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A framework for a comparative study of pre-service elementary teachers' knowledge of rational numbers

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Abstract. This paper presents a framework of a PhD research of the first author about a comparative study of pre-service elementary teachers' knowledge of rational numbers between Indonesia and Denmark. To obtain the data, the authors design a series of hypothetical teacher tasks (HTTs), inspired by a paper of Durand-Guerrier, Winsløw, and Yoshida (2010). Subjects in this research are pre-service elementary teachers from a selection of different University Colleges in Denmark and from the elementary school teacher education study program, Riau University, in Indonesia. The praxeological reference models and the levels of didactic codetermination are used as tools to analyse the result.

Resumen. Este artículo presenta un marco general de un proyecto de investigación doctoral. Se trata de un estudio comparativo sobre el conocimiento acerca de los números racionales en la formación de maestros de enseñanza primaria en Indonesia y Dinamarca. Para obtener los datos los autores han construido una serie de «tareas hipotéticas de enseñanza» (HTTs), inspiradas de un artículo de Durand-Guerrier, Winsløw y Yoshida (2010). Participan en nuestra investigación estudiantes de una selección de diferentes colegios universitarios en Dinamarca y del programa de formación de maestros de primaria de la Universidad de Riau en Indonesia. El análisis utiliza los niveles de codeterminación didácticos y los modelos praxeológicos de referencia.

Résumé. Cet article présente le cadre d'un projet de recherche doctoral. Il s'agit d'une étude comparative sur les connaissances sur les nombres rationnels, chez des étudiants en formation pour enseigner à l'école primaire, en Indonésie et au Danemark. Pour obtenir les données, les auteurs construisent une série de « tâches hypothétiques d'enseignement » (HTTs), inspirés par un article de Durand-Guerrier, Winsløw et Yoshida (2010). Les sujets de notre recherche sont des étudiants d'une sélection de différents collèges universitaires au Danemark et du programme de formation d'enseignants au primaire de l'Université de Riau en Indonésie. Un modèle praxéologique de référence et les niveaux de codétermination didactique sont utilisés pour l'analyse.

1. Introduction

Comparative studies on teaching and learning rational numbers have been done by several researchers. For instance, Li (2014) compared British and Taiwanese pupils' conceptual and procedural knowledge of rational numbers, more specifically of adding fractions. Taiwanese pupils performed better than British peers because they were more successful to apply algorithms for adding fractions. British pupils had a tendency to add numerators and denominators, respectively. Similar comparisons have been done among other countries around the world. Lai and Murray (2014) compared Hong Kong Chinese and Australian pupils' understanding of decimal numbers. Even though Hong Kong Chinese pupils performed better than their Australian peers did, they had similar misconceptions about decimal numbers, for instance in comparing two decimal numbers, as pupils struggled with the concept of place value.

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Comparative studies between Western and East Asian countries have indeed become common in recent years and are often motivated by a desire to understand the background for different performance in international tests such as PISA or TIMSS. We consider specific topics, such as the arithmetic of rational numbers in order to understand, in a fine-grained way, the differences at first more coarsely observed. A main motivation to get such an understanding is to assess what factors are important in causing the observed differences. Here, teachers' knowledge has often been advanced as a key factor.

Some researchers have already conducted studies on elementary teachers' knowledge of rational numbers. Ma (1999) compared U.S. and Chinese pre-service and in-service elementary teachers' capability of solving and constructing meaningful problems involving fraction division. She found that U.S. teachers were less successful than Chinese teachers on both kinds of tasks, and most of them did not understand the rationale underlying their calculation and the meaning of division by fractions. Meanwhile, Stacey et al. (2001) investigated Australian pre-service teachers' knowledge about pupils' difficulties with decimal numbers. Their result was that pre-service teachers mainly possessed simple content knowledge about decimals. They could notice pupils' errors with comparing decimal numbers, but they could not explain why these occurred. Both studies have similar approaches to investigate pre-service and in-service elementary teachers' knowledge about rational numbers through simple tests, based on a cognitive paradigm that focuses on individual knowledge.

