COMPLEMENTARITY AND THE ANALOG/DIGITAL DISTINCTION

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Abstract
Niels Bohr, as is well known, introduced the notion of *complementarity* into physics, as a fundamental principle of quantum mechanics. It holds that objects have complementary properties that cannot be measured accurately at the same time. For example, the particle and wave aspects of physical objects are such complementary phenomena. Both concepts are borrowed from classical mechanics, where it is impossible to be both, a particle and a wave at the same time. Particle and Wave represent the complementarity of the *Discrete* and the *Continuous*. Humans reason by means of concepts (meanings) and language, as well as, by means of logical or arithmetic symbolism. Meanings are continua, whereas logic and arithmetic are based on relations of identity and difference.

Keywords: Complementarity. Analytic and Synthetic Method. Peirce. Riemann.

I.

Niels Bohr, as is well known, introduced the notion of *complementarity* into physics, as a fundamental principle of quantum mechanics. It holds that objects have complementary properties that cannot be measured accurately at the same time. For example, the particle and wave aspects of physical objects are such complementary phenomena. Both concepts are borrowed from classical mechanics, where it is impossible to be both, a particle and a wave at the same time. Therefore, it is impossible to measure the *full* properties of the wave and particle at a particular moment.

We are not concerned at this moment with the philosophy of science, but are interested in the fact that waves are continuous entities and particles are distinct or discrete. Or rather, we are interested in the difference between analog and digital representations, and with the different types of generality associated with these.

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Analogy means being concerned with structural similarities. Everything appears similar to everything else in some respect. Therefore, the art consists in finding structural similarities that can be cast into mathematical definitions.

Claude Levi-Strauss once said that in science there are two methods only, the reductionist or the structuralist (Levi-Strauss, C., *Myth and Meaning*, Routledge London, chapter 1). However, the structural relationships considered should be similar too in terms of function. For example "looking in a natural history museum at the skeletons of various mammals, you may find them all frightening. If this is all the similarity, you can find between them you do not see much analogy. Yet you may perceive a wonderful suggestive analogy if you consider the hand of a man, the paw of a cat, the foreleg of a horse, the fin of a whale… these organs so differently used, as composed of similar parts related to each other" (Polya, 1973, p. 13)

Structuralism is synthetic it is connected with constructivism – Piaget did emphasize this over and again – and it determines objects by comparison of structural relations. Functionalism represents, in contrast, an analytical approach. One example: Bernhard Riemann (1826-1866), one of the greatest mathematicians of the 19th century and philosophically the most sublime of all, once wrote a small paper on the "Mechanism of the Ear" (Riemann, 1953), in response to a publication of Hermann Helmholtz of 1863. He did so for essentially methodological reasons, the main point of his study being the clarification of the analytical method.

Helmholtz was a leading proponent of the synthetical approach in the natural sciences or medicine, which begins by presenting the object and then assigning certain functions to it, rather than designing hypotheses on base of variable behavior of the organ and drawing conclusions from this. In the present case, it is the middle ear, which the synthetic approach characterizes anatomically to begin with and then one tries to explain on such grounds how the organ works. Riemann begins his paper with the following words:

"The physiology of a sense organ requires - aside from the general laws of nature - two particular foundations; one psychophysical, that is, the empirical verification of the achievements of the organ, and one anatomical, that is, the investigation of its construction. …. Accordingly, there are two possible ways of gaining knowledge about the organ's functions. Either we can proceed from its construction, or we can begin with what the organ accomplishes and then try and explain these accomplishments" (Riemann, 1953, p. 338).

Riemann calls the first route, synthetic and the second analytic and he prefers the analytical route in opposition to Helmholtz, being aware, however of the fact that any more involved investigation will in the end needs to employ both methods. The first proceeds from causes to effects, whereas the second seeks causes of given effects or conditions of intended goals.

Riemann prefers the analytical approach not least because he is always careful to find out about the necessary premises of an explanation or a mathematical proof. The essential chapter of Riemann's famous Habilitationsschrift, for example, begins with the classic statement: "First then, what do we understand by $\int f(x)dx$?" (Riemann, 1953, p. 239).

