STUDENTS' UNDERSTANDING OF AREA: COMBINING PRACTICAL AND MATHEMATICAL KNOWLEDGE WITH A REAL-WORLD TASK

COMO OS ALUNOS ENTENDEM ÁREA: COMBINANDO CONHECIMENTO PRÁTICO E MATEMÁTICO COM UMA TAREFA REAL

Received: 24 April 2018
Accepted: 21 July 2018

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ABSTRACT

Real-world contexts in mathematics could be used as learning situations that allow students to combine their mathematical and contextual knowledge. This study explores what mathematical and contextual knowledge 15 eighth-graders brought to a packing task. Students worked on the task for three 45-minute periods of class. The classroom teacher led the implementation of the task, and the author, as participant-observer, was a supplemental guide. Students solved the task in different ways: applying the area formula, computing volume, and iterating an area unit. The analysis of students' written work, audio recordings, and researcher's field notes informed that students used their mathematical knowledge first, and after the teacher asked them about the context of the task, students started using their contextual knowledge. Students experienced *disequilibrium* when realizing that the formula for area gave them an answer that was inadequate for the context of the task. A discussion about the meaning of the context and the formula for area emerged. This study illuminates how tasks involving real-world contexts and the teacher's role create opportunities for students to combine their contextual and mathematical knowledge. This article contributes to discussions about the use of real-world tasks in mathematics classrooms and their interaction with students' knowledge.

Keywords: concept of area, real-world context, practical knowledge, high school mathematics education.

RESUMO

Contextos do mundo real em matemática podem ser usados como situações de aprendizagem que permitem aos alunos combinar seus conhecimentos matemáticos e contextuais. Este estudo explora quais conhecimentos matemáticos e contextuais 15 alunos do oitavo ano trouxeram para uma tarefa de empacotamento. Os alunos trabalharam na tarefa por três períodos de 45 minutos de aula. O professor da sala de aula liderou a implementação da tarefa, e o autor, como participante-observador, foi um guia suplementar. Os alunos resolveram a tarefa de diferentes maneiras: aplicando a fórmula de área, computando o volume e com a iteração de uma unidade de área. A análise do trabalho escrito dos alunos, gravações de áudio e notas de campo do pesquisador informaram que os alunos usaram seu conhecimento matemático primeiro e, depois que o professor perguntou sobre o contexto da tarefa, os alunos começaram a usar seu conhecimento contextual. Os alunos experimentaram desequilíbrio ao perceber que a fórmula para a área lhes dava uma resposta inadequada para o contexto da tarefa. Uma discussão sobre o significado do contexto e a fórmula para a área emergiram. Este estudo esclarece como tarefas envolvendo contextos do mundo real e o papel do professor criam oportunidades para os alunos combinarem seus conhecimentos matemáticos e contextuais. Este artigo contribui para discussões sobre o uso de tarefas do mundo real em salas de aula de matemática e sua interação com o conhecimento dos alunos.

Palavras-chaves: conceito de área, contexto do mundo real, conhecimento prático, ensino de matemática no ensino médio.

1. Introduction

> *Kent: So, is it 216 [trees] or 223 [trees] that will fit then? Because the area of 8 times 27 is 216 but what we got dividing the area of the tree by the area of the truck [meant dividing area of truck by area of the tree] was 223.*

Area is a crucial concept within the school curriculum. According to its placement in the Common Core State Standards for Mathematics ([CCSSM], National Governors Association Center for Best Practices and the Council of Chief State School Officers, 2010) area should be introduced as a mathematical concept in third grade. In third grade, students should identify area as an attribute of plane figures, measure area counting area units, and associate area with the operations of multiplication and addition. By sixth grade, students should be able to find areas of different kinds of triangles, special quadrilaterals, and polygons by composing or decomposing. At this point, students are also encouraged to use these techniques to solve problems involving real-world contexts. Finally, in grade seven, the last grade in which the CCSSM refers to the concept of area, students are supposed to "know the formulas for the area" and solve real-world problems. Therefore, what students learn about the concept of area in elementary grades is mainly what they apply in the subsequent courses. As a consequence, it is not a surprise that students' understanding of the area concept is to calculate area through formulas (e.g., Baturo & Nason, 1996; Murphy, 2011). Using the area formulas without considering their meaning can lead to students measuring areas by adding side lengths (e.g., Baturo & Nason, 1996; Nunes, Light, & Mason, 1993), being unsure about the connection between area units and length units (e.g., Kordaki & Potari, 1998; Simon & Blume, 1994), or lacking a dynamic understanding of area (e.g., Baturo & Nason, 1996; Kamii & Kysh, 2006).

