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How do textbooks from Brazil, the United States, and Japan deal with fractions?

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ABSTRACT

Background: Researchers recognise the importance of textbooks for teachers' lesson planning and the importance of fraction knowledge for shaping students' future mathematics performance. **Objectives**: The finding of discrepant achievement by Brazilian, American, and Japanese students in the last three editions of PISA led us to investigate how textbook authors from these countries approach fraction content in elementary education relating to magnitude, flexibility, reasonableness, as well as conceptual and procedural knowledge from both symbolic and nonsymbolic perspectives. **Design**: The quantitative performances in mathematics of Brazilian, American, and Japanese students in the last three PISA editions lack qualitative and exploratory research to understand some reasons presented by the numerical results. **Data collection and analysis**: To achieve the objectives, we selected three textbook series, one each from Brazil, the United States of America, and Japan, that schools in those countries widely use. **Results**: The main results revealed that all textbook series practised flexibility and reasonableness with different emphases, but not the sense of magnitude. Brazilian and U.S. textbooks were based primarily on part-whole interpretation and on a procedural approach. In contrast, Japanese textbooks emphasised the understanding of measurement as the iteration of unit fractions and more conceptual development. **Conclusions**: The fraction knowledge approach in the Japanese textbook series seems to be close to what the mathematics education researchers recommend, which can be an essential differential to explain the Japanese results in PISA.

Keywords: textbooks; fractions; Brazil; United States; Japan.

RESUMO

Contexto: Pesquisadores reconhecem a importância dos livros didáticos para planejamento de aulas por professores, bem como a relevância do estudo de frações por

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moldar o desempenho futuro da matemática de alunos a depender da abordagem. **Objetivos**: A constatação de desempenhos discrepantes de alunos brasileiros, estadunidenses e japoneses nas últimas três edições do PISA nos levou a investigar como autores de livros didáticos daqueles países abordam o conteúdo de frações no Ensino Fundamental no que diz respeito à flexibilidade, razoabilidade, senso de magnitude e com o desenvolvimento conceitual e procedimental, nas perspectivas simbólicas e não-simbólicas. **Design**: Os desempenhos quantitativos em Matemática de estudantes brasileiros, estadunidenses e japoneses nas três últimas edições do PISA carecem de investigação qualitativa e exploratória para se entender algumas razões apresentadas pelos resultados numéricos. **Coleta e análise de dados**: Para atingimento dos objetivos, foram selecionadas três séries de livros - brasileiro, estadunidense e japonês - amplamente utilizados pelas escolas daqueles países. **Resultados**: Os principais resultados revelaram que todos os livros praticaram a flexibilidade e a razoabilidade com ênfases diferenciadas, mas não o senso de magnitude. Os livros brasileiros e estadunidenses basearam-se prioritariamente em interpretação parte-todo e em abordagem procedimental, enquanto os livros japoneses apresentaram ênfase na interpretação de medida como iteração de frações unitárias e no desenvolvimento mais conceitual. **Conclusões**: A abordagem de frações dos livros japoneses parecem estar mais próximos do que recomenda a comunidade científica de Educação Matemática e este pode ser um importante diferencial que explique parte dos resultados no PISA.

Palavras-chave: livros didáticos; frações; Brasil; Estados Unidos; Japão.

INTRODUCTION

As sources for teachers' and students' knowledge of school mathematics, national and local educational authories recommend printed and digital textbooks. They are primary resources for teachers to plan and teach their lessons (Beaton et al., 1997; Schmidt et al., 1997; Escolano & Gairín, 2005; Reys et al., 2007; Alajmi, 2009, 2012). At the same time, textbooks are often the only material students access to learn about mathematical objects and their operations (Watanabe et al., 2017). In parallel assessments, Chingos and Whitehurst (2012), Fan, Zhu, and Miao (2013) note that "researchers generally agree that textbooks are important means of conveying curriculum and play a dominant role in modern education scenes in different school disciplines" (p. 635). Previously, Robitaille and Travers (1992) stated that dependence on textbooks is "perhaps more characteristic of mathematics teaching than of any other discipline" (p. 706, our translation).

Cirillo, Drake, and Herbel-Eisenmann (2009) found that teachers indeed rely on textbooks as their primary resource for teaching mathematics. This reliance on textbooks is caused by the essential utility of visual models to represent abstract mathematical objects and operations on them (Fan et al., 2013). The benefits of printed and physical visual model for students' understanding were researched (e.g., Arcavi, 2003; González-Martín, Nardi & Biza, 2011; Stylianou & Silver, 2004) and subsequently defended in curricular policy documents (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2014).

Mathematics textbooks have specific contents that stem from the desire to understand phenomena or some human material need as occurred with the emergence of fractions (Caraça, 1951; Aleksandrov, 1963). Regardless of their origins, some contents are core and support the development and deepening of others, raising them to a degree of importance that researchers, teachers, and textbook authors should consider. Fractions are one of these central contents in basic education, whose conceptual and operational understanding is predictive of algebra performance and, more broadly, of general mathematics achievement (Kieren, 1976; Siegler et al., 2012) such as in probability, and calculus (Lamon, 2001; Bailey et al., 2012; Booth & Newton, 2012; Siegler & Lortie-Forgues, 2015; Torbeyns et al., 2015). The essential importance of fraction content made mathematics education researchers conclude that its teaching should consider different dimensions - flexibility, reasonableness, and sense of magnitude (Powell & Ali, 2018) – in addition to conceptual and procedural understanding of nonsymbolic and symbolic represention s (Siegler et al., 2012; Powell, 2019b) for broad and deep understanding.

To know these dimensions, awareness or other aspects that have textbooks as a source, Watanabe, Lo, and Son (2017), Charalambous et al. (2010), and Li, Chen, and An (2009) recommend that the study be conducted in two ways: (1) by macroanalysis or horizontal examination and (2) by microanalysis or vertical examination. Macro-analysis [1] (or horizontal examination[2]) has to do with the general structures of books or the unfolding of certain content over school years or education levels and may focus on different educational systems. Microanalysis (or vertical examination) seeks to capture the mathematical process of content development. Examining the two modes - henceforth simply macroanalysis and microanalysis - can minimise unfavourable aspects intrinsic to each when studied alone. Researchers who chose singularly for macroanalysis were limited to the overview of topics

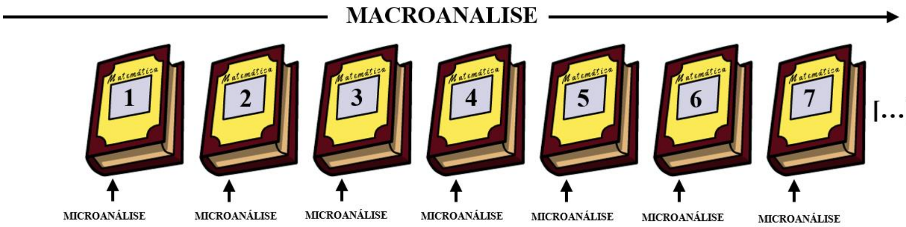
[1]The terms “macroanalysis” and “microanalysis” were denominated by Li, Chen, and An (2009).
[2] The terms “horizontal” and "vertical” were denominated by Charalambous et al. (2010).

throughout schooling but skipped important details of the contents. The opposite also occurred with the microanalysis. (Watanabe, Lo, & Son, 2017).