Riemann, by asking what the symbol $\int f(x)dx$ means, searches for necessary conditions of integrability, rather than starting some more or less traditional and arbitrary sufficient properties of the functions to be integrated, like continuity, as Cauchy did. Riemann was thereby able to generalize the notion of the integral and was beginning to realize its dependency from theories of measure. It seemed thus, "that Riemann had extended the concept of an integrable function to its outermost limits" (Hawkins, 1970, p. 34).

There is more, however, to this complementarity of structure and function. Structure and function are not strictly connected or subordinated to each other, but are complementary and this complementarity becomes effective from an evolutionary perspective. Suppose we have put a coordinate grid on the image of a wolf skeleton, which specifies certain points of the skeleton structure, such as a toe, the ankle, the last rib in the geometric plane. We then try to adapt this grid with its assignments to the analogical picture of the skeleton of some dog, a Great Dane, for example, or a dachshund. The previously straight lines of the grid now appear deformed because, for example, the relationship between head size and leg length might have changed more or less. The degree of deformation measures the degree of evolutionary change or the distance in the relationship line (see: *Wolf to Woof, The Evolution of Dogs*, Nat. Geographic, vol. 201(1), January 2002, pp. 2-11).).

What has been a particular thing before now becomes the sign of the continuous context and of the evolution, it becomes dissolved into a continuous movement, into the context of the continuum and thus it becomes a general. The individual skeleton becomes a variable of a kind. The continuum represents a realm of possibility and thus a different kind of generality, relational generality, than the usual generality of predicates or functions. When we conceive of generalization as the introduction of variables, we can realize that difference by observing that in discrete mathematics and computer science variables are mere placeholders, while in continuous mathematics and the empirical sciences variables are "general", that is, incompletely determined objects, like the general triangle. In addition, in a proposition like "an apple is a fruit" it would be unnatural to interpret "an apple" as a placeholder, because this presupposes that we have given individual names to all the apples in this world (Quine, 1974).

There are ideas of an apple or a triangle in general, but they turn out to be ideas of particular triangles, put to a certain use. On such an account, a general triangle is a free variable, like the terms in axiomatic descriptions, and not a collection of determinate triangles. It is an idea, which governs and produces its particular representations. And which properties are essential to a „general triangle", depends on context, on the activity and its goals. If the task, for instance, is to prove the theorem that the medians of a triangle intersect in one point, the triangle on which the proof is to be based can be assumed to be equilateral, without loss of generality – because the theorem in case is a theorem of affine geometry and any triangle is equivalent to an equilateral triangle under affine transformations. This fact considerably facilitates conducting the proof because of such a triangle's high symmetry. Thus, we end up by choosing a particular triangle, a particular exemplar of a type by considerations of functionality. Bishop Berkeley's discussion of the idea of "general triangle' had already made us aware of these realities.

II.

Communication is a kind of behavior and any behavior is communicative, but may be ambiguous in its message: a clenched fist may communicate excitement, fear, anger, frustration and many more things and a punch to the shoulder may mean to reinforce friendship, or encouragement, or, to the contrary, it may be an assault. Therefore, communication depends on meta-communication. The greeting, "Pleased to meet you!", is as a rule classified by facial expression and gestures as either true or false. Moreover, if the meta-communication lacks, paradox may be the result, as when I say, "I am lying". The switch from the analogical and continuous to the digital and discrete is usually accompanied by a loss in sense, that is, meaningfulness and a gain in information.

All natural systems of communication employ both analog and digital communication at some level in the system and “the question of the analog and the digital is one of relationship, not one of entities” and thus a question of complementarity (Wilden, 1972, p. 188).

An analog computer works by means of an analog between real continuous quantities and some other set of variables. The balance, the flyball governor, the wind-channel or even the accelerator pedal in your car, are examples of analog computers, whereas electronic computers are digitalized systems. They operate on discrete elements, on a set of distinct symbols, like zero and one, or *Yes* and *No,* distinctions made possible by electronic switches or on/off processes. There are no variables in a discrete system and what is called variable is merely a location or a placeholder. Negation in any language depends on syntax, such that the analog computer contains no negation, it cannot say *not-A* and cannot represent nothing. Mathematical knowledge is based on the relations of identity and difference. Otherwise, the law of non-contradiction does not apply. The principle of consistency, according to Kant, only applies if there is an object given. The statement that “a triangle has three angles”, says Kant, “does not enounce that three angles necessary exist, but upon the condition that a triangle exists three angles must necessarily exist in it” (Kant, B 622).