Rather than being reduced to a formula, area is a concept that students could explore using real-world contexts. The National Council of Teachers of Mathematics (NCTM, 2014) encouraged the use of curriculum that exposes the relevance of mathematical practices and their applications to solving real-world problems. However, the use of real-world contexts could present a challenge to teachers and mathematics educators, because students might naturally bring their out-of-school knowledge to solve these tasks (e.g., Lubienski, 2000) or might use their mathematical school knowledge alone (e.g., Silver, Shapiro, & Deutsch, 1993). As is the case in the excerpt above, this could lead to different solutions for the same problem, and some of them could be incorrect according to the context of the task. Therefore, it is important that teachers understand the types of knowledge students bring to bear when working on real-world tasks. Ideally, students are encouraged to use a combination of their personal mathematical knowledge, i.e., school mathematics, and practical/contextual knowledge, i.e., out-of-school knowledge or *mathematical power* (Kastberg, Ambrosio, McDermott, & Saada, 2005).

In this article, I provide insight into the contextual and mathematical knowledge students bring to bear when solving an area task involving a real-world context. I also describe how the teacher's comments motivated students to bring together these two types of knowledge. My goal is to contribute to discussions about the use of real-world tasks in mathematics classrooms and their interaction with students' mathematical and practical knowledge.

2. Background and Theoretical Framework

2.1. Understandings of Area

Baturo and Nason (1996) asserted that students need to understand area from two perspectives: *static* and *dynamic*. The *static* understanding of area consists of a quantitative description of a region enclosed, while a *dynamic* understanding involves thinking about the relation between the boundary and the region enclosed. However, studies have shown that students in elementary school up to and including prospective teachers may not have both types of understanding of area (e.g., Baturo & Nason, 1996; Kamii & Kysh, 2006; Murphy, 2012).

Adults and young learners present a static understanding of area. Murphy (2012) found that prospective teachers were able to quantify area of a region enclosed, but were missing the understanding of area as a relation between the boundary and the region enclosed. Similarly, Baturo and Nason (1996) found that for their participants, the measuring of area was related to teaching formulas without any further application. Simon and Blume (1994) documented that when prospective teachers were identifying the number of rectangular cardboards needed to cover a table, they used the formula "length times width" without considering that the length units from the different sides of the rectangle were different.

When first measuring the side lengths of two rectangles in order to compare their areas, some children add the numbers; however, these same students successfully computed the area when they were provided with an area unit, such as a square (Nunes, Light, & Mason, 1993). Similarly, Kordaki and Potari (1998) found that when 12-year old students were measuring areas of real objects, they could not relate length units with area units. In other words, they could not articulate the relation between lengths of a shape and the measurement of the area of the shape. These results are analogous to those presented by prospective teachers. Kamii and Kysh (2006) also documented that students envisioned small square as discrete, and not as a continuous unit. This made it difficult for the participants to think about modifying the unit to measure the area of an irregular shape.

Overall, the studies described suggest that the teaching of area typically does not include activities that provide opportunities for students to advance their understanding of area from a discrete to a continuous unit (e.g., Baturo & Nason, 1996; Kamii & Kysh, 2006). Moreover, the most popular practice taught in middle school grades is to calculate area through formulas (e.g., Baturo & Nason, 1996; Murphy, 2011). Using formulas, students are encouraged to measure area by counting area units, or splitting areas into familiar shapes and adding up the areas of the parts.

2.2. Types of Knowledge that Students Bring to bear to Solve a Real-World Task

When working with the concept of area in real-world tasks, students have the opportunity to draw on both their personal mathematical knowledge as well as practical or contextual knowledge. Studies have documented the kinds of knowledge students brought to real-world tasks (e.g., Kastberg et al., 2000; Lubienski, 2000; Silver et al., 1993). Below, I describe those studies referring to the kinds of knowledge as *personal mathematical knowledge* and *practical knowledge*.

2.2.1. Students' personal mathematical knowledge

Students apply their mathematical knowledge, or school knowledge, to solve a task when they use their personal meanings of the mathematical knowledge learned in school. For example, students might think that whenever they have a rectangular prism they should compute the volume of it, or whenever they have to compute area, they should use the formula "length times width." Grounded in the philosophy of von Glasersfeld (1984, 1987), this personal mathematical knowledge is not necessarily the same for every student, nor is it the same as the one that the teacher wanted to teach.

2.2.2. Students' practical knowledge

Students' out-of-school experiences are different and might help them to understand better some mathematical contexts over others. For example, if a student has visited the geysers in Yellowstone National Park, he or she might have more ideas about geysers than a student who has never seen one. Therefore, such students might combine their mathematical and practical knowledge to provide a more appropriate solution for a real-world mathematics problem involving geysers than a student who uses his or her mathematical knowledge alone.

2.2.2.1. Real-World Contexts

NCTM (2014) suggested the use of curricula that enhance the importance of the mathematical practices and their application to real-world problems. However, the incorporation of real-world contexts in mathematics adds complexity to mathematics teaching and learning. When students solve real-world problems, they might just use mathematics alone (e.g., Silver et al., 1993) without paying attention to the contexts. Other students might thrive at solving real-world problems but not rely on formal, school-based notation or algorithms (see Rose in Lubienski, 2000). Finally, some students might use a combination of their previous experiences and school mathematical knowledge (e.g., Kastberg et al., 2000). Ideally, students integrate their mathematics knowledge with their practical knowledge while working on real-world tasks.