This research project takes a different approach, based on the *anthropological theory of the didactic* (ATD) introduced by Chevallard (1992, 2006, 2007). In this framework, knowledge is considered as institutionally situated, and it is studied through *praxeological reference models* (PRMs). A comparative study of secondary level teacher students' knowledge was conducted by Winsløw and Durand-Guerrier (2007) and Durand-Guerrier, Winsløw, and Yoshida (2010); our study adopts their notion of *hypothetical teacher tasks* (HTTs) and associated PRMs. The mathematical focus of the present study is the order structures and arithmetic of rational numbers.

Our aim in this paper is to develop a framework to study pre-service elementary teachers' shared mathematical and didactic knowledge about rational numbers. The framework will be applied to a comparative study of pre-service elementary teachers (PsETs) from Indonesia and Denmark. We would like to do this in both countries because one of the researchers comes from Indonesia and is doing his PhD program in Denmark. We hope that this research can address the gap between knowledge development by teachers at universities and their subsequent resources for teaching pupils at schools. The results of this study will also contribute to develop our knowledge about teaching didactic knowledge of rational numbers to PsETs in both countries. To clarify the goals of our research, we formulate specific research questions for the entire PhD program of the first author as follows:

1. How can the ATD function as a framework to study pre-service elementary teachers' mathematical and didactic knowledge of rational numbers?
2. In particular, how could HTTs be used to study pre-service elementary teachers' mathematical and didactic knowledge of rational numbers?
3. What similarities and differences can be identified between Danish and Indonesian pre-service elementary teachers' knowledge of rational numbers?

4. At what *levels of didactic codetermination* the origin of these differences can be identified?

In this paper, we focus on the first research question by describing mathematical and didactic praxeologies.

2. The ATD and the levels of didactic codetermination

The ATD provides a detailed model of the levels of didactic codetermination which may help explain the sources of differences in PsETs' knowledge of rational numbers, as shown in figure 1 (Artigue & Winsløw, 2010; Bosch & Gascón, 2006, 2014). In general, the levels are divided into nine categories. Some general educational studies only focus on the levels above *discipline*, while specific subject didactic studies, such as didactic of mathematics, are mostly concerned by the levels at or below the level of the discipline (Bosch & Gascón, 2006, 2014).

The first five levels of analysis cover both *mathematical organisations* (MOs) and *didactic organisations* (DOs) that can be directly observed in teaching and learning practices, as well as in tests or documents such as textbooks, curriculum, etc. The MOs are linked to the mathematical contents that teachers should teach and, thus, are supposed to be highly competent on. For instance, a teacher who gives a task to pupils such as adding and subtracting two fractions should be able to activate an MO that provides one or two techniques for solving the task. S/he should be able to explain a variety of techniques given, relate them to other tasks, and provide some justifications based on appropriate technological-theoretical discourses. The DO refers to teaching and learning praxeologies related to the MO. To design a lesson plan to teach addition and subtraction of two fractions is an example of DOs.

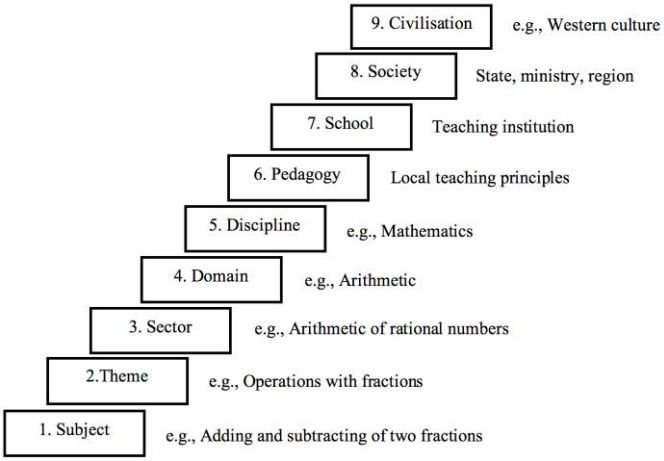


Figure 1. The level of the didactic codetermination.