With modern mathematics things are exactly the opposite way, that is, the law of non-contradiction comes first and the whole universe of possible “objects”, the entire ontology comes to be based on this principle. The first condition a system of axiomatical definitions must fulfill, in order to function as a sign or representation at all, is consistency

If we think of numbers, we realize that zero, 0, is not a number, according to the Frege-Russell definition, but is a sign to organize numerical representation. Zero is “a meta-integer, a rule about integers and their relationships” (Wilden, 1972). It is, generally conceived of, for example, as the balancing out of a positive and a negative number of equal value. Equality is after all the most important relationship of arithmetic and algebra and it should be used to generalize. The rules for calculating with fractions, so difficult for children to remember or understand, come out completely natural, if we represent for example the fractional number 3/7 by the equation $7x = 3$!

By the same token the “empty set” is not a set, but is essentially the sack into which elements could be put. This view of the notion of set, as “collection-as-one” draws a categorical distinction between a set and the collection of its elements.

Moreover, by putting the sack into another sack, and so forth, one creates a conceptual hierarchy of great complexity. Frege did not adopt this view of sets, arguing, "if we burn all the trees of a forest, so we burn the forest (Wenn wir sämtliche Bäume eines Waldes verbrennen, so verbrennen wir den Wald)" (Frege, 1996, p. 93). The paradoxes of set theory resulted therefore "not from an inconsistency of our intuitive notion of set, but from a conflation of two or more incompatible notions (set-as-one, set-as-many)" (Potter, 1990, p. 10).

The somewhat specific character of Frege's view is obvious because a forest is, in fact, more than a set of trees, and human society something more than the sum of individuals. In Frege's universe, there are only two things, objects and functions. And functions are not objects, or at any rate, they are not objects of the same type, as the arguments falling under them. Therefore, we have second-level functions, etc. thereby creating a logical system of enormous complexity, which can be handled by the computers, but not by man - in computer programming it is common to insert programs into programs as subroutines.

To avoid this complexity Frege, supposed that a function of any level determines a set of ground-level objects, called its extension. Frege in particular interpreted the natural numbers as concept-extensions, because the number concept has to be universally applicable. Russell informed Frege in 1902 that this construction produces the so-called semantic paradoxes of set theory.

Logical argumentation is based on some assumption of self-evidence. If somebody does not understand what you are saying, you are inclined to respond: But, it is logical, isn't it! On such an assumption, Hilbert has called formal logic self-evident. The starting point for this problem is the assumption that every proposition immediately implies itself. If I say "p" this implies "p". And this in turn means, "p is true". The predicate "is true" does not really add something to the status of the original affirmation, although "p" and "p is true" are in general different sentences. Truth is undefinable, as Frege did say already. Thus, one might want to settle with the view that "p is true", really implies "p is true". Therefrom results the 'immediacy assumption' for formal systems, which is a kind of minimum loop principle.

Churchman asks, "What's wrong with the minimum loop principle? …. What can disturb the balance of logical perfection? Now a Cretan can. This Cretan – we may call him Epimenides - says all Cretans are liars". And this inevitably leads to the abyss of paradox. If we do not prohibit the Cretans to talk about his own talk, we end up in a paradox (Churchman, 1968, p. 113-114).

As a reaction, one might try and adopt as maximum loop principle. "The maximum-loop principle is "fantastic. It says that self-reflection is possible only if one returns to the self after the longest possible journey. It is exemplified in the great myths of the heroes: Ulysses must go through every deep experience of human life before he can come to his resting point. …. The maximum-loop principle is based on a monistic philosophy: There is one world of interconnected entities, not many. The most distant galaxies and the most menial worker somehow have a connection. The principle is also teleological. For the mind to know itself; it must also know the destiny of all minds as well as all matters" (Churchman, 1968, p. 113-114).