In this study, on the one hand, we seek to understand that the dimensions of flexibility, reasonableness, and sense of magnitude presented in textbooks used sequentially in early school grades, precisely because those dimensions involve the articulation of different interpretations that should be introduced in teaching gradually and related to each other, justifying a macroanalysis. On the other hand, conceptual, procedural, nonsymbolic and symbolic understandings are (or should be) revealed at the heart of the teaching of each interpretation. In other words, each interpretation should value the diversification of the understandings, aiming at extending and deepening students' understanding, as we will detail in a later session. Thus, the study of the conceptual, procedural, nonsymbolic, and symbolic understandings of fractions is justified by a microanalysis, amid the explanations and activities proposed in each textbook, as shown in Figure 1.

Figure 1

Macroanalysis and microanalysis



Given the above, it is important to understand how textbooks deal with the presentation of ideas about fractional numbers. We present the results of an exploratory study of textbooks focused on an academic-scientific agenda on the teaching of fractions supported by the international mathematics education community. The following are research questions that guide our study of elementary school textbooks:

1. How do textbooks treat the magnitude, representational flexibility and computational reasonableness of fraction?
2. How do textbooks attend to conceptual and procedural development of nonsymbolic and symbolic fractions?

The first research question reflects a macroanalysis, whereas the second reflects a microanalysis. We examined mathematics textbooks from three countries: Brazil, the U.S., and Japan. We choose those countries not randomly but rather based on their students' mathematics performances in Programme for International Student Assessment (PISA). PISA measures knowledge in reading, mathematics and sciences of around 15-year-old students of participating countries. In its last three editions (Organization for Economic Co-operation and Development, 2012, 2015, 2 018), in mathematics, Brazilian students ranked in the last place, U.S. students mesasured in a middle position, and Japanese students achieved first place (Table 1).

The *ranking* alone does not mean much, as the performances of the three countries could be close and the absolute differences are not quantitatively relevant. It is useful, therefore, to seek subsidies in statistical measures that reveal how far those countries are in terms of performance in PISA. The three countries were relatively stable in their performances in relation to the arithmetic mean of all participants in the 2012, 2015, and 2018 editions. It is noteworthy, however, a strong negative discrepancy ($X-\bar{X}$) for Brazilians, slightly negative to positive discrepancy for the U.S., and a high positive discrepancy for the Japanese respondents. The positions of these groups are confirmed by z-scores, revealing that the three countries, in fact, are significantly distant in their relative positions in the three editions of PISA. It remains for us to understand the reasons for those quantitative results, thus referring us to qualitative studies broken down in the methodological course of research.

Table 1

Mathematics performance, ranking, discrepancy in relation to the arithmetic mean, and z-score of the results of students from three countries in the 2012, 2015, and 2018 PISA. (Organisation for Economic Co-operation and Development, 2012, 2015, 2018)

Country	PISA 2012 - Mathematics ($n = 65; \bar{X} = 473; s = 55$)				PISA 2015 - Mathematics ($n = 70; \bar{X} = 462; s = 55$)				PISA 2018 - Mathematics ($n = 79; \bar{X} = 459; s = 55$)			
	X_1[a]	*Ranking*	X_1-$\bar{X}$	z	X_2[a]	*Ranking*	X_2-$\bar{X}$	z	X_3[a]	*Ranking*	X_3-$\bar{X}$	z
Brazil	391	58/65	-103	-1.50	377	68/70	-113	-1.55	384	71/79	-75	-1.36
United States	481	36/65	-13	+0.14	470	40/70	-20	+0.15	478	38/79	+19	0.34
Japan	536	7/65	+42	+1.15	532	5/70	+42	+1.27	527	6/79	+68	1.23

[a]X_1, X_2, X_3 are the mathematical performances in each country; n is the number of countries participating in PISA in a given edition; X-$\bar{X}$ is the discrepancy of the country's performance in relation to the arithmetic mean in a given edition of the PISA; z score is the number of standard deviations above or below the arithmetic mean of the population.

Given this context, we introduce the next topic with the conceptual and theoretical structure of fractions in the dimensions of magnitude, representational flexibility, and computational reasonableness, as well as the conceptual and procedural development of nonsymbolic and symbolic fractions.

RELEVANT LITERATURE

Flexibility

Representational flexibility is characterised by the mental resourcefulness in connecting different interpretations of fractions: part-whole, quotient, ratio, operator, measure, and the act of measuring. No one interpretation alone satisfies all nuances of the fraction concept (Lamon, 2007; Watanabe, 2006) or is a powerful tool alone for achieving flexibility. Another aspect of flexibility is the use of equivalent fractional expressions in specific situations to convey particular meanings.

Concerning different intrerpretation of fractions, the meaning of part-whole emerges from the division of divisible things. This interpretation has no ontological basis (Escolano & Sallán, 2005; Bishop, 1999) but has pedagogical value when, for example, it exceeds the simple shading of one of four equal parts to represent 1/4. Besides the area-based representativeness, it is worth understanding this meaning as length (1/4 of the distance between Recife and Rio de Janeiro) and set (1/4 of the train passengers), connecting them. Adding to these three strands - area, length, and set – we must consider their representational diversity in each one. The area or region model can be visualised with rectangles, circles, and other geometric figures, regions of a geoboard, the Cartesian plane, and so on. The length model can be explored with items such as Cuisenaire rods, paper strips, string, or the number line. The set model should highlight the number of equal subsets in a set rather than the size of the set. "For example, if 12 counters form a whole, then a set of 4 counters is a third, not a fourth, since three equal sets form the whole" (Van de Walle, Karp, & Bay-Williams, 2010, p. 290, our translation).

In addition to the part-whole interpretation, there is the idea of fraction as a quotient that arises in situations of two variables, discrete or continuous, one in the numerator and the other in the denominator: equally distribute $1 to 4 people, equally divide 1 pizza among 4 people, or evenly distribute the weight of containers all along a ship.