Defining *mathematical power* as, "the integration of knowledge from various content areas" (p. 15), Kastberg et al. (2005) analyzed students' responses to the National Assessment of Educational Progress (NAEP) assessment items released between 1994 and 2001. According to the authors, 11 of these items involved real-world context situations in which students had to apply mathematics to solve the problems. The authors also found that some students used mathematics to solve the problem without paying attention to the real-world context. Other students used their personal experiences to solve the problem. For example, in one of the items, students were shown three possible routes for a railroad line and asked to select the least expensive. Some students selected the shortest route (i.e., a straight line) even when this one passed *through* a river. The authors described that students might have used their personal experiences of riding a train over rivers, or they might also have used the mathematical idea that the shortest distance between two points is a straight line.

Silver et al. (1993) described the students' answers to a "bus" problem. The problem involves finding the number of buses needed to transport 540 people, where each bus can transport 40 people. In this study, 58% of the students used the division algorithm to solve the problem and 43% provided the correct answer: 14 buses. An interesting result of this research is the varied ways in which students interpreted the remainder. Some students did not provide a real-word interpretation, while others considered getting another bus, a van, cab, or minibus. Curcio and DeFranco (as cited in Silver et al., 1993) implemented a second version of "bus" problem in which they asked students to make a telephone call to order the transportation. In this new version, more students were successful in answering and interpreting the problem in a practical way. It seems that exposing students to the practical situation, making the call to order the buses, helped them to combine their mathematical and practical knowledge. In Lubienski (2000), students solved tasks involving real-world contexts. The author, who played the dual role of researcher and teacher, asserted that one of her students, Rose, solved the problems in a "contextualized matter" (p. 469). According to the author, this impeded Rose's understanding of the important mathematical ideas discussed in classes.

By paying attention to the knowledge students are bringing to the task, teachers and researchers can identify areas in their mathematical thinking to foster and can encourage them to combine both types of knowledge to find an appropriate answer according to the context of the task. In this study, my purpose was to explore what knowledge or meaning learners brought to a task involving a real-world context, what kinds of discussions, or disequilibrium, emerge from this, and how these discussions are resolved, or the equilibrium is re-established. Deliberately, I will not be focusing on how learners perform in those tasks per se, but I want to communicate their understandings or meanings associated with the task. Specifically, I focus on the following research questions: What mathematical and practical/contextual knowledge do students bring to a task involving a real-world context? How does students' work evolve through interactions with the teacher?

3. Methods

3.1. Participants and Setting

In the year of the study, the enrollment at the American Midwestern junior high school was 1,030 students (50.4% White, 29.9% Hispanic, 13% Black, 0.2% American Indian, and 5.6% Multiracial). In the same period, 70.5% of the students received free or reduced lunch benefits. The students in the school were grouped according to their academic level in three groups, from the lowest to the highest: inclusion, academic, and honors classes. The counselors made these groups by looking at the standardized testing results and the teachers' recommendations from the previous academic year.

The classroom teacher involved in this study was a senior teacher that taught two eighth grade honors algebra classes and four academic mathematics classes at the time of this study. Two of the four academic classes were recruited to participate in the study, a total of 42 students. These two classes were selected because the author had been working with these students one day per week throughout the entire academic year. Fifteen students consented to participate in this study. Although all of the students worked on the task in groups, the participants were grouped together by the teacher, so the author could observe them and ask/answer possible questions. In the first class, there were two groups: one of four and the other of five students. In the second class, there were two groups of three students.

It is important to note that the author had a collaborative relationship with this eighth grade mathematics teacher. Previous to this study, the author and the teacher had implemented other lessons over two years of collaboration in the same school (Suazo-Flores, 2016).

3.2. Problem Solving Task

The study focused on students working on a problem-solving task over three 45-minute class periods. The author developed the task with the goal to create a situation in which students had to integrate mathematics to solve a real-world problem. In the task, students were asked to provide a design for the cargo hold of a truck to allow for as many of two plants as possible: White Cedar trees and Petunia plants. Students received the task information as a letter written by the owner of the garden (see Figure 1), and they were asked to solve his problem. In the problem, students were told that the cargo space in the truck measured 220 by 740 by 220 centimeters, the Cedar trees' dimensions were 27 by 27 by 54 centimeters, and the Petunia flowers had dimensions 9 by 9 by 18 centimeters. Students' final packing designs were presented to the whole class.

Dear Students:

I am the owner of a garden center, and I need your help to improve the packing system for my truck. The cargo space in the truck measures 220 by 220 by 740 centimeters (cm). I would like you to design a method of packing two types of plants in the truck so that I can make the most profit.

There are two kinds of plants that I will be transporting: the Petunia flower and White Cedar tree. Both have a square base. The potted Petunia flower has dimensions 9 cm by 9 cm by 18 cm, and each one weighs 1 pound. The potted White Cedar tree has dimensions 27 cm by 27 cm by 54 cm. I receive $1 in profit for every potted flower I sell and $10 profit for every potted tree I sell.
You should also know that I am willing to modify the truck if needed with wood boards to transport a larger number of items. These boards are 2 cm by 30 cm by 220 cm, and each board can hold up to 50 pounds.