This maximum principle fails us too, however, because it is actually unrealizable, because the world as a whole would put an infinite resistance to our knowledge process, therefore science has to try and limit its contexts of investigation. The complementarity of maximum and minimum principle presents us another example of the complementarity the continuous and the discrete. In mathematics continuous geometry one the one side and discrete arithmetic on the other represent the analog/digital difference.

What about algebra? Algebra is a kind of meta-arithmetic combining the analog and the digital. Algebraic diagrams are essentially icons of structural relationships. Now, all mathematical reasoning is diagrammatic and "all necessary reasoning is mathematical reasoning, no matter how simple it may be" (Peirce, CP 5.148).

Diagrams are essentially icons, and icons are particularly well suited to make graspable and conceivable the possible and general, rather than the actual and existent. Diagrams, however, must also include *indices*, signs that indicate, or denote or are actually connected to, some particular thing in order to fix references. The indices occurring in pure mathematics refer to entities or objects that belong to a model, rather than to "the real world", that is, they indicate objects in constructed semantic universes.

Indexicality is what in particular makes the semiotic approach to mathematics unavoidable and Peirce saw as no one before him had, that indication (pointing, ostension, deixis) is a mode of signification as indispensable as it is irreducible. Frege missed this point and adopted a theory of reference based on descriptions. This brought about the troubles of self-reference that created paradox.

III.

A simple illustration of the complementarity between the continuous and the discrete and the between the related differences in argumentation may be obtained by representing proofs of the incommensurability of the side and the diagonal of the square or the regular pentagon, respectively. Looking at the common diagrams which represent traditional visualizations of the familiar incommensurability of side and diagonal in the case of the square respectively the regular pentagon (Otte, 1990, p. 37), interpretation can be done in two ways.

First, I may concentrate on the method or way of "finishing" the picture, that is, the recursive sequence of the picture within the picture, within the picture …... Doing this, I find out that this method corresponds to the Euclidean algorithm. As a result, I obtain, based on the visual representation, insight into the recursive structure of this algorithm, as well as the insight that this algorithm can be used for a proof of incommensurability without recurring to the natural numbers.

The diagrams of the regular polygons also require us to disregard scales. The invariance under geometric similarity or “application to the problem of the incommensurability of side and diagonal of self-similarity” then demonstrates directly that the algorithm does not lead to the desired goal, i.e. to a division without residue. Side and diagonal are thus incommensurable.

Second, I may also, following a different approach replace the geometrical quantities by their numerical measures, with respect to a fundamental unit that measures both the side and the diagonal in whole numbers. The recursivity or self-similarity leads to an infinitely decreasing progression of natural numbers, as all figures have sides composed of "whole numbers". This gives a contradiction. In doing this, I may incidentally note that the geometrical method of construction of the sequence of polygons can be interpreted in terms of the Euclidean algorithm (Otte, 1990).

If I consider the algorithm, however, only in die field of arithmetic, only using it together with the numbers, and not interpreting it geometrically, visually, it will not appear in itself, it will not show its recursive structure. To attain this, I have to use visualization and interpret them directly that is "an image within an image within an image...“ With the concept of recursion, I shall then simultaneously obtain a means to describe a large number of algorithms.

This tells us that there are two types of generality, Fregean or Platonic predicative generality of functions, on the one hand, and continuity, in the sense of Aristotle or

Peirce, on the other hand. Peirce, calling himself “an Aristotelian of the scholastic wing”, describes them thus:

“The old definition of a general is *Generale est quod natum aptum est dici de multis*. This recognizes that the general is essentially predicative and therefore of the nature of a representamen. ... In another respect, however, the definition represents a very degenerate sort of generality. None of the scholastic logics fails to explain that sol is a general term; because although there happens to be but one sun yet the term sol *aptum natum est dici de multis*. But that is most inadequately expressed. If sol is apt to be predicated of many, it is apt to be predicated of any multitude however great, and since there is no maximum multitude, those objects, of which it is fit to be predicated, form an aggregate that exceeds all multitude. Take any two possible objects that might be called suns and, however much alike they may be, any multitude whatsoever of intermediate suns are alternatively possible, and therefore as before these intermediate possible suns transcend all multitude. In short, the idea of a general involves the idea of possible variations”, of free variables, or of continuity (Peirce, CP 5.102-103).