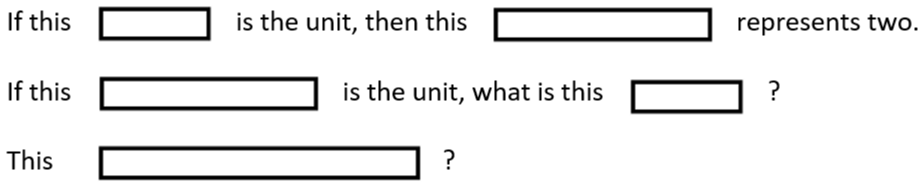
Another interpretation of fraction is ratio, meaning a proportion or comparison. For instance, the fraction 1/4 may signal a ratio of likes and dislikes: 1 who likes and 4 who do not like a musical style. Hart (1987) understands ratio as a statement of a numerical relationship between two entities, while Behr et al. (1983) argue that a ratio is a comparative index and not as a number. Lamon (1994) sees fraction 1/4 as one object for every four to receive. Our understanding is that every ratio represents a comparative index between two quantities of different or the same kind. When it represents a comparison between quantities of the same kind such as the ratio between the ages of two people, the ration expresses a fraction. However, for example, the ratio between distance and time (i.e., average velocity), where the comparison involves two quantities of different kinds, the ratio is not a fraction. Therefore, not every ratio is a fraction, but all fractions are ratios.

A fourth interpretation of fraction is the notion of operator. It indicates that a fraction operating on quantity's magnitude may transform it. For instance, in a class 3/4 of the students passed an examination means that the quantity that represents the number of students in the class a smaller number passed the exam. Whereas, this year's production is 5/3 of last year's, means that this year's production is greater than last year's by 2/3.

A fifth meaning of fraction is as a measurement, signifying "how much" rather than "how many parts" (Behr et al., 1983; Martinie, 2007). For Behr et al. (1983), the measure means a reconceptualisation of the notion of fractions as part-whole, and this contributes to the expansion of the understanding of fractions as a single quantity and not as two isolated integers. Van de Walle, Karp, and Bay-Williams (2010) exemplify the idea of measurement with fraction 1/8 being able to assume a unit of length to measure 5/8, i.e., it will take 5 units of 1/8 to measure 5/8, which is an iteration of the unit fraction, 1/8. Kieren (1976) emphasizes that a unit, once chosen, can be divided into any number of equal parts. Kieren's view falls on the idea of Van de Walle, Karp, and Bay-Williams (2010) when exemplifying the measure as in Figure 2:

Figure 2

Fraction as measure in Kieren's view (1976, p.125)



The five interpretations by Kieren (1976) and Behr et al. (1983) - part-whole (or part-part), quotient, ratio, operator, and measure - are based on the notion of the partitioning a quantity. In contrast, based on the history of mathematics by Aleksandrov (1963) and Caraça (1951), Powell (2019a) presents the notion of fraction from another perspective - of measuring: a fraction is a multiplicative comparison between two quantities of the same kind measured by the same unit. In this perspective, in the fraction a/b, the numerator a represents the measure of one quantity by the unit, and the denominator b the measure of the other. From the measuring perspective, the fraction 3/4 means the multiplicative relationship between two quantities, where measure of one quantity is three fourths. This view of a fraction is also represented in the work of Venenciano et al. (2021), based on the work of Davydov (1991).

Despite the varying interpretations, most mathematics education researchers' agree that the definitional meanings of a fraction are exemplified in Table 2, which we will adopt in our study.

Table 2

Interpretations for fraction 1/4.

Interpretations	Examples for fraction 1/4
Part-whole (part-part)	1 of 4 equal parts; 1 girl for every 4 boys.
Quotient	1 divided by 4.
Ratio	1 of something compared to 4 of something else, in a multiplicative sense.
Operator	1/4 of something that is the unit, of a quantity.
Measure	The length of 1/4 of a unit in a number line, which can be iterated *n* times to obtain the fraction *n*/4.
Measuring	The measure of one quantity is 1/4 of the measure of another.

Reasonableness

Reasonableness is prompted by the plausible reasoning generated by flexibility (Powell & Ali, 2018). Reasonability has to do with the recognition of equivalent fractions, with understandings about the correct positioning of a fraction in the numerical line, with the calculation of arithmetic operations - when the fractional number is presented in the mixed, proper, improper form, and denominators are equal or not -, and with the comparison of fractions by their measurements, for example. It is not an isolated concept; on the contrary, it is drawn amid the breadth and depth of the versatile construction of the different interpretations of the fraction. Therefore, there is no boundary line between flexibility and reasonableness precisely because their senses inform each other and also facilitate the apprehension of the concept of magnitude.

Magnitude

The size of a quantity—something that can be increased or decreased—is its magnitude (Euler,1765/1822; De Morgan, 1836/2010)). It is represented by a number that results from measuring, a proportional comparison of one quantity with another of the same kind considered the unit of measurement (Powell, 2019). The measurement of the quantity is its magnitude. An example of a quantity is length, and measuring it determines its magnitude.

Nonsymbolic and symbolic conceptual, nonsymbolic and symbolic procedural development of fractions

The flexibility, reasonableness, and sense of magnitude of fractions occur not only from the conceptual and procedural perspectives, but also through nonsymbolic and symbolic knowledge. Regarding these latter ideas, Siegler et al. (2012) state that to understand fractions, we must consider two crucial differences: (1) the distinction between conceptual and procedural knowledge and (2) nonsymbolic and symbolic knowledge. Conceptual knowledge - either nonsymbolic or symbolic - has to do with understanding the properties of fractions, that is, their magnitudes, principles, and notations (Table 3). This seems to be a sensitive point in the teaching and learning of this mathematical content. For example, eighth-graders of the U.S. elementary school do not order the proper fractions correctly (Siegler et al., 2012). Others apply properties of natural numbers to any numbers.

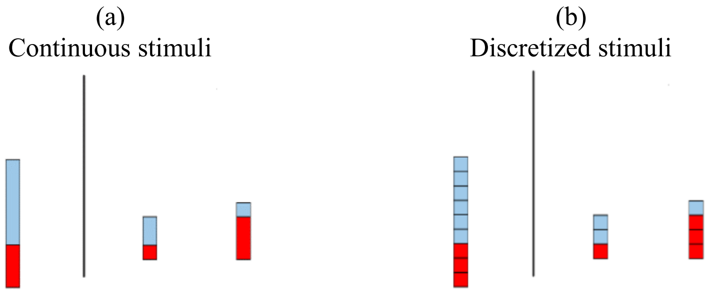
Natural and fractional numbers have common and noncommon properties. Researchers (e.g., Siegler et al., 2012; Powell, 2019b) report that students seem to apply properties of the natural numbers indiscriminately to fractional numbers. In natural numbers, the more digits, the greater the number (e.g., 93>5). For fractions, this property does not hold true(e.g., $\frac{93}{100} < \frac{5}{4}$). Each natural number has a single immediate predecessor (e.g., 3 is the immediate predecessor of 4). Due to their density, between any two fractions, there are infinitely many other fractions (e.g., $\frac{3}{5}$ between $\frac{4}{5}$ there are these fractions: $\frac{7}{10}$, $\frac{71}{100}$, $\frac{701}{1000}$, $\frac{7001}{10000}$, ….). With regard to operations, the multiplication of two natural numbers always yields a product larger than each of the two factors. However, the same is not always true of multiplying fractions (e.g., $3 \text{ x } \frac{4}{5} = \frac{4}{5} + \frac{4}{5} + \frac{4}{5} = \frac{12}{5}$, $\frac{4}{5} < \frac{12}{5}$) does not always occur with the multiplication of fractions (e.g., $\frac{4}{5} \text{x} \frac{1}{2} = \frac{4}{10}$, while $\frac{4}{5} > \frac{4}{10}$ and $\frac{1}{2} > \frac{4}{10}$). Finally, the symbolic representation of the magnitude of a natural number is unique (e.g., a set with 3 elements is symbolised only with the numeral 3), while for fractions, there are a infinitely many symbolic representations for the same magnitude (e.g., $\frac{4}{5} = \frac{8}{10} = \frac{12}{15} = \cdots$).