Let me know if you have questions! Thank you for helping me,

Figure 1. Letter from the owner of the garden to the students

3.3. Role of Instructor and Author

The author and teacher had implemented the task the year before with other students. Therefore, both were familiar with potential questions and reasoning approaches within the task. Students were encouraged to take notes individually, discuss in their groups, and verbalize their thinking as much as they could. The teacher explained to the students that we wanted to know how they were thinking and to do that we could only focus on their talk and drawings. The classroom teacher led the implementation of the task, and the author was a supplemental guide. During the lessons, the author and teacher discussed the students' questions and reactions to the task. When the teacher or author observed a student struggling with the task, or the majority of the students were struggling, then both talked about how to guide the students' thinking to move them forward. After the teacher and author agreed on a teaching strategy, the teacher implemented it with the class or group.

3.4. Instruction

Students were asked to say their names aloud before working on the task. In this way, the author was able to recognize their voices in the audio recordings. One voice recorder was located on each table of the target groups for the complete time that they were working on the task. The author sat close to the participants to write field notes, provide facilitation if needed, and ask clarifying questions. Students worked on the task for three class periods.

3.4.1. Day 1

Students were introduced to the task. Pictures of a real truck with the same dimensions were shown to the students (see Figure 2), so they could have an idea of one possible design of the cargo part of the truck. This design consisted of parallel boards that are used as shelves in several rows within the cargo part of the truck. Students were welcome to use the same design, but they still had to present a calculation for the total number of plants and trees, and where they would be located. The teacher and author provided opportunities to ask questions, making sure that every student understood the task description, context, and the problem to be solved. After reading the owner of the garden's letter and analysing the information paragraph by paragraph, students were encouraged to draw a representation of the plant and tree to pack in the truck. Students and the teacher agreed to use a vertical rectangular prism to represent the space that a plant and tree needed in the cargo part of the truck. The given dimensions of the plant and tree were applied to the dimensions of these imaginary rectangular prisms. A considerable part of the first lesson was used to understand the problem and the meaning of the given dimensions. For example, the dimensions of the cargo part of the truck were 220 cm by 740 cm by 220 cm. Students were asked to make a sketch of the cargo part of the truck and locate the given dimensions in the drawing. This was important to understand the problem because there was the possibility that a student would draw a vertical rectangular prism with base 220 cm by 220 cm, which does not represent a "real" cargo part of a truck. Using the students' contextual knowledge about trucks, they were able to translate the dimensions to the new representation of the horizontal rectangular prism with base 740 cm by 220 cm.

Blank pieces of paper were provided to the groups. Before starting to work within the groups, students were asked individually to write notes, questions, or drawings for the problem on the given piece of paper. This was used as a record for the initial individual students' thinking. The author assumed that that reasoning could change after sharing ideas with the other members of the group. After that, students shared their work with their group mates and made notes on the same or a new piece of paper.

Figure 2. Cargo part of the truck empty on the left and full on the right

3.4.2. Day 2

Students worked on the task and the teacher walked around the room answering or asking questions. Meanwhile, as on the previous day, the author sat close to the participants to write field notes and solve or ask possible questions. The author took notes about students' gestures and the order in which they were solving the problem. For example, the author registered the initial and final time of the lesson, students' gestures, what students did first, second, and so on to solve the task, and paid special attention if students began to discuss/defend different strategies to solve the task. The author also kept a record of the students' questions as well as the questions asked by the teacher when she approached the group. The author asked clarifying questions to check her understanding of the students' thinking. Examples questions are: Can you repeat what you just said? Why did you write that? Could you tell me what are you thinking? Can you draw or make a representation of what are you thinking?

3.4.3. Day 3

Students finished working on their solutions to the problem and spent time preparing their informal presentations to the whole class.

Data consisted of (a) the author's field notes; (b) the participants' written work, both their notes and their final solution to the problem; and (c) transcriptions of the participants' conversations associated with the task.

4. Analysis

4.1. Analysis of the students' written work

The author was less interested in the accuracy of the students' responses, but rather in evidence of the students' thinking and strategies used by them to make sense of the task (Confrey, 1994). The goal of the task was to find the total number of boards and plants to be located in the truck. Although the problem included several high level calculations, for this paper the analysis focused on the students' strategies to find the total number of trees to be located on the floor of the truck. To identify the number of trees, students brought to bear the concept of area, specifically the use of the formula "length times width," which motivated the writing of this manuscript. The author made copies of the students' written work after the end of every day. Students' written work across days was compared to each other and classified according to the strategy implemented. The author also looked for changes in the students' notes.

After analysing the student's written work, the author and another mathematics educator identified the following categories: 1) the use of the formula for area, 2) the use of the formula for volume, or 3) iterating a unit to compute the total numbers of trees to be located on the floor of the truck. Specifically, the author focused on the students' meanings and the mathematics concepts that they brought to bear to solve the task. Two of the categories were predominant: the use of the formula for area and iterating a unit strategy. These categories are described later in the results. Since both strategies produced different results for the total number of trees to be located, the author was interested in how students faced and solved their disequilibrium (Piaget, 1972) when encountering this in their work.

