DISCOVERING THE PYTHAGOREAN THEOREM USING GEOGEBRA SOFTWARE[1]

DESCOBRINDO O TEOREMA DE PITÁGORAS AO USAR
O SOFTWARE GEOGEBRA

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ABSTRACT

This paper presents some of the activities carried out during a master's study, in which the role of digital technologies was investigated in a group of humans-with-media for the production of knowledge about the Pythagorean Theorem. The research subjects were elementary school students. This qualitative study is theoretically based on the construct of humans-with-media, and the role of dialogue is defined as a conversation that aims at learning. The results indicate that the GeoGebra software contributed to the creation of a learning environment that favored the students' actions in the construction of mathematical knowledge, providing rich possibilities for the visualization of concepts and properties, enhanced by the dynamism of the trials from the constructions performed using GeoGebra.

Keywords: Mathematics Education; Pythagorean Theorem; Dialogues; GeoGebra; Humans-with-Media.

RESUMO

Este artigo apresenta algumas das atividades realizadas durante um estudo de mestrado, no qual o papel das tecnologias digitais foi investigado em um grupo de humanos-com-mídia para a produção de conhecimento sobre o Teorema de Pitágoras. Os sujeitos da pesquisa foram alunos do ensino fundamental. Este estudo qualitativo é teoricamente baseado no construto humano-com-mídia, e o papel do diálogo é definido como uma conversa que visa à aprendizagem. Os resultados indicam que o software GeoGebra contribuiu para a criação de um ambiente de aprendizagem que favoreceu as ações dos alunos na construção do conhecimento matemático, proporcionando ricas possibilidades para a visualização de conceitos e propriedades, potencializadas pelo dinamismo dos ensaios das construções. usando o GeoGebra.

[1] This article deals with a redeveloped version of a study presented at the 12th International Conference on Technology in Mathematics Teaching, entitled the Production of Knowledge about the Pythagorean Theorem in a Technological Environment.

Palavras-chave: Educação Matemática; Teorema de Pitágoras; Diálogos; GeoGebra; Humanos com Mídia.

1. Introduction

Digital technologies are frequently used in research in mathematics education. The integration of information technology into the schools, as well as its contributions to the teaching and learning processes, is a reality. This reality leads those involved in the school context to rethink their ideas about education and teaching strategies.

In this article, we present part of a study conducted by the first author, guided by the second, in which we sought to understand how the production of knowledge of the Pythagorean Theorem takes place in a humans-with-media unit, using information technology and with students in groups as the actors.

The content was chosen based on authors who emphasize the importance of geometry "as historical constructions, requested and legitimized by the demands of social practice and shaped by cultural criteria" (Fonseca, Lopes, Barbosa, Gomes, Dayrell, 2011, p. 115), and who support the teaching of geometry from two aspects: the practical and the formative. Thus, we chose to work with the Pythagorean Theorem, which is of great importance to students' learning, since it plays an important role in the study of mathematics as well as other sciences. The understanding and application of the Theorem is essential to the study of various physical phenomena, for example. We also considered the emphasis given by the authors to the importance of geometry in researching regularities and the possibilities of using information technology for this.

To carry out the study, we developed, applied and analyzed activities using information technology, specifically the GeoGebra software, with a group of 15 students in the last grade of elementary school in a public school in Minas Gerais, Brazil. The activities took place in nine meetings of 90 minutes each. Initially, we attempted to review mathematical concepts of the Theorem, such as the classification of triangles according to their angles and the concept of area, as well as to familiarize students with the GeoGebra software. In the fourth and fifth meetings, we developed activities aiming to explore the relationships between the areas of squares constructed on the sides of any triangle, in order to discover the relationships that characterize the Pythagorean Theorem, as well as the Theorem statement. We also developed activities related to some geometric demonstrations of the Theorem and to a generalization to the Pythagorean Theorem, considering areas of other geometrical figures built on the sides of the right-angled triangle.

We designed activities considering, as does Borba (2001), that knowledge is produced using a given media or using an intelligent technology, and not by single or collective comprising only humans. This is the theoretical perspective of humans-with-media (Borba & Villarreal, 2005; Villarreal & Borba, 2010). Computers can be used to create experimental environments, interactions, dialogues and conjectures about mathematical knowledge. The technology resources, such as numerical and graphical manipulation and dynamic visualization tools, can help students make predictions and simulations, challenge their initial ideas, and lead to the verification of relations, regularities or properties.

