartido de la investigación y el desarrollo profesional: una experiencia en matemáticas con maestras. *Enseñanza de las Ciencias*, *21*(3), 387-404.

Climent, N., Romero, J.M., Carrillo, J., Muñoz Catalán, M.C.& Contreras, L.C. (2013). ¿Qué conocimientos y concepciones movilizan futuros maestros analizando un vídeo de aula? *Relime: Revista latinoamericana de investigación en matemática educativa*, *16*(1), 13-36.

Davis, B. & Renert, M. (2009). Mathematics-for-Teaching as shared dynamic participation. *For the Learning of Mathematics.* 29(3), 37-43.

Depaepe, F., Verschaffel, L., & Kelchtermans, G. (2013). Pedagogical content knowledge: A systematic review of the way in which the concept has pervaded mathematics educational research. *Teaching and Teacher Education*, *34*, 12-25.

Dias, C., & Santos, L. (2016). Reflective portfolio of mathematics as a tool for self-regulation of learning for high school students. *RELIME*, *19*(2), 187-216.

Even, R., & Ball, D. L. (Eds.). (2009). *The professional education and development of teachers of mathematics: The 15th ICMI Study*. New York: Springer.

Figueiras, L., Ribeiro, C. M., Carrillo, J., Fernández, S., & Deulofeu, J. (2011). Teacher's advanced mathematical knowledge for solving mathematics teaching challenges: a response to Zazkis and Mamolo. *For the Learning of Mathematics 31*(3), 26-28.

Fiorentini, D., & Crecci, V.M. (2017). Metassíntese de pesquisas sobre conhecimentos/saberes na formação continuada de professores que ensinam matemática. *Zetetike*, *25*(1), 164-185.

Givvin, K.B., Jacobs, J., Hollingsworth, H., & Hiebert, J. (2009). What is Effective Mathematics Teaching? International Educators' Judgments of Mathematics Lessons from the TIMSS 1999 Video Study. In J. Cai, G. Kaiser, B. Perry and N.Y. Wong (Eds.), *Effective Mathematics Teaching from Teachers' Perspectives. National and Cross-National Studies* (pp. 37-70). Rotterdam/Taipei: Sense Publishers.

Kilpatrick, J., & Spangler, D.A. (2016). Educating future mathematics education professors. In L.D. English, & D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education* (pp. 297-309). New York: Routledge.

König, J., Blömeke, S., Paine, L., Schmidt, W. H., & Hsieh, F. J. (2011). General pedagogical knowledge of future middle school teachers: On the complex ecology of teacher education in the United States, Germany, and Taiwan. *Journal of Teacher Education*, *62*(2), 188-201.

Krauss, S., Baumert, J., & Blum, W. (2008). Secondary mathematics teachers' pedagogical content knowledge and content knowledge: validation of the COACTIV constructs. *ZDM*, *40*(5), 873-892.

Li, Y., & Kaiser, G. (Eds.). *Expertise in mathematics instruction. An international perspective*. New York: Springer.

Ma, L. (1999). *Knowing and Teaching Elementary Mathematics*. Mahwah, NJ.: Lawrence Erlbaum Associates.

Marks, R. (1991). *When Should Teachers Learn Pedagogical Content Knowledge?* East Lansing, MI: National Center for Research on Teacher Learning. (ERIC Document Reproduction Service No. ED335338).

Mason, J., Burton, L., & Stacey, K. (1988). *Pensar matemáticamente*. Madrid: Labor (Original: 1982. Thinking mathematically. Reading, Massachusetts: Addison-Wesley).

Montes, M., & Carrillo, J. (2017). Conocimiento Especializado del Profesor de Matemáticas acerca del Infinito. *Bolema*, *31*(57), 114-134.

Oliveira, H., Menezes, L., & Canavarro, A. P. (2012). The use of classroom videos as a context for research on teachers'practice and teacher education. *Proceedings of ICME 12*, 4280-4289.

Ponte, J. P. (2012). A practice-oriented professional development programme to support the introduction of a new mathematics curriculum in Portugal. *Journal of Mathematics Teacher Education*, *15*(4), 317-327.

Ponte, J.P., & Chapman, O. (2006). Mathematics teachers' knowledge and practices. In A. Gutiérrez and P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future*. Rotterdam: Sense Publishers.

Schoenfeld, A. H., & Kilpatrick, J. (2008). Toward a theory of proficiency in teaching mathematics. *International handbook of mathematics teacher education*, *2*, 321-354.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(*2*), 4-14.

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education*, *11*(6), 499-511.

Speer, N. M., King, K. D., & Howell, H. (2015). Definitions of mathematical knowledge for teaching: using these constructs in research on secondary and college mathematics teachers. *Journal of Mathematics Teacher Education*, *18*(2), 105-122.

Tardif, M. (2002). *Saberes docentes e formação profissional*. Petrópolis: Vozes.

Tatto, M. T., Peck, R., Schwille, J., Senk, S. L., Ingvarson, L., Rowley, G., Peck, R., Bankov, K., Rodriguez, M. & Reckase, M. (2012). *Policy, Practice, and Readiness to Teach Primary and Secondary Mathematics in 17 Countries: Findings from the IEA Teacher Education and Development Study in Mathematics (TEDS-MM)*. Amsterdam: International Association for the Evaluation of Educational Achievement.

Trivilin, L. R., & Ribeiro, A. J. (2015). Mathematical Knowledge for Teaching Different Meanings of the Equal Sign: a study carried out with Elementary School Teachers. *Bolema*, *29*(51), 38-59.

Schwab, J.J. (1978). Education and the structure of the disciplines. In I. Westbury & N.J. Wilkof (Eds.*). Science, curriculum and liberal education* (pp. 229-272), Chicago: University of Chicago Press.

Wasserman, N., & Stockton, J. (2013). Horizon content knowledge in the work of teaching: A focus on planning. *For the Learning of Mathematics*, *33*(3), 20-22.