On the procedural side, either nonsymbolic or symbolic knowledge primarily involves fluency in the calculation of fractions with the four arithmetic operations and definitions (Table 3). The focus is much more on “how” rather than “what.” In this case, researchers report that students have difficulties to know when and why they should keep the denominator and add the numerators and when they should multiply numerators and denominators. One origin of those difficulties lies in the exclusive view of fractions as parts of a whole (Siegler et al., 2012; Sophian, 2007) and inattention to the unit in question (Cramer et al., 2002). In these contexts, students struggle to conceive improper fractions since, for example, the fraction $\frac{5}{3}$ would indicate five parts of an object divided into three equal parts.

Conceptual or procedural nonsymbolic knowledge demands understanding with concrete stimuli (Table 3). For example, recognising the greater or lesser proportion of objects in one and the other set, and counting elements or measuring parts of an equally divided circle (half a circle). In this regard, it is worth noting that Boyer et al. (2008) told a story to 6-year-olds that had as its essence the importance of maintaining the proportion between quantities sometimes shown in a continuous form (Figure 3a), sometimes in a discrete form (Figure 3b). They found that children’s proportional reasoning skills vary depending on the structure of the representations that are given. Children respond correctly to continuous stimuli but not discretized ones. When subjected to discretized stimuli, they count the quantity of coloured parts of the start bar (on the left side) and opt for the right sidebar that contains the same amount of coloured parts of the start bar, rather than associating the ratio of the start bar to the bar to be chosen.

Figure 3

Example of continuous stimuli and discretized stimuli for proportional reasoning (Boyer et al., 2008, p.19, part of the figure; adapted)



Conceptual or procedural symbolic knowledge concerns numerical representations and notations of fractions (Table 3). This is the case, for example, of the magnitude comparison between 2/3 and 3/5. Moreover, in symbolic form, errors occur with the sum of fractions (e.g., $\frac{2}{3}+\frac{1}{2}=\frac{3}{5}$) and in the multiplication of fractions (e.g., $\frac{1}{5}\ .\frac{2}{5}=\frac{3}{5}$). These errors reflect challenges with both symbolic and conceptual understanding.

Table 3

Examples of knowledge of nonsymbolic conceptual, symbolic conceptual, nonsymbolic procedural and symbolic procedural fractions.

Knowledge	**Conceptual**	**Procedural**
Nonsymbolic	Magnitude recognition. 	Recognising $\frac{1}{4}$ a circle.
Symbolic	Types of fractions. $\frac{1}{7},\frac{5}{6}$ e $\frac{4}{3}$	Operation $\frac{2}{3}+\frac{1}{2}$.

Given our review of relavant literature, we argue that the teaching of fractions should take into account the four types of knowledge: nonsymbolic conceptual, symbolic conceptual, nonsymbolic procedural, and symbolic procedural. Furthermore, teaching should attend to developing students fraction sense, including flexibility, reasonableness, and sense of magnitude. These knowledge categories and elements of fraction sense will guide our analysis of textbooks that we will describe in the following sections.

METHOD

The investigation

The transnational textbook analysis is a relatively new field of research (Watanabe, Lo, & Son, 2017). The existing literature recommends that such investigations be based on macroanalysis and microanalysis. As we mentioned in the Introduction, our macroanalysis focuses on the unfolding of fraction content regarding flexibility, reasonableness, and sense of magnitude. Our microanalysis concerns the conceptual, procedural, nonsymbolic and symbolic understandings of fractions in textbooks for each grade of elementary education (Figure 1). We chose these variables since the assessments (see Table 1) of the mathematics performance results of Brazilian, U.S., and Japanese students in the last three editions of PISA (2012, 2015, and 2018) lack qualitative descriptions that reveal reasons for the discrepant numerical results.

The sample

For our study, we chose three elementary education textbook series, one each from Brazil, the United States, and Japan. We selected the Brazilian textbook, entitled *A Conquista da Matemática* [*The Conquest of Mathematics*][3], edited published by FTD, for two reasons. First, it was approved in the most recent edition of the 2019 National Book and Textbook Programme (PNLD-2019) of the Brazilian Federal Government. Second, this textbook series is one of the most adopted in schools. This series contains nine textbooks. The first five are intended for the elementary-school years (grades 1 to 5) and the middle-school years (grades 6-8). The formal study of fractions is in the 4th to 8th textbooks of the series.

[3] Giovanni Jr., J. R. (2018). *A Conquista da Matemática*. 4th and 5th grades. São Paulo: FTD. e Giovanni Jr, J. R. & Castrucci, B. (2018). *A Conquista da Matemática*. 6th to 8thgrades . São Paulo: FTD.

For the U.S. textbook series, we chose *Eureka Math*[4] since it is popularity among teachers. According to Ed Reports Org and Rand Corporation, 57% of American teachers use this series in the elementary-school years (grades 1 to 5) and 47% in the middle-school years (grades 6-8). The series is available on the Internet and is free for *download* and non-commercial use. *Eureka Math* consists of fourteen textbooks, each subdivided from 5 to 8 modules. In the first to seventh textbooks, we analyzed the content of fractions instruction.

For the Japanese textbook series, we chose is *Study with Your Friends: Mathematics for Elementary School*. It was officially approved by the Ministry of Education, Culture, Sport, Science, and Technology of Japan. The textbooks were edited by Gakko Tosho Co., Tokyo's second-largest publisher, and schools in Japan and abroad (in the English language version) adhered to those mathematics textbooks[5]. The series consists of eleven textbooks, one for the first year and two textbooks for each grade from the second to the sixth. The series also contains a twelfth volume that concludes the first half of basic education by reviewing and relating the main ideas studied in isolation from the first to the sixth year, highlighting important points of the contents, stimulating thinking through problems and challenging students about aspects that they have not yet studied.