4.2. Analysis of the students' conversations

The author evaluated the students' conversations and reasoning while they were working on the task. To have evidence of the students' thinking, the author constantly asked questions that encouraged them to justify or articulate their thinking (Steffe & Kieren, 1994; von Glasersfeld, 1987). Using the transcripts of the audio recordings and the field notes, the author registered and identified moments in which students were engaged working on the task. Comments such as, "Yesterday I went to watch a movie," were not considered part of the task. Moreover, following the study of Kastberg et al. (2005), the author focused on identifying the participants' mathematical and practical knowledge. Because this research was embedded in a real-world context, the author paid attention to the students' interpretations of the context of the task (i.e., practical knowledge) and the mathematical content that they brought to bear to solve the task (i.e., personal mathematical knowledge).

The transcripts of the audio recordings were also used to support students' written work or provide evidence of the students' thinking. Evidence of students' thinking was identified and some examples are the following: "I would divide the area of the floor of the truck by the area of the base of one tree," or "In my opinion we should compute the volume because the shape of the plants are rectangular prisms." Students' conversations were used to confirm students' written work. The conversations were used to provide evidence of the categories described: use of the formula for area, volume, and iterating a unit. In the case that a student did not draw any pictures, his or her conversation was considered evidence of the student's thinking. If the student did not talk at all during the lessons, his or her written work was considered evidence of the student's thinking. Since not all the students articulated their thinking, wrote notes, and drew pictures, the researcher's field notes were used as a backup to construct a narrative of how students engaged with the task.

4.3. Analysis of the teacher and students' interaction

The author assumed that learning is a result of a mutual interaction among students and between teacher-student interactions (Confrey, 1995). Students might increase their confidence after realising that the teacher, or other teammate, valued their methods to approach a task. Students might also improve their calculations after being asked to justify or rethink their solutions. The audio recordings and field notes were analysed to find evidence of the teacher's questions or comments to the group that had impacted the group's work towards the task. Moreover, at the end of every day, the author made copies and carried out a preliminary analysis of the students' written work and field notes. This analysis was shared with the teacher. If the teacher and author considered that students needed some guidance to resolve a conflict, the same day or the next day, the teacher intervened in the group by asking questions about their work. The goal of the questions was to get them to justify their thinking. At the end of the day, the student-teacher conversations and written work were analysed again. The author looked for evidence of a shift in the students' thinking after the teacher intervened the group.

5. Results and Discussion

Three of the four groups initially used the *formula for area* to compute the number of trees to be located on the floor of the truck. These participants computed the area of the floor of the truck (i.e., $220cm$ ×

$740cm$=$162{,}800cm^2$) and divided that number by the area of the base of the tree (i.e., $27\ cm \times 27cm$=$729cm^2$). Using this method, the students found that 223 trees could be located on the floor part of the truck. The other group considered the squared base of the tree and iterated it to obtain the number of trees to be located across and along the floor of the truck. After these participants computed these numbers (i.e., eight rows of 27 trees), they multiplied them to obtain a total of 216 trees. The author referred to this strategy as *iterating a unit*. In two of the four groups, the participants also computed the volume of the truck, plant, and tree, but they did not use these numbers to provide a final answer. After the teacher or the author asked the three groups who used the formula for area to describe their current solution method and to justify it, two of these groups attempted solving the task using the iterating a unit strategy as a way to justify it. The next paragraph presents a deeper description of how students in one of those two groups engaged with the task. The group was selected because, at the beginning, they solved the problem using the formula for area and after the teacher's intervention they used the iterating a unit strategy.

5.1. Day 1: Using the formula for area

Paul worked with Kent and Ann on the task (all names are pseudonyms). Paul was highly engaged in the task the first day. During the introduction to the task, he answered aloud three of the teacher's questions to the whole class. As students began to work within their groups, Paul led the speaking. His teammates recognised him as the only member of the group that understood the task. Kent said, "Paul is the only one [that] has been saying anything," to which Ann replied, "Paul is smart." Paul described his reasoning of the problem aloud and his teammates added or asked questions, showing they were interested in understanding and following Paul's thinking. During the first 15 minutes, Paul's group brought to bear their real-world experiences to understand the task.

> *Paul:* Ok, so for the shelves, someone has to be able to get in the truck, walks in, and get the other plants. So they can't be like just one giant board.
> *Ann:* So, now we can start.
> *Paul:* Maybe what we can do is to put the trees on the bottom part of the truck. Since they probably will be taller and on the shelves the flowers right above of the trees.
> *Ann:* Yes, great idea.
> *Kent:* And the trees probably will weigh more too. It does not say the weight, does it?

After talking, writing notes to understand the context of the task and negotiating the location for the trees, the students continued:

> *Kent:* Do we need the area not the volume, don't we?
> *Paul:* We need everything. No, probably we need the area for the bottom of the truck.
> *Kent:* Yeah, for the trees, [using the calculator] 220 by 220 by 740. What side is which measurement?
> *Paul:* So, I think, I think 220 is going up into the side while 740 is the long ways of the truck.
> *Kent:* Mmm, yeah, right, right, right.