A dynamic geometry environment can be defined as a software whose main characteristic is the possibility of "dragging" geometric constructions with the mouse, while its measurements are simultaneously updated. Goldenberg, Scher and Feurzeig (2008) state that such environments “allows students to create geometric constructions and then manipulate them easily. Dragging […] allows user to move certain elements of a drawing freely and to observe other elements responding to the altered conditions” (p. 53). Therefore, the computer screen gives the impression that the geometric construction is being deformed continuously throughout the dragging process, while maintaining the relationships specified as essential in the original construction (Silva & Penteado, 2013).

We chose GeoGebra, a dynamic geometry program with a user-friendly interface. It allows varying parameters, giving a dynamic character to the activities. In addition, we explored the potential for experimentation that this software offers, making the specific properties of the physical representation of the object change, but in such a way that the actual geometric properties of the construction were maintained.

The manipulation and visualization possibilities of GeoGebra were essential for exploring concepts in the activities conducted, as well as the possibilities for interaction and dialogue among the participants. The importance of the collective is also highlighted by Araújo (2002; 2004) who, based on the ideas of Alro and Skovsmose (1996), states that mathematical meaning emerges among the participants in their interactions during the process of teaching and learning, not being passed down from teacher to student or constructed by each student individually. Furthermore, by means of dialogue, participants in an educational environment can negotiate their perspectives, to try to understand and share them in order to negotiate the meanings of the activities, concepts and results. Thus, we consider as the main theoretical references for this study the construct of humans-with-media (Borba, 2011; Borba & Villarreal, 2005; Levy, 1993), visualization (Arcavi, 2015; Borba & Villarreal, 2005; Gúzman, 2002), and dialogues (Alro & Skovsmose, 2010; Araújo, 2002; 2004).

In this qualitative study, we proposed to analyze knowledge about the Pythagorean Theorem produced by the collective consisting of humans and media. We attempted to observe if this collective favored the production of knowledge about the theorem and other concepts related to it, such as the classification of triangles according to their angles. Activities were initially designed and developed based on the methodology chosen. After the development of the first activity, the following activities were discussed in each meeting, enabling modifications based on what was observed. To collect the data, we used audio and video recordings of the activities, written productions of the participants and daily field notes of the researcher. We tried to narrate the classroom situations in a descriptive and detailed way. Thus, it was possible to focus on the process from which the data emerged and not simply on the results achieved by the end of a specific intervention or even during the whole observation time. We present below the theoretical references used for the design and analysis of the activities.

2. Digital technologies, visualization and dialogues for the production of knowledge

The present study, either for the conception or the analysis of activities, was based on the production of knowledge from the collective of humans-with-media. The focus was especially on the visualization and experimentation provided by the GeoGebra software, and on the dialogues that emerged within this collective in which students work in groups.

Borba (2001) presents the theoretical construct humans-with-media, supported by Tikhomirov's notions of the reorganization of thinking (1981) and by Levy's (1993) relationship between technique, knowledge and history. For Borba and Villarreal (2005), "human beings are made up of technologies that transform and shape their thinking and, at the same time, these humans are constantly transforming these technologies". A common thread between the work of Tikhomirov (1981) and Levy (1993) is that a dichotomy between technique and human being should not exist. Levy (1999) supports this idea when he states "that not only the techniques are devised, produced and reinterpreted during their use by men, but it is also the intensive use of tools that makes mankind as such" (p. 21).

Therefore, knowledge is always built, linked to a type of media, based on the notion that knowledge is produced by a collective consisting of humans-with-media as the basic unit of knowledge. If every medium enables the production of qualitatively different knowledge, there is no point in doing an analysis of the improvement in education based on comparisons among them (Borba, 2001) but, rather, in identifying the changes that they made possible in practice, exploring their possibilities because, as Levy says (1999, p. 26), "a technique is neither good, nor bad, [...] nor neutral."

A key point to produce knowledge is visualization. To Arcavi (2015):

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown and advancing understandings. (p.145)

A collective of humans-with-media can favor visualization in the sense proposed by Arcavi (2015), since it constitutes a space for creation and reflection on images and representations using the different media, with possibilities for developing understandings and producing knowledge.