Structure and coding scheme

We developed a structure and coding scheme to describe and analyse the textbooks' the flexibility, reasonableness, and sense of magnitude, as well as nonsymbolic and symbolic developments of conceptual and procedural aspects of fractions (Table 4).

Based on the literature, we analyzed flexibility from different interpretations of fractions: part-whole, quotient, ratio, operator, measurement as the iteration of unit fractions and measuring as the multiplicative comparison of two quantities. We analyzed reasonableness in the form of the equivalence of fractions, positioning fractional numbers on the numerical line, and arithmetic operations with proper and improper fractions and mixed numbers. The sense of magnitude considered discretized and continuous approaches that represent the size of a fraction. Finally, we examined the conceptual and

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[4] https://greatminds.org/math;https://www.bcsdk12.org/apps/pages/index.jsp?uREC_ID=1050328&type=d&pREC_ID=1348636

[5] The Japan Times; https://www.pref.kanagawa.jp/docs/v3p/cnt/f6670/documents/366367.pdf.

procedural symbolic or nonsymbolic aspects both in the initial approaches of each interpretation and in the tasks proposed in the textbooks.

Table 4

Structure and coding scheme for qualitative data analysis.

Item	Qualitative analysis	Description	Coding
1	macroanalysis	flexibility	PT – Part-whole QT – quotient RZ – ratio OP – operator MD – measure ME - measurement
2	macroanalysis	Reasonableness	EQU – equivalence (P-proper fractions, I-improper fractions, M- mixed numbers) RNU - numerical line (“x” there was; “-”there was not”) OPE - operation (P-proper fractions, I-improper fractions, M- mixed numbers)
3	macroanalysis	sense of magnitude	DIS - association with a number representing its size in discrete form. CON - association with a number representing its size in a continous form.
4	microanalysis	conceptual, procedural, symbolic, nonsymbolic development	CNS - nonsymbolic conceptual CSI - symbolic conceptual PNS - nonsymbolic procedural PSI - symbolic procedural

In more detail, in item 1 of Table 4, flexibility was analysed by the diversity of interpretations throughout the textbook series. In this regard, we did not take into account the depth and extent to which each interpretation was developed. However, we did consider connections between different interpretations. To measure item 1, we appraised how the textbook series fostered the understanding of the same situation, or in related situations, by two or more interpretations of fraction. In item 2, we investigated the equivalence of fractions and arithmetic operations by the diversity of the study with the same or different denominators being performed with proper or improper fractions or mixed numbers. We judged reasonableness by how the textbook series encouraged positioning of fractions on the number line. In item 3, we examined how the sense of magnitude was treated by how the textbook series associated the measure with a number's size, both in a discretized and continuous format. In item 4, the conceptual and procedural development of the fractions was inferred from the unfolding of the discourses and activities offered by the textbook series, discriminating them regarding the nonsymbolic and symbolic approach options. In other words, this question was concentrated on the paths chosen by the authors of the textbooks regarding the development of the conceptual or procedural side and the preferences of concrete stimuli or mathematical notations of fractions. In general, the codifications for item 4 were attributed to the tendency or greater emphasis the authors offered on the conceptual or procedural and nonsymbolic or symbolic side. Thus, at some point, the textbooks may have given greater emphasis on one or the other development and one or the other stimulus, despite having used both. In this case, we brought to the fore the greatest emphasis given in the textbooks, whose results and analyses are as follows.

RESULTS AND ANALYSES

General context of textbooks

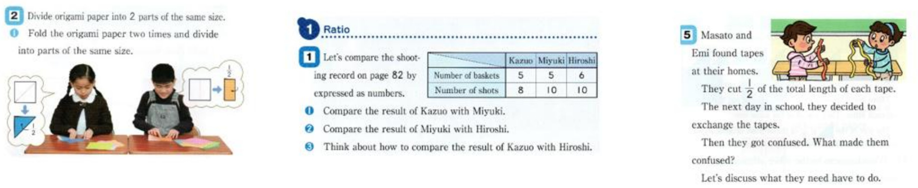
Before presenting the analyses and results, we provide an overview of the approaches in each series. The Brazilian textbooks begin each chapter with a comic strip or an illustrated situation that intends to introduce students to the first notions of a new topic, and it supports subsequent initial activities. These notions are presented without referencing the experience or interactions among the students. In nearly each chapter, the development of the content is supported by questions and activities to "fill gaps" in procedural knowledge. There are more pages devoted to drill exercises than to the conceptual development of ideas.

The U.S. textbooks starts each new topic with some appeal to students' reasoning, either through a written dialogue or through a model to be replicated later in drills. The textbook authors suggest to teachers material to reinforce the ideas to be developed. The textbooks contains many pages of drill exercises, almost always based on solved models. After the drill exercises, the textbooks offer problems related to the topic. The U.S. textbooks seem to prescribe teachers' actions with many attitudinal and procedural details.

The Japanese textbooks, in general, began new content with some conceptual development on different fronts and representations (Figure 4a). They guided teachers on arguments and interactions among students whenever possible. These supportive guides are followed by discussions of different views or interpretations of the objects under study.

Figure 4

(a) different views on the same object; (b) invitation to compare and expand reasoning; (c) invitation to interact with fractions amid a measurement activity (Elementary School - (a) textbook2, p. 84; (b) textbook 2, p. 86; (c) textbook 5, p. 84.)



The Japanese textbooks do not contain a large number of drill exercises and seem to leave it up to teachers to develop activities according to their intuitions, creativities, and experiences. Those textbooks often pose questions that led students to reflect on some activity beyond the limits of what they did at that time. Furthermore, the textbooks often invite students to shared work and compare (Figure 4b) their reasoning and solutions, including reinforcing this idea with figures of students performing the tasks together (Figure 4c). The use of pictorial aspects is evident in Japanese textbooks.

Breakdowns of the four items under study

Flexibility

In a macroanalysis, the result of the study with mental resourcefulness through the interpretations of fractions in the three textbook series is summarised in Table 5.

Table 5

Interpretations of fractions treated in textbooks.

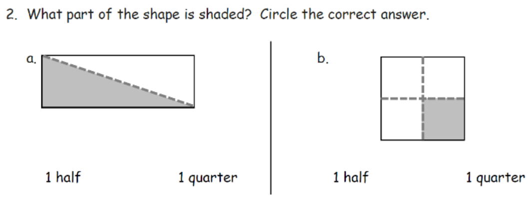
Book	Brazil	United States	Japan
1	-	PT	-
2	-	-	PT/MD
3	-	PT	PT/MD
4	PT	PT	PT/MD
5	PT/QT/OP/MD	PT	PT/QT/RZ/OP/MD
6	PT/QT/RZ/OP	PT/RZ	PT/QT/RZ/OP/MD
7	PT	PT/QT/RZ/OP	-
8	PT/OP	-	-

The Brazilian textbook series focused on the part-whole interpretation of fractions. There was only one incidence of the use of measurement interpretation to compare lengths of bars, strings, and toothpicks to introduce the idea of fraction in textbook 5. The other four interpretations - quotient, ratio, operator and measure - were offered primarily in drill exercises and word problems.