The author intervened in the group when she realized that students were not considering the flower with a squared base. This was important to address because students could transfer this method to find the number of trees to be located on the floor of the truck.

> *Ann:* 162 centimetres is the area.
> *Author:* The area of what?
> *Paul:* Of each flower.
> *Author:* So how you computed that number?
> *Ann:* I did 9 times 18.
> *Author:* 9 times 18. So, how the flowers look like? Can you draw a picture?

As a result of the author's intervention, students computed the numbers again but now considering the flowers with a squared base (9x9). Paul and Kent also made jokes about "Ann's mistake" on understanding the dimensions of the flower. Then, students found the number of trees to be located in the truck by using their mathematical knowledge of area: dividing the area of the floor of the truck by the area of the base of a tree. The author asked students, at two different times, to check or justify the obtained number of trees. Students first re-typed the numbers in the calculator, obtaining the same result. The second time, the author asked the group why they thought their method was right; Paul and Kent replied, "Because that is how many trees would fit in the area of the bottom of the truck." Then, the author, hoping to encourage students to think in another approach to find the total number of trees, asked the students to think about how physically they were going to locate the trees on the floor of the truck. Paul replied, "Put it side by side" to which Kent began to draw squares side by side in a big rectangle. At this point, the class was dismissed.

5.2. Day 2: Confusion around the number of trees

During the second day, students began to compute the number of boards to be located in the truck. They forgot about the last drawing done the day before. The author asked the group to describe their work and reminded them about the last drawing done at the end of the previous day. Paul decided to compute the numbers of trees to be located across and along the floor of the truck, different from his first day computation. He did not share his computations with the other teammates, probably because he was confused or because the whole group work was concentrated on finding the total number of boards. Therefore, the author and teacher decided to intervene again, because the group was relying just on their mathematical knowledge. The goal of the intervention was to create a situation in which the group realised that they had two different numbers for the quantity of trees to be located along the floor of the truck: the number obtained the day before using just their mathematical knowledge, 223, and the new number computed by Paul using the iterating a unit strategy, 216. The teacher, who had not interacted with the students the day before, asked them to explain their solutions to her. Students began explaining their thinking and criteria to find the number of boards (see Figure 3) and flowers on each board. The teacher drew Figure 4 while students were explaining to her how to find the number of flowers per board. Then the teacher asked them about the trees.

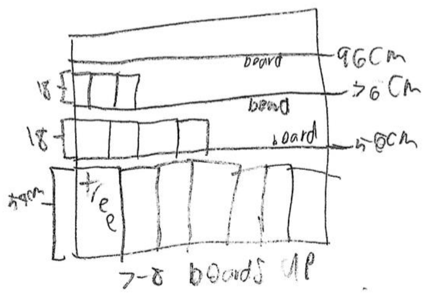

Figure 3. Kent's drawing for the boards.

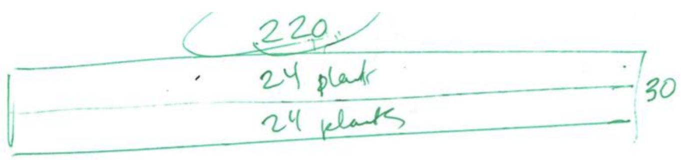

Figure 4. Teacher's drawing of the plants to be located on each board.

> *Teacher:* Can you give me a picture like this for trees? Did you do a picture like that? [Showing Figure 4]
> *Paul:* Sort of.
> *Teacher:* Kind of.
> *Paul:* We can fit 8 trees going in that way and 26 but 27 if you count the eighth.
> *Teacher:* So, really, it will be 27, so, you are saying 27 rows of eight, is that true? Does that math work out? Can you try that? Can you prove that to me? I am all about making sure it works. [Correcting the numbers in 45]
> [Students enter the numbers in the calculator]
> *Kent:* 216.

There was a silence after Kent said that number, so the teacher began to express her thinking aloud, trying to help the students to understand Paul's thinking. Using a piece of paper, she reproduced his drawing (see Figure 6), so that Ann and Kent could join in the reasoning. Then, the equilibrium was disrupted when students, especially Paul, realised that they had two different numbers for the quantity of trees.

> *Teacher:* Ok, how many trees can I put in there?
> *Paul:* 223.
> *Teacher:* How did you get that?
> *Kent:* For the total.
> *Teacher:* Because you just told me that.
> *Paul:* 216.
> *Kent:* How we get that?
> *Teacher:* I am not following you guys.
> *Kent:* It would be 216 because.
> *Paul:* I don't know.
> *Teacher:* Wait a minute, why are you changing it? [She looks at Paul's notes]
> *Paul:* Because I was wrong yesterday.

Figure 5. Paul's drawing including the teacher's numbers.