Level one up to three, *subject*, *theme*, and *sector*, are the main levels we use to design HTTs to investigate PsETs' knowledge of rational numbers. Those levels are approached through praxeological reference models. The subject corresponds to a *type of tasks* (*T*) and a *technique* (τ) (see also Artigue & Winsløw, 2010; Winsløw et al., 2014). To assess teachers at the subject level, the HTTs include mathematical tasks that are designed to uncover characteristic difficulties among pupils, identified in the research literature on teaching rational numbers. As an example, the type of tasks (*T*) can be to add or subtract two fractions with different denominators. To solve such tasks, a technique (τ) is needed, such as changing each fraction into fractions with a common denominator. Meanwhile, a *technology* (θ) and a *theory* (Θ) occur

at the level of the theme and sector respectively. The explanations of the techniques are contained within a wider technology about operations with fractions (a discourse on how to calculate with). A theory behind that technology contains more or less formal definitions, rules, and proofs which justify the technology. It is developed from the arithmetic of rational numbers.

The next two levels, *domain* and *discipline*, refer to more global MOs. Arithmetic is the domain for the school praxeological organisation of addition and subtraction of fractions. Mathematics is the discipline in a given school institution.

The last four levels are *pedagogy, school, society,* and *civilisation.* The pedagogy is proper to school institutions and implemented by teachers as a professional body. Also a school has rules and regulations, for instance concerning the autonomy of teachers, and a school institution is situated with in the rest of a society, along with superior institutions such as the Ministry of Education, which has the power to regulate the school through a national curriculum, funding, and national assessment of pupils. As an example, certain systematic differences among the teacher education systems in France, Denmark, and Japan may be observed through the differences in the teacher students' performance on the mathematical tasks (Durand-Guerrier et al., 2010). Meanwhile, the civilisation may also influence the teachers' and pupils' performance on mathematics, schools' level of autonomy, etc.

The levels of didactic codetermination have been used by Artigue and Winsløw (2010) to compare and analyse studies such as PISA and TIMSS. They showed that the comparative studies could rely on a horizontal comparison (between two contexts at the same level) or on comparing certain vertical relations between different levels within each context (Figure 2). Differences between two contexts at the same level could be claimed to be caused by other higher-level differences. We apply the latter method in our study starting by comparing mathematical and didactic praxeologies, specifically mathematical and didactic techniques (τ). Then, we also investigate some factors that affect the differences among teachers' praxeologies in the two countries, through comparing factors at higher levels, such as the curriculum and textbooks used by schools and by teacher education institutions.

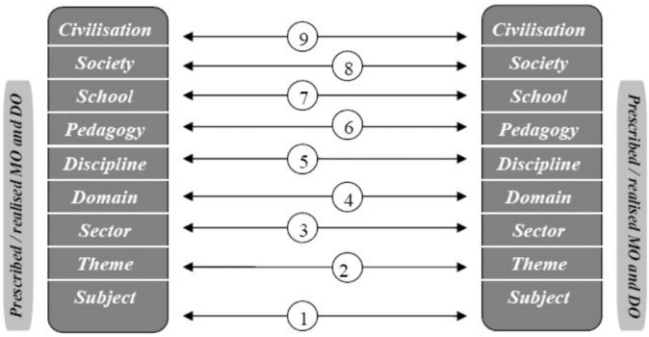


Figure 2. Possible levels of comparison of the didactic codetermination (Artigue & Winsløw, 2010).

3. Hypothetical teacher tasks (HTTs)

The notion of hypothetical teacher tasks (HTTs) first appeared in the study of pre-service lower secondary mathematics teachers' knowledge of teaching similarity and proportion, and the multiplication of two negative numbers (Durand-Guerrier et al., 2010; Winsløw & Durand-Guerrier, 2007). The HTTs are constructed so as to introduce a teaching situation which could conceivably appear in a classroom setting, and where teachers would have to invest both

mathematical and didactic knowledge, in order to act properly in the situation. The HTTs thus initially enable us to study pre-service teachers' mathematical and didactic knowledge. Each HTT consists of a mathematical and a didactic task. The mathematical task is a standard task given to pupils at schools, but the task is set for teachers within a situation where pupils struggle to find a correct answer. Therefore, the teachers have to provide various mathematical techniques. Meanwhile, the didactic task is a task for teachers to handle in a *didactic situation* (Brousseau, 1997), and they must suggest some didactic techniques to further pupils' learning. The didactic tasks are strongly related to the mathematical tasks.