The first textbook of the U.S. series used the idea of equality-inequality and half and quarters of diversified geometric figures (Figure 5) based on a part-whole interpretation.

Figure 5

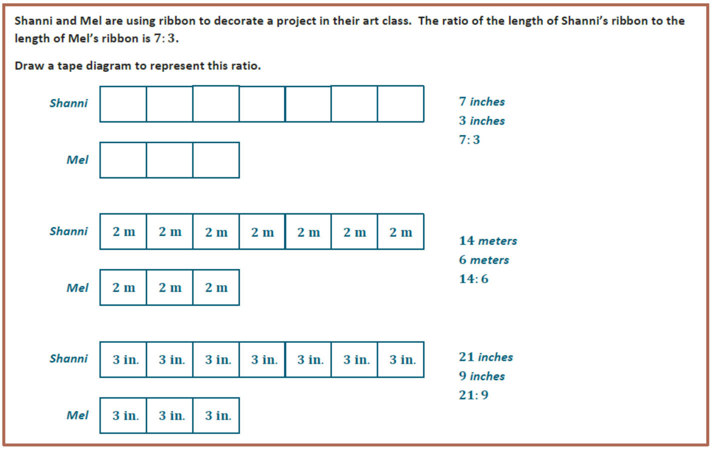
Recognition of halves and quarters in the U.S. textbook 1 (Eureka, textbook 1, p. 121)



The U.S.textbook series work with the part-whole interpretation only at the end of textbook 3, next to the beginning of positioning fractions on the numerical line. In general, the U.S. textbook emphasized the part-whole interpretation with rectangles and other geometric figures and placing fractions on the numerical line. Textbook 6 initiated work with ratios consistent with the part-whole interpretation presented in the previous textbooks (Figure 6).

Figure 6

Ratio in the U.S. textbook (Eureka, textbook 6, p. 30)



The Japanese textbooks series worked intensively on part-whole and measurement interpretations in textbooks 2 to 4 (Figure 3c), inserting the others in textbooks 5 and 6. This series contained more pages concerning the conceptual development of the notions under study than exercises and problems.

About flexibility, as Table 2 shows, all textbook series incorporated the first five interpretations of fraction (part-whole), not addressing measurement. A more accurate analysis, however, reveals different ways of implementing those interpretations. The Japanese textbooks are closer to what mathematics educators, cognitive psychologists and neuroscientists recommend for the teaching of fractions, which is a diversified treatment of the various interpretations of fractions, despite not making use of measurement. On the contrary, the Brazilian and U.S. textbooks approached the part-whole almost exclusively, employing other interpretations sparingly and in isolation. This approach may impair a broad comprehension of fractions , considering that the part-whole does not seem to promote the necessary breath and depth understanding to fertilise other spheres of mathematics.

Reasonableness

In our macroanalysis, we also considered reasonableness in the three textebook series. The results are shown in Table 6.

Table 6

Equivalence, numerical line and arithmetic operations with fractions.

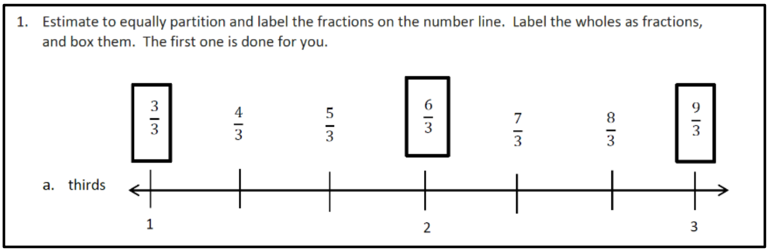
Book	Brazil			United States			Japan		
	EQU	RNU	OPE	EQU	RNU	OPE	EQU	RNU	OPE
2	-	-	-	-	-	-	-	-	-
3	-	-	-	P/I	x	-	-	x	P[a]
4	-	x	-	P/I/M	x	P/I/M	P/I/M	x	P/I/M
5	P/I/M	x	-	P/I/M	x	P/I/M	P/I/M	x	P/I/M
6	P/I/M	-	P/I/M	P/I/M	x	P/I/M	P/I/M	x	P/I/M
7	P/I	x	P/I/M	-	-	-	-	-	-
8	P/I/M	x	P/I/M	-	-	-	-	-	-

[a] only arithmetic operations with equal denominators.

The Brazilian textbook series introduced the study of fraction equivalence through comparisons of part-whole figures, followed by a definition. The number line was used to position the fractions to evaluate their relative magnitudes. There was an approach to proper, improper and mixed fractions, mainly the proper ones. The Brazilian textbook series began operations with fractions in the sixth textbook. The U.S. textbooks followed the same line of treatment as the Japanese, except for the option of positioning fractions on the numerical line disconnected from the understanding of measurement (Figure 7).

Figure 7

Positioning of fractions on the numerical line (Eureka, textbook 3, p. 197)



The Japanese textbooks present and with the number line from the beginning to the end of the fractions approach from the perspective of the measure as an iteration of the unitary fraction. This perspective is given to support the part-whole interpretation. Proper and improper fraction and mixed numbers are treated in textbook 4 to 6 along with the basic arithmetic operations.

From these results, we inferred that the three series worked reasonableness, differing only in the *modus operandi* of the uses of equivalence of fractions, the number line, and basic arithmetic operations.

Sense of magnitude

Table 7 summarises the macroanalysis on the sense of magnitude expressed in the mathematical comparisons of quantities with the aid of a number representing its size, either in a continuous (CON) or discrete (DIS) form.

Table 7

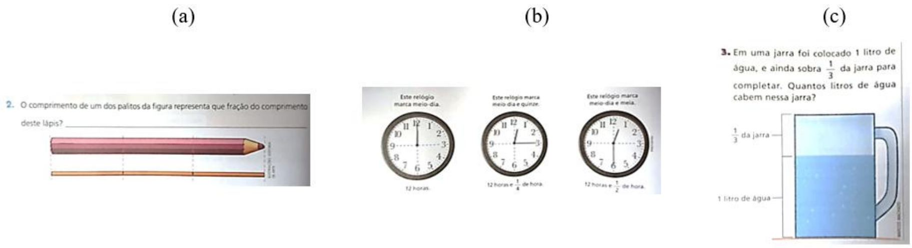
Sense of magnitude - measure unit in textbooks.