Guzmán (2002) also emphasizes the importance of visualization. According to the author, mathematical visualization consists, basically, of the attention devoted to the possibilities of concrete representation of objects which are manipulated, in order to approach abstract relations more effectively. The author also states that visualization is a very important aspect of mathematics, taking into consideration the meaning of mathematical activity itself and the structure of the human mind. For him, visualization does not consist of the immediate view of a relationship, but of the possibility of interpretation from the contemplation of a situation.

According to Borba (2011), educational software can assign an important role to visualization, enhancing the visual component of mathematics. In this way, "in this collective, the media acquires new status, goes beyond showing an image. More than that, it is possible to say that the software becomes an actor in the process of doing Mathematics" (p.3). Visualization seems to be the primary feedback provided by the computers (Borba & Villarreal, 2005). By manipulating images and experimenting, it is possible to follow paths not always foreseen initially, causing the subject to interpret the answers and images generated during the process, which is in line with Guzmán's (2002) idea, discussed above.

It is important to give students opportunities to explore situations in pursuit of knowledge. We believe that a good tool for this exploration is the dynamic geometry software, GeoGebra.

According to Stylianides & Stylianides (2005), the use of construction tasks in dynamic geometry environments (DGEs) can favor the access of the theoretical world of geometry. A relevant feature of this type of software is the treatment of "drawings in motion", causing the specificities of the physical representation of the object to change, keeping the invariants, that is, the actual geometric properties of construction (Gravina, 1996). The moving figures are obtained by dragging the points that build them (as the vertices of a polygon). It is also possible to get measurements related to the objects manipulated, such as the amplitudes of angles, segment lengths, areas of flat figures, among others.

Bairral (2015) points to the observation of different forms of non-static representation of the objects studied, and the survey and verification of conjectures as important contributions of dynamic geometry. To Olive et al. (2010) "by observing properties of invariance simultaneously with manipulation of the object, there is potential to bridge the gap between experimental and theoretical
mathematics as well as the transition from conjecturing to formalizing" (p.150).

> In DGEs, the validation criterion most often used to date has been that a solution of a construction problem is valid if and only if it passes the 'drag test' (Jones, 2000; Mariotti, 2001).[…] Specifically, the drag test may permit the validity of constructions created using measurement tools (such as angle measures, calculations, and rotations using numerically-specified angles). (Stylianides & Stylianides, 2005, p.32)

In these environments, the manipulation of robust constructions, by dragging some of their elements and the use of measuring tools, allows observing regularities and the making of conjectures about the properties of the objects constructed.

Other important aspects to consider are the interaction and dialogue between the participants of the activities. Santos (2009) states that "in the teaching and learning of Mathematics, linguistic aspects need to be considered inseparable from the conceptual aspects so that communication and, therefore, learning happens" (p. 119). This idea is consistent with the theoretical construct of humans-with-media. As stated by Alro and Skovsmose (2010), "humans" appears in the plural form because it is important to consider learning as a process of interaction of various persons, which presupposes dialogue and communication. For these authors, a dialogue is understood as "a conversation that focuses on learning", not being conceived as ordinary conversation. It is only through dialogue that the real communication is established, and the most important element in the dialogue is the "nature of the conversation and the relationship among the participants" (Alro & Skovsmose, 2010).

Alro and Skovsmose (2010) interpret the dialogue related to learning, focusing on terms of ideal elements. Those elements are to investigate, to take risks, and to promote equality. They discuss the notion of investigation from the collectivity and the collaboration, in which dialogue participants can express their views with no room for complacency. The negotiation of meanings takes place when students let go of some of their views, even for a few moments, enabling them to explore new assumptions from new angles so that, in some situations, they can build new perspectives. According to Carvalho (2009), the negotiation of meanings "cannot be understood as a pre-existing agreement between two persons with the goal of solving a proposed task, but as a dynamic and complex activity (p. 19). For the author, in general, in a dynamic negotiation of meanings, students try to find a solution individually. Then, in an interactive sequence triggered by the resolution strategy proposal, the colleague may react and resolve the problem in different ways: in an impasse between the points of view; in the acceptance of one of the solutions proposed by them; or, even, with a new solution drawn up jointly by them.