The HTTs developed in this project aim to investigate the knowledge of PsETs about rational numbers, and the teaching of such knowledge. We first study MOs of rational numbers from punctual to global organisations. The *punctual organisations* contain just one type of tasks such as to find a fraction equivalent to $\frac{3}{4}$ (HTT 1). Various types of tasks that employ a common technology (such as the equivalence of two equivalent fractions) are unified as *local organisations* of specific themes. Several technologies may be justified by a theory (e.g. a theory of order structures of rational numbers) and a family of *praxis* sharing one theory is known as a *regional organisation*. Some regional organisations may be further unified in a *global organisation* of specific domains (e.g. rational numbers). In fact, MOs are structured and stratified in mathematical domains or knowledge to be learnt, while in teaching practice, they are often established only at the punctual or local level (Durand-Guerrier et al., 2010).

In order to study PsETs' knowledge on rational numbers, we designed five HTTs (Figure 3). The first three tasks, HTT 1, HTT 2, and HTT 3 are all linked to the order structures of rational numbers. Techniques related to the equivalence of rational numbers can be applied to solve the type of tasks of HTT 2, HTT 3, and also HTT 4. Meanwhile, HTT 4 and HTT 5 concern the arithmetic of rational numbers. In HTT 5 the main point is that multiplication of rational numbers cannot, in general, be explained as "repeated addition".

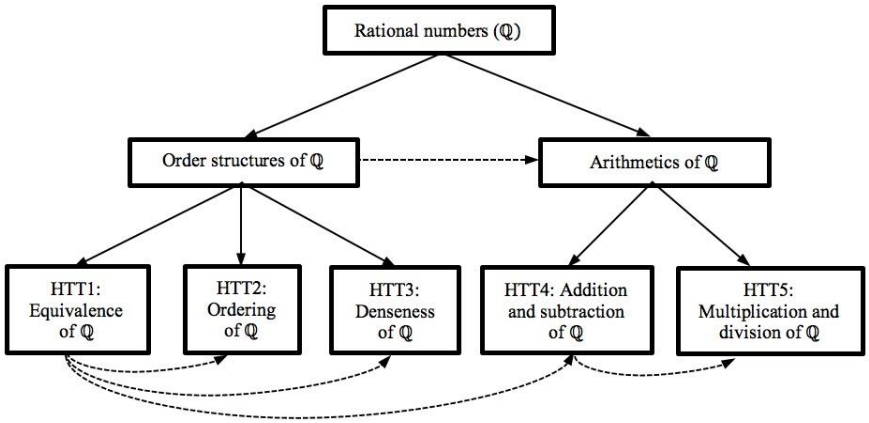


Figure 3. A mathematical organisation of rational numbers.

4. Praxeologies reference models (PRMs)

To study PsETs' knowledge in a systematic way, we have constructed a reference model for each HTT, specifying the corresponding mathematical and didactic praxeologies. We focus mostly on the reference model for practical blocks i.e. types of tasks and techniques. In this paper, we only describe the detail models of HTT 1, and we assume readers can figure out how it is done to the other four HTTs.

HTT 1 is about equivalent fractions, and the problem given to pairs of PsETs is presented as follows:

> You ask fourth grade pupils to find fractions which are equal to $\frac{3}{4}$.
> *A pupil claims that* $\frac{3}{4} = \frac{8}{9}$ *because if you add 5 to both the top and the bottom of a fraction, the new fraction must be equal to the original.*
> a. What do you think about this answer? Please explain! (*to be solved individually within 3 minutes*).
> b. What would you do as a teacher to help the pupils from this case to understand the concept of equal fractions better? (*to be discussed and solved in pair within 5 minutes*).

Figure 4. HTT 1 about equivalence of fractions.

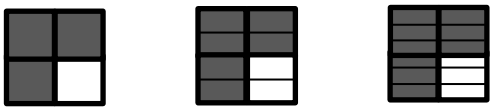
A priori analysis of HTT 1 consists of mathematical and didactic praxeologies. A mathematical task given to pupils can be described on the following type:

T_1: given a positive fraction, $\frac{a}{b}$, determine other fractions that are equal to it.