Book	Brazil	United States	Japan
2	-	-	CON
3	-	DIS	CON
4	DIS	DIS	CON
5	DIS	DIS	DIS-CON
6	DIS	DIS	DIS-CON
7	DIS	DIS	-
8	DIS	-	-

The Brazilian textbook series primarily presented counting situations in all textbooks, except for rare circumstances, as shown in Figure 8. Due to this scarcity, we consider that the series presents magnitude mainly in discrete cases.

Figure 8

Sense of continuous magnitude in Brazilian textbooks (A Conquista da Matemática: (a) textbook 5, p. 148; (b) textbook 6, p. 134; (c) textbook 7, p. 128)

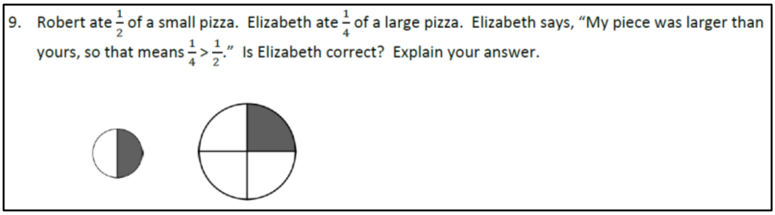


The U.S. textbook 3 began its treatment of magnitude with fractions represented in circles and bars (Figure 9a) in a problem situation. The extensive explanation treate discrete cases, with only one case with liquids and, for this reason, we considered it to have been primarily a discrete representation (Figure 9b). Size representation of fractions was rare in the U.S. textbook series, and when it occurred, it was discrete (Figure 9c).

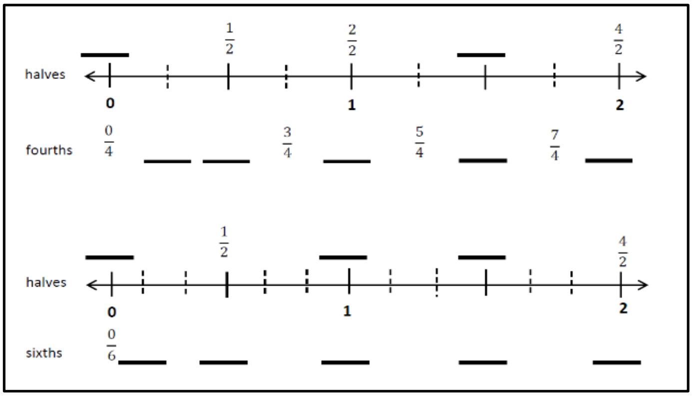
Figure 9

(a) Sense of magnitude through a problem; (b) Positioning of fractions on the numerical line; (c) Sense of magnitude in the U.S. textbook (Eureka textbook 3 - (a) p. 128; (b) p. 249; (c) p. 334) - (continuous)

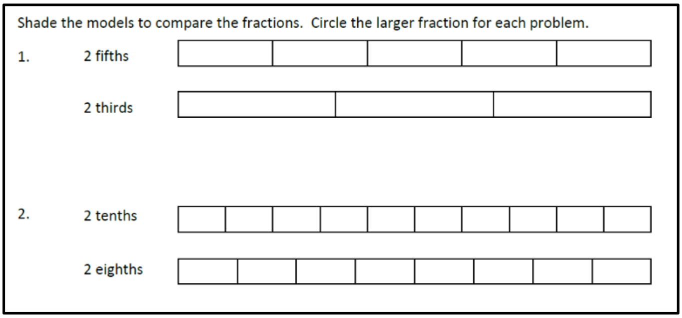
(a)



(b)



(c)



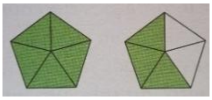
It is worth mentioning that the Brazilian and U.S. textbook series proposed exercises without attention to determining the unit. The Brazilian textbook 5 proposed the writing of the fraction that represents the coloured part of the figure (Figure 10a) without determining the unit. In this case, the response could vary at least between 8/5 or 8/10, depending on what was considered as a unit. textbook 3, in turn, did not consider the possibility that the magnitude of 1/4 could be greater than, less than, or equal to 1/2, depending on the unit of the fractions (Figure 10b). This fact occurred several times throughout the textbook series.

Figure 10

Fractions without determination of unit - (a) Brazilian; (b) U.S. (a- A Conquista da Matemática, textbook 5, p. 154; b- Eureka, textbook 3, p. 118)

(a)

Write the fraction that represents the part coloured in green in each case.



(b)

2. Circle *less than* or *greater than*. Whisper the complete sentence.

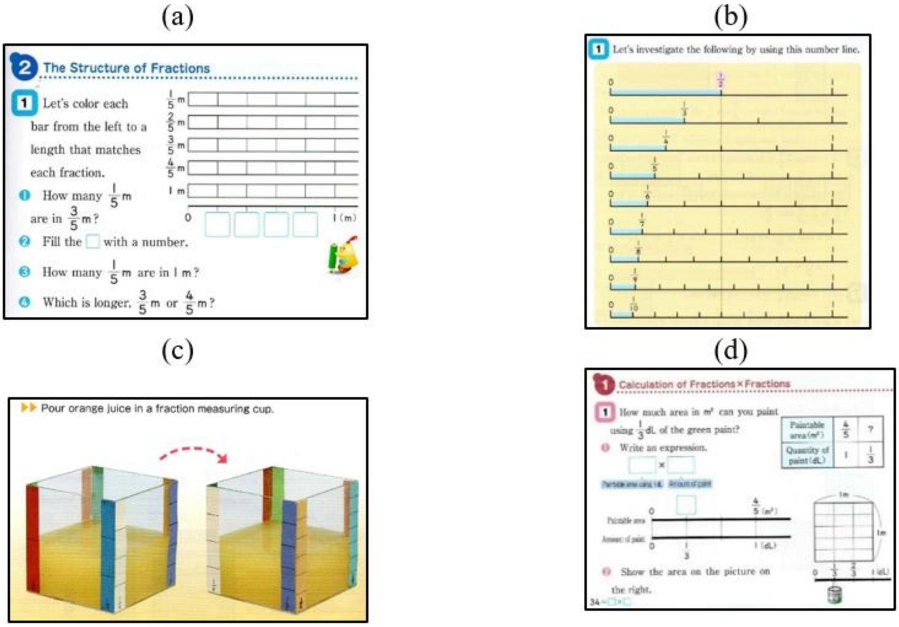
a. $\frac{1}{2}$ is less than / greater than $\frac{1}{4}$

b. $\frac{1}{6}$ is less than / greater than $\frac{1}{2}$

The second textbook of the Japanese series introduces the idea of fraction magnitude with an invitation to reflect on the comparison of different lengths of two tapes by measure $\frac{1}{2}$ (Figure 4c). This introduction may lead to a discussion about the consideration of different fractional units. Subsequent textbook in the series expanded and deepened this idea, as the theme gained new ingredients: In the third Japanese textbook, fractions of the same denominator were compared with(Figure 11a) continuous models discretized. The fourth textbook expanded this comparison with different denominators (Figure 11b); while the fifth textbook compared fractions using capacity measurements (Figure 11c) with discrete and continuous models. Finally, the sixth textbook presented discrete and continuous comparisons with fraction arithmetic operations (Figure 11d).