We see dialogue, as do Alro and Skovsmose (2010), as something unpredictable since, by its own nature, it sometimes enables the rejection of perspectives and the creation of new ones. Presenting perspectives is a way to take risks, because it often leaves students vulnerable to criticism, causing discomfort. It is important that the discomfort is not exaggerated to the point of neglecting the participation of a member in the investigation.

Therefore, the dialogue must be grounded on the principle of equality in which all participants are equal, having the right to present their positions and to be respected for doing so. "Promoting equality does not mean promoting agreement" (Alro & Skovsmose, 2010, p. 133). In short, in a dialogue one must seek to be consistent, to understand and respect the view of the other, to argue and build knowledge from the clash of ideas.

After a brief presentation of the theoretical references of the study, we will show, below, the activities relating to the section of the study aimed at leading the students to understand and state the Pythagorean Theorem, based on the visualization and experimentation offered by the GeoGebra software and the dialogues among the participants.

3. Activities aiming at the perception and statement of the pythagorean theorem

We deal here with activities that were developed in two 90-minute meetings to lead students to identify the relationships that characterize the Pythagorean Theorem, and to express them by stating the Theorem.

The scripts of the activities sought to enable the manipulation and investigation of images constructed in GeoGebra, as well as the dialogue among the participants. We chose to present figures previously built by the researcher, giving special attention to findings from the experimentation and the manipulation of these objects using the dynamic geometry software, allowing the development and testing of conjectures about the relationships (Gravina, 1996).

In the first meeting, we made a construction (Figure 1) available to the students in which there were a triangle and squares built on the sides of this triangle, highlighting the amplitudes of the angles of the triangle and the measurements of the areas of the squares. We asked the students to move the vertices of the triangle to get different types of triangles, and to observe the corresponding changes in the values of the areas of the squares. We wanted the students to establish a relationship between the values of the areas of the squares for each type of triangle: acute, right-angled and obtuse.

In our activities, associating dynamic manipulation with obtaining new numerical values of the areas gave the students the possibility to change the position of the vertex of a triangle and to observe the transformations of the angles, the measures of the sides and the numerical values referring to the areas of the squares built on the sides of the triangle. Thus, in a short time, we obtained a large number of numerical values from the areas of the squares constructed for the acute, obtuse and right triangles, and we were able to stimulate the students to observe the relations. Such experimentation would not be possible without the resources of GeoGebra.

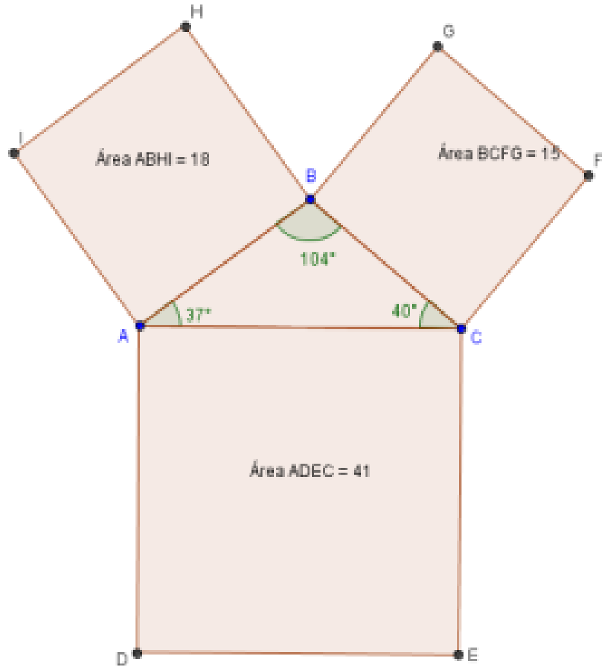


Figure 1. A triangle and squares built on its sides

The values obtained for the different triangles were recorded in a table that we supplied (Figure 2).