We can describe some possible mathematical techniques to solve the tasks of type T_1:

τ_{11}: compute correct equal fractions of $\frac{a}{b}$ by multiplying/dividing each numerator and denominator by the same positive integer.

τ_{12}: first represent $\frac{a}{b}$ in a model such as a rectangle or a circle diagram, then draw another model for $\frac{a}{b}$ by dividing it into 2, 3, or more parts. Finally, it can be shown that both models generate equal fractions, e.g. as follows:



$\frac{3}{4}$ $\quad$ $\frac{3 \cdot 2}{4 \cdot 2} = \frac{6}{8}$ $\quad$ $\frac{3 \cdot 3}{4 \cdot 3} = \frac{9}{12}$

τ_{13}: first change $\frac{a}{b}$ into a decimal, then find another fraction that is equal to that decimal.

There are three possible techniques to change fractions into decimals: a division algorithm, specific fractions memorised as decimals (e.g. $\frac{1}{4} = 0.25$), and finally using calculators, computers, or other electronic devices to divide a by b. Meanwhile, we also predict that some teachers probably use addition and subtraction of numerator and denominator by the same number. This technique is called as τ_{14}^{-} (a minus indicates the technique is incorrect).

τ_{14}^{-}: compute equal/equivalent fractions of $\frac{a}{b}$ by adding or subtracting the same positive integers to/from the numerator and the denominator, which amounts to the (wrong) claim that $\frac{a}{b} = \frac{a \pm n}{b \pm n}$.

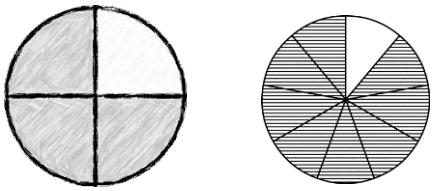
A second type of mathematical tasks is implicit in question (a) as follows:

T_2: given two positive fractions, $\frac{a}{b}$ and $\frac{c}{d}$, decide if they are equal.

There are some possible mathematical techniques to solve such tasks. The first technique is to change both fractions into the same denominator and then compare numerators. We state this technique as τ_{21}.

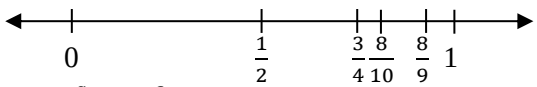
τ_{21}: first change both fractions into an equal denominator and then compare numerators, e.g. $\frac{3}{4} = \frac{3 \cdot 9}{4 \cdot 9} = \frac{27}{36}$ and $\frac{8}{9} = \frac{8 \cdot 4}{9 \cdot 4} = \frac{32}{36}$, then since $27 \neq 32$, we conclude $\frac{3}{4} \neq \frac{8}{9}$.

τ_{22}: represent both fractions into rectangle or circle diagrams (sometimes called pizza diagrams) and compare their areas or sizes, for instance as follows:



τ_{23}: change fractions into decimals to show both fractions are not equal (use one of these techniques: a division algorithm, specific fractions memorised as decimals, or using calculators, computers, or other electronic devices), e.g. $\frac{3}{4} = 0.75$ and $\frac{8}{9} = 0.88$ ..., so $\frac{3}{4} \neq \frac{8}{9}$.

τ_{24}: represent both fractions on a number line, and show that the numbers are positioned at different points, for instance as follows:



τ_{25}: for fraction $\frac{a}{b}$ and $\frac{c}{d}$, divide c by a and d by b, or multiply a by d and b by c, when the results are equal, the fractions are equal.

τ_{26}^{-}: for fraction $\frac{a}{b}$ and $\frac{c}{d}$, compute $b - a$ and $d - c$ or $c - a$ and $d - b$, when $b - a = d - c$ or $c - a = d - b$ concludes that the fractions are equal.

In general, the fundamental law of fractions (for any fraction $\frac{a}{b}$ and any integer $n \neq 0, \frac{a}{b} = \frac{na}{nb}$) and the definition of equivalence of fractions (two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if and only if $ad = bc$) can be the main technological-theoretical justifications for these mathematical techniques for the tasks of types T_1 and T_2, respectively (Long & De Temple, 2003 pp. 351). Multiplicative or proportional reasoning and multiple representations of rational numbers (e.g. fractions, decimals, percentages, or diagrams) can be the other possible technological-theoretical blocks to justify other mathematical techniques.