Picture 11

(a) comparison of fractions with the same denominator; (b) comparison of fractions with different denominators; (c) comparison of fractions with capacity measures; (d) comparison of fractions with arithmetic operations (Elementary School - (a) textbook 3, p. 93; (b) textbook 4, p. 78; (c) textbook 2, p. 118; (d) textbook 6, p. 34)



Conceptual, procedural, symbolic, nonsymbolic development

Table 8 summarises the microanalysis on the conceptual, procedural, symbolic or nonsymbolic development found in Brazilian, Japanese and U.S. textbooks, for each year of basic education.

Table 8

Conceptual, procedural, nonsymbolic, symbolic development.

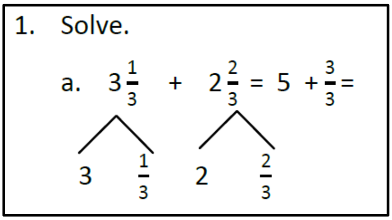
Book	Brazil	United States	Japan
1	-	PNS	-
2	-	-	CNS/PNS
3	-	CSI/PNS/PSI	CNS/CSI/PNS/PSI
4	CNS/PNS/PSI	PSI	CNS/CSI/PNS/PSI
5	PNS/PSI	CSI/PSI	CNS/CSI/PNS/PSI
6	CNS/PNS/PSI	CSI/PSI	CNS/CSI/PNS/PSI
7	PNS/PSI	-	-
8	PNS/PSI	-	-

The Brazilian series began each chapter with some story or situation to contextualise the content. The first notions of fractions were developed with folds in textbook 4. Soon after, it favoured more procedural than conceptual approaches, in the sense that we defined earlier. In general, the brief introduction of the content was followed by some model of solved exercise accompanied by problems proposed. Nonsymbolic procedures were presented - and not developed or stimulated - and were almost always associated with symbolic procedures.

The U.S. textbooks, despite developing some conceptual aspects, whether symbolic or not, emphasised procedural developments followed by a large number of exercises with a model that preceded them. For example, the sum of fractions with the same denominators was studied from a model of decomposition of the parts into mixed numbers (Figure 12). Conceptual stimulus in the U.S. textbooks was suggested at the beginning of each new topic as a strategy given to teachers.

Picture 12

Decomposition of fractions for the sum operation in the U.S. textbook
(Eureka, textbook4, p. 421)



The Japanese series has always associated conceptual and procedural aspects. In the beginning, only nonsymbolic aspects (book 2) were developed, expanding to symbolic situations from textbook3 to 6.

In general, all textbooks that we reveiwed endeavoured to work on conceptual, procedural, symbolic and nonsymbolic aspects of fraction knowledge. Between the textbook series, the emphasis was on the differential. The Brazilian and U.S. textbooks focused on procedural to the detriment of conceptual aspects. This practice contrasts with the construction of concepts that may influence future mathematics, as revealed by Siegler et al. (2012). Focus on procedural fraction knowledge without attention to conceptual understanding is likely to influence future mathematics learning negatively. Using definitions prematurely, regardless of the actualisation of the main notions may compromise the understanding of what fractions are and, consequently, justify the fact that some students consider 2/5 as a result of 1/2 + 1/3, for example. The number of pages intended for exercises and problems to the detriment of the work with conceptual aspects seems to indicate the authors' preference for one or the other approach.

CONCLUSIONS

Our investigation aimed to understand in qualitative macroanalysis and microanalysis how three series of textbooks from Brazil, the United States, and Japan deal with the flexibility, reasonableness, and sense of magnitude of fractions, and with conceptual and procedural development, from symbolic and nonsymbolic perspectives, supported mainly by the following scientific findings: (1) researchers recognise the relevance of fraction study by shaping the future performance of student mathematics, depending on the approach chosen by teachers; (2) teachers rely primarily on textbooks to plan classes; and (3) PISA presents discrepant performances in mathematics of Brazilian, U.S., and Japanese students in the last three editions.

The macroanalysis examined the dimensions of flexibility, reasonableness, and sense of magnitude. Flexibility was practiced in the three textbook series, yet with different emphasis. The Japanese textbooks chose to base the study on the measure (as an iteration of unit fractions) and part-whole interpretations, while the Brazilian and U.S. textbooks based their approaches on the part-whole interpretation. Reasonableness, likewise, was covered in the three series regarding the use of equivalence of proper and improper fractions and mixed numbers, positioning fractions on the numerical line, and basic arithmetic operations. Again, inequality focused on the intrinsic *modus operandi* of the approaches. In a divergent way, the sense of magnitude was initially treated with continuous models in the first three books of the Japanese textbook series, progressing to discrete models. This fact is in line with the emphasis of the study of fractions as a measure, although it was not also as a comparison of different quantities (as measuring). The Brazilian and U.S. textbooks emphasised discrete models and rarely continuous models. Furthemore, these two series did not always focus students' attention on the unit as the basis to understand what they should do.

The microanalysis examined the conceptual, procedural, symbolic and nonsymbolic development of fractions. The Brazilian and U.S. textbooks highlighted the procedural skills to the detriment of conceptual understanding. The Japanese textybook series invested most of it pages to conceptual knowledge of fractions.

In summary, the situation presented in this investigation confirms the claims of the scientific community about the study of fractions. The Brazilian and U.S. contexts focused on part-whole interpretation and contained procedure-based approach, while the Japanese textbooks highlighted the interpretation of measure as an iteration of unit fractions, part-whole, and conceptual understanding, thus being closer to what is recommended by mathematics education researchers.

None of the three textbook series introduced the measuring perspective of fraction knowledge (Powell, 2019a). The lack of this fundamental perspective deserves empirical further research to determine how it shapes students' fraction understanding along the dimensions of flexibility, reasonableness, and sense of magnitude. On the question of magnitude, empirical evidence suggests that students as young as seven years old do develop this dimension when learning about fractions from a measuring perpective (Powell, 2019b). Implementing this perspective in textbook series may improve the discrepent panorama presented in PISA.

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AUTHORSHIP CONTRIBUTION STATEMENT

The authors MAVFS and ABP jointly discussed the entire scope of the work, including results, analyses, and conclusions. The first author MAVFS collected and analysed the data in collaboration with the second author ABP.

DATA AVAILABILITY STATEMENT

The data supporting the results of this study are available from the authors upon reasonable request at the discretion of the authors.

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