	A	B	C	D	E
1	**Biggest Angle**	**Type of Triangle**	**Smallest Area**	**Median Area**	**Greatest Area**
2	**83º**	**Acute triangle**	**16**	**18**	**30**
3					

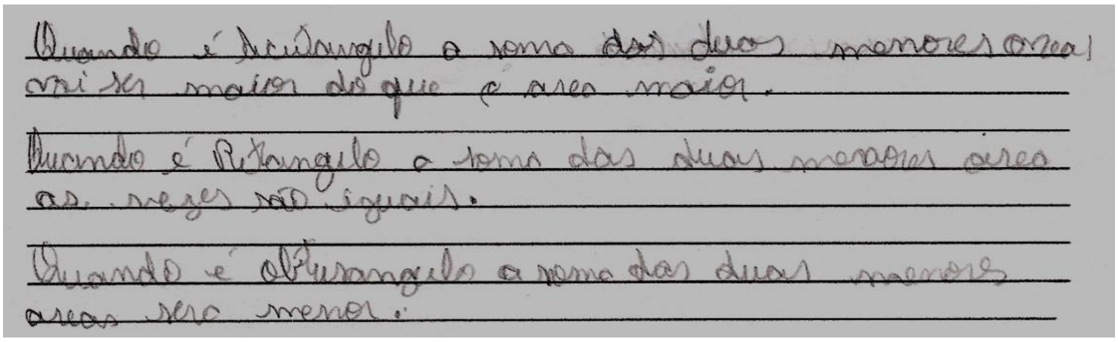
Figure 2. Table of activity for the first meeting

Initially, the students had some difficulty handling objects in GeoGebra. This is, possibly, because they were used to working only with static objects, usually with pencil and paper. Later, they managed to do what was asked. To encourage them to think about the different values they recorded, we asked them to observe if there was any relationship between the measurements of the areas of the squares for each type of triangle and to record their findings. We expected that students would dialogue in an attempt to find answers. The dialogue didn't happen immediately, and it was possible to observe a certain anxiety among the students to find quick solutions. It is possible that this type of passive student behavior originates from school experiences in which the student remains in the role of receiver, and the teacher is the one who answers all the questions.

On the other hand, the teacher is not used to not responding immediately to students' questions. This also raises teachers' anxiety for not knowing how to conduct the activities in order to guide their thinking, but without giving too much guidance. In the situation in question, we asked students to add the values of the two areas and compare the total to the third area, also to describe the relationship with the kind of triangle. Some, but not all, of the students managed to understand and verbalize the relationship satisfactorily.

Reflecting on the situation, we believed it necessary to resume this subject in the following meeting. We built a table similar to that of Figure 2 on the board, filling it with values provided orally by the students of the different groups. The table data were discussed with the students and some of them were able to understand the relationships. Regarding non-right-angled triangles there was no doubt: the students concluded that, for acute triangles, the area of the largest square is always less than the sum of the areas of the other two squares; and, for obtuse triangles, the area of the largest square is always greater than the sum of the areas of the other two squares.

A very interesting fact was observed with the right-angled triangles. Due to the rounding applied to the amplitudes of the angles and the measurements of the areas by the software, our construction did not allow us to draw a conclusion about the relation between the areas. This happened because, in some cases, the area of the largest square was smaller, in others it was greater, and in others it was equal to the sum of the smaller areas. Figure 3 shows some of the students' answers in the original language and Figure 4 shows the translation of these answers.


Quando é Acutângulo a soma das duas menores área vai ser maior do que a área maior.

Quando é Retângulo a soma das duas menores área as vezes são iguais.

Quando é obtusângulo a soma das duas menores áreas será menor.

Figure 3. Answers of the A group in the original language

In an acute triangle, the sum of the two smaller areas is going to be greater than the largest area.

In a right-angled triangle, the sum of the two smaller areas, sometimes, is the same.

When the triangle is obtuse, the sum of the two smaller areas is smaller.

Figure 4 .Translations of the answers of the A group.

At this moment of writing the conclusions, one of the students (student J) was curious as he realized that, for two of the right triangles the relations were apparently different with respect to the areas of the squares. This justifies his writing that, "When it is a right rectangle the sum of the two smaller areas are **sometimes** equal." Thus, a whole-class discussion appeared about the approximations and roundings made by the software. The dialogue was like this:

> Researcher: My question is: if the triangle is a right triangle, can you say that the largest area is always equal to the sum of the other two areas?
> All: no!
> Student J: Sometimes! (As he had written on the activity sheet)
> Researcher: Sometimes, J said! When? How to identify these times? Because, look: in the acute triangle everything was bigger … bigger, bigger, bigger .... and here everything was smaller, smaller, smaller. … but in the right triangle something is different .... I wonder why.
> Student D: It's because it's not exactly 90 degrees! It can be 90 degrees and a half... maybe that's why!
> Researcher: That is ...... this 90-degree angle that is written here...
> Student D: It's a false 90!
> Researcher: Maybe it's not exactly 90 degrees?
> Student D: um-hum.
> Students suspected that, in the case of the right-angled triangle, this relationship would be equivalent. However, the need to understand what was going on and to use other ways to prove their conjectures became apparent. There was a discussion with the whole class about the approximations and rounding applied by the software. We asked them to allow a larger number of decimal places in GeoGebra. At this point, the students realized, and were surprised, that the apparently straight angles of some of the triangles obtained had amplitudes either less than or greater than 90°, as student D had imagined.
> In the following activity, the students explored another construction in GeoGebra in which there was a right-angled triangle, and where the right angle was set by the software. This was planned so that, regardless of the moves performed on its vertices, the triangle always kept a right angle, changing only the amplitudes of the acute angles and the measurements of the sides and, consequently, of the areas of the squares.

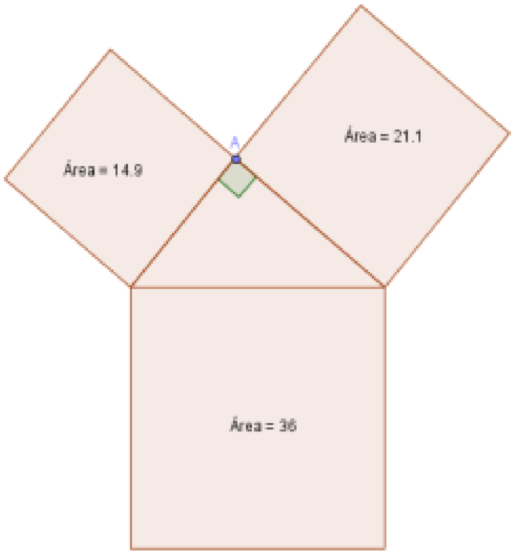
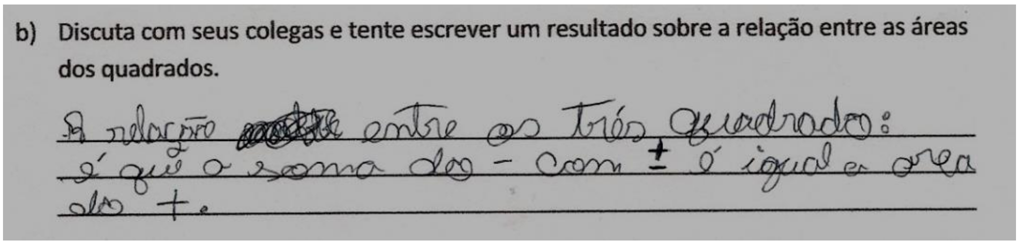


Figure 5. Right triangle with squares on its sides

The students were asked to submit in writing the relationship they had observed, in an organized and clear way and in consensus with all group members. The goal was for the students to discuss with colleagues the conjecture they had drawn up, and to find a way to express the result obtained. We encouraged them to write, even if they did not use the symbolic language of mathematics. We also wanted to encourage collective work and dialogue between the members of the groups, which did not take place in the previous activity. Some of the responses of the groups are shown here:


b) Discuta com seus colegas e tente escrever um resultado sobre a relação entre as áreas dos quadrados.

A relação entre os três quadrados: é que a soma dos – com ± é igual a área do +.

Figure 6. **Answers of the A group in the original language**

The relationship among the three squares is that the sum of the – with the ± is the same as the area of the +.

Figure 7. Translations of the answers of the A group

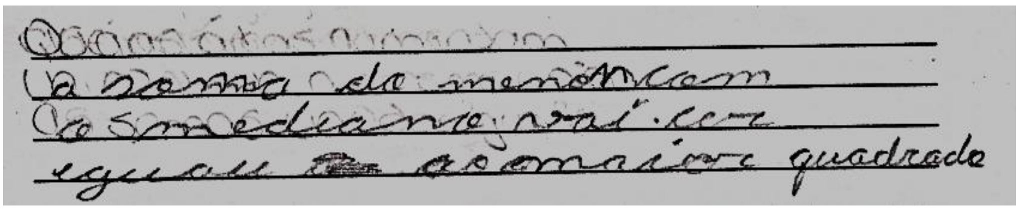


Figure 8. **Answers of the D group in the original language**

The sum of the smaller with the median is going to be equal to the larger square.