Rather than speaking about suns, we might think of the general triangle or similar variables (see above).

This twofold character of the general is expressed in the history of mathematics by two different interpretations of the *Continuity Principle*, two interpretations over which Cauchy and Poncelet quarreled (Belhoste, 1991), when the idea of pure mathematics was at stake, although they had been present since Antiquity. It seems, indeed, that these interpretations occurred in two different kinds of proof in Greek mathematics.

During the first phase of Greek mathematics, a proof consisted in showing or making visible the truth of a statement. This was the *epagogic* method. This first phase was followed by an *apagogic* or deductive phase. During this phase, visual evidence was rejected and Greek mathematics became a deductive system (Koetsier, 1991, and the bibliographic reference given there).

Now epagoge and apagoge, apart from being distinguished, roughly according to the modern distinction between inductive – or rather: abductive or hypothetical - and deductive procedures were also identified on account of the conception of generality as continuity. Epistemology of mathematics today only remembers the distinction, forgetting where they agreed, in this manner not only destroying the unity of the perceptual and conceptual, but also forgetting what could be gained from Aristotelian demonstrative science.

Even the diagrams of Euclid could be interpreted in two complementary ways. Ian Mueller, for example has described the situation in relation to Euclid's diagrammatical proofs as follows: "The Euclidean derivation is a thought experiment. [...] the major obstacle to an acceptance of the interpretation of Euclid's arguments as thought experiments is the belief that such arguments cannot be conclusive proofs. In particular, one might ask how consideration of a single object can establish a general assertion about all objects of a given kind. Part of the difficulty is due, I think, to failure to distinguish angles equal. Under one interpretation the statement refers to a definite totality [...] and it says something about each one of them. Under the other interpretation no such definite totality is presupposed, and the sentence has much more conditional character – 'If a triangle is isosceles, its two base angles are equal'. A person who interprets a generalization in the second way may hold that the phrase 'the class of isosceles triangles' is meaningless because the number of isosceles triangles is absolutely indeterminate" (Mueller, 1969, p. 291-292, 299-300).

Mathematics then reasons starting from the meanings of certain representations, rather than from supposed characteristics of a class of objects. Theoretical concepts on such accounts are not empirical abstractions, but are operative schemata, like in modern axiomatics in the sense of Hilbert or Peano.

Mathematics becomes intensional and it must be complemented by some intended applications. In a dynamic view, the intensions and extensions assume greater autonomy and independence from each other, and the problem of the so-called. "impredicative definitions" loses its threatening character (Smirnov, p. 223-232).

Epagoge is often translated by induction, but is not really induction, but is more like what Peirce calls abduction, or reasoning from intuited hypotheses. It proceeds by taking one individual as prototypical for the whole kind. However, one has to choose the type or kind. Whewell, arguing against Mill's positivism, expresses in a quite charming manner:

"Induction is familiarly spoken of as the process by which we collect a General proposition from a number of particular cases: and it appears frequently imagined that the general proposition results from a mere juxta-position of the cases. … But if we consider the process more closely … we shall perceive that this is an inadequate account of the matter. …. The pearls are there, but they will not hang together till someone provides the string. ... Hence in every inference by Induction, there is some conception *superinduced* upon the facts: and we may henceforth perceive this to be the peculiar import of the term *Induction*" (Whewell, 1847, vol. 2, pp. 46-48).

IV.

Let us come back to the problem of incommensurability, making clear that there is no possibility of positively telling what an irrational number is, without employing some notion of continuity, like the continuum of real numbers, which can be presented only axiomatically, that is, conceptually, because of the fact that the set of real numbers is not countable or enumerable.