The type of didactic task of question (b) can be described as follows:

T_1^*: propose strategies to help pupils to solve a task of type T_1.

A common didactic technique to solve the task of type T_1^* is to simply explain a correct mathematical technique for the tasks of type T_1 or even T_2. For instance, a teacher shows to pupils how to find an equal fraction of $\frac{3}{4}$ by multiplying 2 to the numerator and to the denominator to get $\frac{6}{8}$ (τ_{11}). This didactic technique is coded τ_{11}^*. Hence, we get four possible different didactic techniques as τ_{11}^*, τ_{12}^*, τ_{13}^* and τ_{14x}^* corresponding respectively to $\tau_{11,}$ τ_{12}, τ_{13} and τ_{14}^- (adding x to represent a didactic technique based on an incorrect mathematical technique). Meanwhile, the didactic techniques τ_{21}^*, τ_{22}^*, τ_{23}^* τ_{24}^*, τ_{25}^*, and τ_{26x}^* correspond respectively to $\tau_{21,}$ τ_{22}, τ_{23}, τ_{24}, τ_{25}, and τ_{26}^-. There are also other possible didactic techniques that some of them can be variants of those techniques (coded by adding a letter):

τ_{12a}^*: represent both fractions into one (two different) number line(s) and show pupils that both fractions stand in the same point.

τ_{15}^*: present and explain the mathematical task T_1 into an appropriate contextual or real life situation, e.g. a task related to share pizzas or cakes.

τ_{15x}^*: presents inappropriate contextual or real life situation for the mathematical task T_1 or suggest to teach pupils through a contextual or real life situation but do not know how to do that. (adding *x* to represent an inappropriate didactic technique)

τ_{16}^{*}: use a simple fraction such as $\frac{1}{4}$ and $\frac{1}{2}$ as a starting point to explain a mathematical technique for the task of type T_1.

τ_{17}^{*}: organize a class discussion of different pupils' answers.

τ_{22x}^{*}: show to pupils that both fractions are not equal through wrong rectangles or circle diagram representations.

τ_{27}^{*}: show a counter example to the claim that adding the same numbers to the numerator and denominator give an equal fraction, because then you should also be able to subtract the same ("going back"), but $\frac{3-2}{4-2} \neq \frac{3}{4}$ is obvious.

Actually, the lists of techniques mentioned above are not exclusive. During their discussion, PsETs could suggest other possible techniques or even technologies. In any case, they might not offer a model for the technological and theoretical discourses upon working with the HTT.

5. A methodological approach to empirical studies

After we designed and analysed the HTTs, we conducted the first empirical study in January 2016, with 11 pre-service teachers (prepared to teach pupils at grade 4 to 10 or approximately age 9 to 15) at the Metropolitan University College (MUC). From February to March 2016, we tried the HTTs with 32 PsETs (prepared to teach grade 1 to 6 or approximately age 6 to 12) from the Elementary School Teacher Education study program, Riau University, Indonesia. Finally, we tested the HTTs from December 2016 to March 2017, with 20 PsETs (also prepared to teach grade 1 to 6 or approximately age 6 to 12) from other three university colleges in Denmark. Most of them worked in pairs except for one group consisting of three pre-service teachers.

During the first data collection, we focus on whether pre-service teachers could understand and solve the HTTs, and what constraints they have when they are working individually and in pairs. In general, they were able to solve all HTTs except for the HTT 5 (see appendix 1) that was really challenging. For instance, when we had a short conversation after the test, a pre-service teacher said that they could solve the mathematical task, but they lacked didactic techniques such as to explain and justify the mathematical techniques to pupils. They were not able to construct an appropriate situation or context related to that mathematical task. It seems that the HTTs are relevant to study teachers' knowledge since they have various levels of difficulty. The other obstacle was the number of pre-service teachers in a group. Since there were three pre-service teachers in one group, the group had more difficulties to share their ideas. For instance, when a pre-service teacher shared a technique to solve a task, another pre-service teacher sometimes seemed to dismiss it, by proposing another technique. They easily moved from one technique to another before they had developed a clear idea for the previous technique. Therefore, we decide for the main study that PsETs should work in pairs.