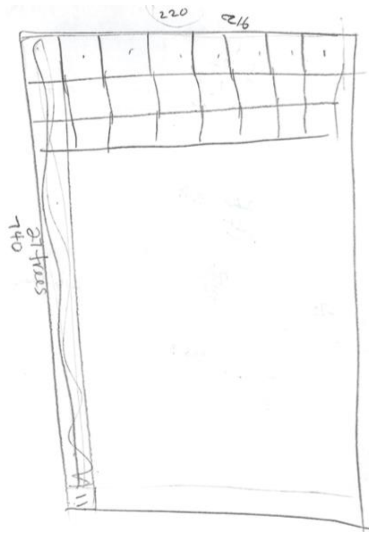

Figure 6. Teacher's explanation of Paul's thinking.

The teacher encouraged Paul to describe his thinking aloud. Kent also interacted with the teacher and, with Paul, described to her their previous thinking to obtain 223. The teacher indicated to them that their thinking made sense to her and asked them which number they thought was the right answer for this problem. Paul began to think aloud.

> *Teacher:* Which one do you think is right?
> *Paul:* 216.
> *Teacher:* You don't think this works [showing the number 223 obtained using the formula for area]. [Silence]. But wait, did you do anything wrong? I am ok; I think you punched them right? [She referred to the numbers used in the formula for area and entered in the calculator] The area of the tree and the area of the truck, and divide them. Why doesn't it work?
> *Paul:* Maybe because it is adding the area that we have left but don't have enough room for a tree.
> *Teacher:* Oh ok, wait; show me, can you explain that?
> *Paul:* There is an end where the 27 trees go and maybe it is counting all the area the trees will not fit it [showing Figure 5].

Kent, who was not aware of Paul's new strategy, was computing the numbers of trees along and across the floor of the truck by himself and obtained the new number, 216. Using Figure 6, the teacher asked them to compare the numbers of trees with the length and width of the floor of the truck so they could represent the space that remained in a diagram. Students helped her to compute the numbers. Particularly, Ann computed one dimension of the leftover space along the floor of the truck, i.e., 27 cm times 27 trees subtracted from 740 cm. This was a contribution to the group because Paul struggled to compute that number. Before the teacher came back to the group, Ann, excited, told to the group, "I did something!" The group congratulated her and then Kent asked the group:

> *Kent:* I am confused now. She just confused me.
> *Ann:* She messed my brain even more with this.
> *Kent:* So, is it 216 or 223 that will fit then? Because the area of 8 times 27 is 216 but what we got dividing the area of the tree by the area of the truck was 223.

There was a silence and the teacher approached the group again, students shared their confusion and even accused her of tricking them.

> *Teacher:* What are you thinking?
> *Paul:* I don't know. You just confused me.
> *Teacher:* Did I?
> *Kent and Paul:* Yeah.
> *Paul:* I think we were wrong, but we thought we were right, in the way that you talk made us do it like.
> *Teacher:* I did not try to make you to do it; it was your idea.
> *Kent:* Yeah.
> *Paul:* But you push in us ideas.
> *Kent:* Yeah.

Then, the teacher explained to the group that she needed to know the number of trees to be located on the floor of the truck. She referred to the context of the task so that students could use their practical knowledge to decide which solution was more appropriate for the problem.

> *Teacher:* So, here is the thing, all I'm asking you, I need you to prove to me that my workers are going to be able to put that many plants in the truck, ok? Because, I can't get them to the loading dock and just leave them there because they are not going to fit.

Kent asserted that the right number was 216, but they could not justify why: "We did the actual numbers and we get 8 times 27 is 216. We don't know how to out which number is right." After this declaration and looking at the students' concern, the teacher encouraged students to think of a way to prove that 216 was the right number of trees by thinking about how they were physically going to place the trees on the truck. She recapped the iteration of a unit strategy to find the total number of trees and helped them to visualize on a piece of paper the remaining space that Paul referred to before.

> *Teacher:* Ok. Now we know that these are going to fit because the math says so, right? You know that if you put those eight in there that is how much is going to take up and that is how much we have [Showing Figure 6 widthwise]. We have the same here [Showing Figure 6 lengthwise], but we can't fit another one there. So my question to you is, which one is right? Do you think you can squeeze… What number do you had before? 223, or what was the other number?
> *Students:* 216
> *Teacher:* The question is kind of, which one is going to work? Ok, so what do you think? [Silence] What do you think Ann?
> *Ann:* I don't know.
> *Teacher:* Based on our picture, what do you think?
> *Ann:* I would say 216.
> *Teacher:* Ok, can we prove it pretty quickly? How many are going across here? [Showing Figure 6]

Paul: 8.
Teacher: And how many across here?
Ann: 27.
Teacher: How many is that?
Paul: [Using calculator] 216. 27 times 8 is 216.