Figure 9. Translations of the answers of the D group

At the end of the lesson, the answers of all groups were written on the board. We discussed the relationship they expressed, and the students concluded that, even though written in different ways, all showed the same result. This result was called the "Pythagorean Theorem". We looked for a more precise way to write the result and the theorem was stated as: "The square of the length of the hypotenuse of a right-angled triangle equals the sum of the squares of the lengths of the other two sides".

These activities were followed by other activities and dialogues, aiming to state the Theorem using algebraic notation. The goal was to prove the Theorem from the equivalence of the areas of the squares and, possibly, to generalize the Theorem considering the areas of other types of polygons constructed on the sides of the triangle. These will not be presented here due to the proposed focus for this article.

The analysis of the results concerning the activities is presented below.

4. The Geogebra software, the dialogues and the production of knowledge about the theorem

In this article, we discuss, in a broader way, two aspects considered in the present study: the influence of features of the GeoGebra software, and the dialogues in the production of knowledge about the Pythagorean Theorem in the collective of humans-with-media. The visualization and experimentation had an important role in the development of activities with GeoGebra, enabling dialogue in which knowledge was produced. According to Arcavi (2015), the visualization can be a facilitator for intellectual experiences that leads us to say 'this make sense to me'. "It may refer to what some describe as an 'aha! moment', an insight, in which we fell how pieces fall into place, how ideas suddenly cohere and connect to each other" (p.147). We believe, as do Borba and Villarreal (2005), that the basic unit of knowledge is formed by humans and media working together.

We highlight two features of GeoGebra which contributed to making the knowledge produced significantly different from that which would have been possible without this media. First, the possibilities for handling figures; and second, for obtaining numerical values related to the geometric figures constructed. For example, the measures of the segments that make up the sides of the triangles and the values of the areas of the squares built on these sides. In the case that we analyzed, not only the visualization but also the values of the corresponding areas of the figure itself were essential. In our activities, associating dynamic manipulation with obtaining the numerical values of the areas gave the students the possibility of changing the position of the vertex of a triangle and of observing the transformations of the angles, the measures of the sides, and the numerical values referring to the areas of the squares. Therefore, in a short amount of time, we obtained a great amount of numerical values of the areas of the squares built on acute, obtuse and right-angled triangles. Furthermore, we were able to encourage students to observe the relationships. Something similar could be done with pencil and paper; however, changing the figure using dynamic geometry software seems significantly different than drawing various figures. Making changes by using the software can lead to conjectures such as: what happens with the areas of squares built on the sides when we change the angles and consequently the type of triangle? This does not seem to be made easier with the observation of different static figures.

GeoGebra allows the treatment of "drawings in motion", making it possible to modify the physical representation of the object and maintain the actual geometric properties of the construction. It was important to be able to discuss the apparently different relationships of the right-angled triangles, arising from the approximations made by the software in the amplitudes of the angles. We were able to explore right-angled triangles set by the software by moving the vertices and keeping its main feature as that of a right-angled triangle. So, the issue of rounding was discussed and the relationship between the areas was observed.

We have seen that the manipulation of the vertices of right triangles by construction (understood as robust constructions) and the obtaining of the measurements of the areas of the squares built on the sides of the triangles have made it possible to verify the perceived relation with the manipulation of any triangles constructed by the students. Therefore, the Pythagorean Theorem was validated using the so-called "drag test" (Stylianides & Stylianides, 2005).

In this study we define the understanding of the relationship of the Pythagorean Theorem as the fact that the students realize what is unique to the right-angled triangle, with regard to the areas of the squares built on its sides, namely: the sum of the areas of the squares built on the sides is equal to the area of the square built on the hypotenuse.