The main data collected from the work of Indonesian and Danish PsETs consist of written answers and video recordings. The written answers can be coded directly based on mathematical praxeologies, specifically the techniques, while the video recordings are transcribed using NVivo version 11.0.0 (1497) computer programming. When we find difficulties to code some texts into a specific technique, we plan to discuss them with other experts in this area. The mathematical and didactic praxeologies discussed by both parties will be compared qualitatively. We also consider giving attributed points to the answers, similar to what was done

by Durand-Guerrier et al., (2010): 0 point for an inappropriate technique which could not support pupils learning process; 1 point for a reasonable technique which might support pupils learning process but lack of reasoning; and 2 points for an appropriate technique which involves adequate justifications of the techniques. Then, we will compare the points obtained by pre-service elementary teachers in Indonesia and Denmark. We expect that these results could provide overall trends related to the research questions. Finally, the levels of didactic codetermination will be used to explain similarities and differences between Danish and Indonesian school systems in relation to the teaching of fractions.

6. Summary

In this paper, we have explained how the ATD can be used to study PsETs' mathematical and didactic knowledge about rational numbers. The idea is to use a specific kind of items, the HTTs, and to construct PRMs that predict PsETs' mathematical and didactic techniques when they solve the HTTs. In this paper, we presented and analysed HTT 1 in details, as we consider that this suffices to give readers an impression of how such items can be analysed and used. We have mainly presented the analysis at the level of techniques, but in analysing actual PsETs' work, a more explicit analysis of technology and theory evidenced in that work will be of capital importance. We have only outlined the general tools for such an analysis (e.g. in Figure 3).

Acknowledgment

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Appendix 1: Hypothetical teacher tasks

HTT2 (Comparing decimals)

Fifth-grade pupils are asked to compare the size of 0.5 and 0.45.

Some pupils answer that 0.45 is greater than 0.5, while others say that 0.5 is greater than 0.45.

a. Analyse the pupils' answers. Explain your ideas to handle the situation in this class? (*to be solved individually in 3 minutes*)
b. How do you use this situation to further the pupils' learning? (*to be discussed and solved in pair within 5 minutes*)

HTT3 (Denseness of rational numbers)

You first ask fifth-grade pupils to discuss how many numbers there are between $\frac{2}{5}$ and $\frac{4}{5}$, and how many numbers there are between 0.4 and 0.8.

Then, they say that there is only one number between $\frac{2}{5}$ and $\frac{4}{5}$ namely $\frac{3}{5}$; they also say 3 numbers between 0.4 and 0.8.

a. How do you interpret this claims? (*to be solved individually within 3 minutes*)
b. Explain your ideas to teach these pupils? (*to be discussed and solved in pairs within 5*)

HTT4 (Addition and subtraction of fractions)

You ask sixth-grade pupils to solve $\frac{2}{3}+\frac{1}{2}=\cdots$, and $\frac{4}{7}-\frac{1}{3}=\cdots$

a. How do you solve these problems? (*to be solved individually within 3 minutes*)

You find that many pupils add and subtract fractions in the following way $\frac{2}{3}+\frac{1}{2}=\frac{3}{5}$, and $\frac{4}{7}-\frac{1}{3}=\frac{3}{4}$.

b. How do you interpret the pupils' methods? (*to be solved individually within 3 minutes*)
c. What strategies can you propose to teach these pupils? (*to be discussed and solved in pair, 5 minutes*)

HTT5 (Multiplication and division of decimals, using calculators)

As a teacher, you ask pupils to compute the following as homework:

a) $0.25 \cdot 8 = \cdots$, b) $8 \div 0.25 = \cdots$.

At the next meeting in the class, a pupil notices that when he enters $0.25 \cdot 8$ into a calculator, the answer is smaller than 8, and when he enters $8 \div 0.25$, the answer is bigger than 8. He is confused with this answer and thinks that the calculator must be broken.

b) What can you do to help such pupils understand this result? (*discuss in pairs in 8 minutes, use the space below if necessary, and write your ideas to support the discussion*)