A couple of minutes before the class was dismissed, the teacher asked the group to think about the difference of trees using the area formula and the iterating a unit strategy, i.e., seven trees, and whether they could locate those trees on the floor of the truck

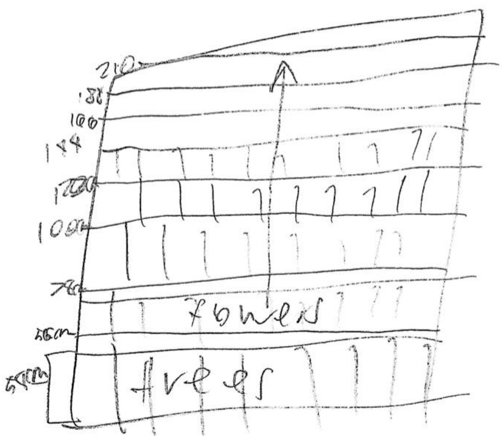

Figure 7. Kent's drawing of one view of the packed truck.

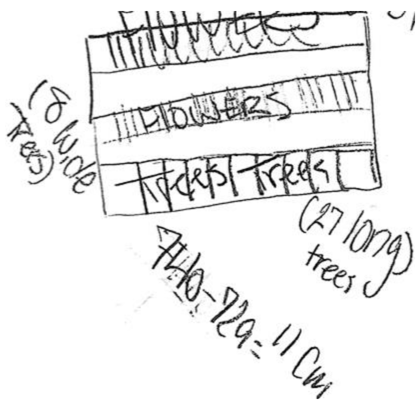

Figure 8. Ann's drawing and computations day 2.

5.3. Day 3: Resolution

On day three, the students seemed to have restored equilibrium in their minds. They looked enthusiastic again. As soon as the teacher finished providing the instruction for the day, they began to work on finishing the solution for the task. Paul was not the only one speaking and doing computations; Kent and Ann (Figure 8 shows part of Ann's work) participated actively, finishing the group's solution for the task. Ann appreciated Kent's drawing of one view of the packed truck (see Figure 7). Kent led the computation of the number of boards and total number of flowers, and Paul tested the numbers. The author approached the group, asking them how they were doing and what happened with the two numbers related to the quantity of trees obtained the day before.

Author: Why do you think there is a difference between these two numbers?
Paul: Because the whole area would count this [showing the whole space in Figure 6] and this is the space that we can't use because it is not big enough [showing the leftover space in Figure 6]. It is like 11 centimetres and a tree is 27 centimetres. So, I think we were counting as we were going to put another tree right there.

6. Conclusion and Discussion

Over the three days of working on this task, students were encouraged to use their personal knowledge. The first day, students brought to bear their contextual, or practical, knowledge to understand the task. For example, students discussed ideas about where to locate the trees and how to locate the boards so someone could get in and out of the cargo part of the truck. They also used their knowledge about plants, trees, and trucks to make sense of the given dimensions. The second day, students conveyed mathematical knowledge to the task and did not explicitly talk about the real-world context. In relation to area, as shown in previous studies, students' mathematical knowledge was aligned with a static understanding of area (e.g., Baturo & Nason, 1996; Kamii & Kysh, 2006). Paul's group first computed the area of the bottom part of the truck and divided that by the area of the bottom part of one tree. To revise the number obtained, Paul's group iterated the bottom part of the tree along and across the bottom part of the truck. Paul's group experienced disequilibrium when they realized that the numbers obtained were different and that the number obtained by using the area formula was not appropriate for the context of the task. It was particularly difficult for the group, especially for Paul, to describe why the numbers from the two methods were different. In general, the group relied on the number obtained by Paul using the formula for area (i.e., personal mathematical knowledge); they did not realise the inadequacy of the number until the teacher asked them to think about the context of the task: how they were going to locate the trees physically on the floor of the truck (i.e., incorporating the practical knowledge with personal mathematical knowledge). That comment made Paul think about the iterating a unit strategy to get a more appropriate answer for the problem. As seen in other studies (e.g., Kastberg et al., 2005; Silver et al., 1993), the students did not bring their experiential and mathematical knowledge together until the teacher asked them to think about it. In this study, the teacher's role was key to encourage students to use their mathematical and contextual knowledge to solve the task.

The use of a task involving real-world contexts might create learning situations in which students bring to bear their mathematical and practical knowledge. They also present a window for teachers and mathematics educators to understand students' mathematical and contextual knowledge. This study shows that students might start using only their mathematical knowledge to solve a task, but later, when an adult such as the teacher asks them to integrate the context of the task to the mathematics procedure, students bring together their mathematical and practical knowledge to the task. This also generated discussions about the meaning of the formula for area. Therefore, mathematics teachers play an important role in encouraging students to combine their mathematical and practical knowledge to solve a task involving a real-world context.

Teachers and/or mathematics educators could enhance this task by discussing with students about the meaning of the area units. Since the static understanding of area has been privileged in K-12 schools, it might be beneficial for students to have a discussion about the dynamic understanding of area. An example of a task that could enhance the dynamic understanding of area is described in King (2015). Moreover, as shown in Silver et al. (1993), it might be beneficial for students to respond to the owner of the garden or to call people to provide them with specific instructions for loading the truck. This might encourage students to bring to bear their practical knowledge of the task and combined it with their mathematical knowledge to provide a more adequate answer for the problem.

7. Acknowledgements
The author thank Lisa Roetker, and her mathematics education fellows at Purdue, for supporting her and sharing their ways of knowing during this study.

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