Some psychologists have tried to avoid the continuum and have failed. Let us look at the following example of two different explications of the notion incommensurability and irrational number:

1) Commensurable line segments

In comparing the magnitudes of two line segments *a* and *b*, it may happen that *a* is contained in *b* an exact integral number *r* of times. In this case we can express the measure of the segment *b* in terms of that of *a* by saying that the length of *b* is *r* times that of *a*. Or it may turn out that while no integral multiple of *a* equals *b*, we can divide *a* into, say, *n* equal segments, each of length *a/n*, such that some integral multiple *m* of the segment *a/n* is equal to *b*: $b = (a/n).m = (m/n).a$

When an equation of the form above holds, we say that the two segments *a* and *b* are *commensurable*, since they have as a common measure the segment *a/n* which goes *n* times into *a* and *m* times into *b*. (Courant & Robbins, 1941, p. 58)

2) Commensurable line segments

It is said that two line segments are commensurable if they have a common measure. What does it mean to have a common measure? Let us assume that one line segment is 3 cm long and another, 9 cm. The two line segments are commensurable: The common measure is 3 cm. It fits once into the first line segment and exactly three times into the second. Let us assume that one line segment is 6 cm long and another, 10 cm. These, too, are commensurable. Their common measure is 2 cm: It fits three times into the first line segment, and five times into the second. Even for two line segments of length, say, 1.67 cm and 4.31 cm, it is easy to find a common measure: 0.01 cm. It fits 167 times into the first line segment and 431 times into the second. What do these examples tell us? Two line segments are commensurable if one line segment (or a fraction of it) is contained within the other without remainder.

The second quotation above was written by two psychologists who wanted to "improve the original mathematical text" by Courant and Robbins. "Avoiding variables, formulae and diagrams" was noted as a typical feature of improvement (and the revision omitted a geometrical diagram that was in the original). Indeed, the revision has its merits from the perspective of "pure" readability, which is conceived of as being neutral with regard to a cognitive use of the text. The second text seems clear and straightforward, like a simple calculation.

On the other hand, the revisers do not seem to have realized that the mathematical subject matter itself has in a way disappeared after the variables and diagrams have been eliminated. If one replaces the relations between line segments by relations between decimal numbers from the very outset, one of course always has a common measure. The object of the original text does not simply consist of a defining circumscription of commensurability; it was occasioned by the problem of incommensurability, which has continued to cause astonishment, speculation, and contemplation since antiquity. This is the question at issue mathematically, and not the verification that 1.67 is a rational number, as the second text leads one to believe. One might even suppose, reading this text, that all numbers are rational and that there is no incommensurability.

The subject matter in question, namely incommensurability, appears then as the unknown, or at least, as the territory not yet described and mapped out. The irrational is characterized in merely negative terms, that is, as that which is not rational. This is exactly what teachers usually tell their students when trying to explain what irrational numbers are. The question cannot be treated without the continuum. The continuum per se is without units and the term "incommensurability" does in fact say that we cannot in certain situations find a common unit.

Bolzano, Frege and Russell opted in favor of an arithmetization of mathematics because the thought that universal mathematics must be digital. Geometry became considered a field of intended applications of discrete or arithmetized mathematics and the Aristotelian continuum became replaced by an arithmetical model of it. Mathematics had thus to become meta-mathematics, taking its own activities and procedures as new objects. Piaget, for instance, describes the process of mathematical development in terms of hypostatic abstractions, which he calls "reflective abstractions".

The essence of hypostatic abstraction is the recursive nature of thought, which is expressed by the fact that a thought or an action can be made the object of another thought. The infinite recursive process of abstraction is a feature of the mathematics of modernity.

"In Greek mathematics, whatever its originality and reputation, symbolization ... did not advance beyond a first stage, namely, beyond the process of idealization, which is a process of abstraction from direct actuality, ... However ... full scale symbolization is much more than mere idealization. It involves, in particular, untrammeled escalation of abstraction, that is, abstraction from abstraction, abstraction from abstraction from abstraction, and so forth; and, all importantly, the general abstract objects thus arising, if viewed as instances of symbols, must be eligible for the exercise of certain productive manipulations and operations, if they are mathematically meaningful" (Bochner, 1966, p. 18).

A look at the history of mathematics, actually, teaches us that the problem of universals, the problem of generalization by hypostatic abstractions and the transformation of contents of thought into objects of contemplation, on the one hand, had been the basis of the dynamics of the development of mathematics in modern times, and that, on the other, have created resistances to the tentatives to clarify the foundations of pure mathematics.

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