The search for patterns and regularities is one of the goals of the teaching of geometry and, in our case, we sought a pattern for each type of triangle. Students did not perceive immediately the relationship and the initially planned activities were not enough; they had to be supplemented by the development of a worksheet by the teacher-researcher, along with the students. The dialogues took place, initially among students, and later, between teacher and students. At the end of this stage, we realized that the students saw the regularity for the right-angled triangle and, in the excerpt below, we observe the student D dialoguing with her colleague and expressing orally this relationship:

> Student J: What's the relationship between the three squares in the right-angled triangle? None is greater than the other?
>
> Student D: No… but we are supposed to write about the relationship between the areas of the squares. Ahhh ... I got it… the sum of the biggest square with the median is identical to the sum of ... [rephrasing] the smallest with the median is identical to the sum of the biggest.
>
> To be able to express the relationships observed is what we are calling to state what was perceived. Students demonstrated some difficulties in writing, however, they managed to do it by using the language they judged appropriate and that, though not always entirely correct from the mathematical point of view, made their understanding clear. Group A (Figure 5), for example, used the + signs to indicate the biggest square and the – signs to indicate the smallest square in an attempt to simplify the writing of the relationship.
>
> We emphasize the importance of the interactions and the dialogues for the production of knowledge about the Pythagorean Theorem in the collective of humans-with-media. We understand that knowledge about the theorem was produced not only in terms of its statement or demonstration, but also in terms of the related mathematical concepts.

We take, as an example, a situation in which student D takes the initiative to help student J to understand acute triangles. It is believed that this student's initiative was possible because of the good relationship between her and the colleague, that is, the respect and consideration they had for each other. And this is one of the qualities that Alro and Skovsmose (2010) attribute to dialogue aiming at the quality of learning, that is, the dialogue "can only take place through its own dynamic sources, perspectives, emotions, intentions, reflections and actions of partners **in the most egalitarian positions possible**" (p.133, emphasis added). This was the dialogue:

Student D: Yours isn't an acute angle!
Student J: Why not?
Student D: Because your angle is 90 ... now it is!
Student J: And now if I do this ...
Student D: Yes! no ... no ... no ... not yet .... it has to be less than 90 degrees ... this way, you are going to increase it (the angle) … not yet .... you have to go up a little more ... a little more ... (indicating the movement to be made with the vertex of the triangle) [exchange of ideas]
Student J: This is an acute angle!
Student D: It's an acute angle... but 90 degrees and 91 is not an acute angle!
Student J was able to clarify the concept of an acute triangle, from the dialogue with his colleague.

D. Martinho (2007) states that only through the practice of group work can students evolve in the task of sharing their ideas and "it is after this stage [...] that [the students] are prepared for the most complex stage involving the ability to explain their ideas, to argue and try to convince colleagues of their opinions, as well as to listen and counter-argue" (p. 30). This author also affirms that this evolution takes place through practice.

Throughout the activities we could notice a gradual change in the students regarding the initiative and autonomy to work with the program, attitudes of collaboration with the members of the group, and especially the dialogues stimulating the production of mathematical knowledge in the collective.

5. Final considerations

The activities developed in this study were conducted in a collective of humans-with-media (Borba & Villarreal, 2005), where students in groups had the opportunity to dialogue (Alro & Skovsmose, 2010) in a collaborative environment.

For Borba and Villarreal (2005), the basic unit of knowledge is the collective of humans-with-media. For them, there is no dichotomy between humans and technologies but a constant transformation between these elements in such a way that it can be said that the computer reorganizes human thinking. The use of GeoGebra software, through the visualization linked to the dynamics of the software, allowed the students to question the results in rich mathematical discussions

The software tools enabled the dialogue, as claimed by Alro & Skovsmose (2010), in a relationship of respect among participants, in which the students assumed the risks that the dialogue offers and thus, carried out investigations that led to the production of knowledge of the Pythagorean Theorem.

Through the manipulation activities, as well as the dialogues encouraged by the scripts of the activities in the 4th and 5th meetings, the participants showed their perception of the relationships between the areas of the rectangles built on the sides of the acute, obtuse and, specifically, the right-angled triangles. We thus interpret what they discovered of Pythagorean Theorem. Having been asked to verbalize and present their findings in writing, the participants stated the Theorem, even though they frequently did not use conventional mathematical language.

For these and other reasons analyzed in this study, in a broader way, we are led to believe that the activities conducted in a collective of humans-with-media may give the students several opportunities to dialogue with their peers. Furthermore, the use of technology based on experimentation may give students the opportunity to investigate, test possibilities and raise hypotheses. The moment of theorizing and generalizing hypotheses, not least, may come after the experimentation. The presentation and discussion of the other activities mentioned which were not presented in this study, as well as the script with all the activities conducted, may be found in Sette (2013).

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