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What algebraic knowledge may not be learned with CAS -a praxeological analysis of Faroese exam exercises

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Abstract. We are interested in the potentials and pitfalls of introducing computer algebra systems in lower secondary school, investigating the case of the Faroese Islands. In order to identify what algebraic knowledge is tested in the final written exam in mathematics after the ninth grade, and how this would change if computer algebra systems were allowed at that exam, we analyse all exam exercises from the past 10 years in terms of the techniques required to solve the exercises both with and without symbolic tools. The comparison suggests that fundamental algebraic structures may not be learned if students consistently use computer algebra systems for the tasks given in the exam.

Résumé. Nous sommes intéressés par les potentiels et les risques liés à l'introduction de logiciels symboliques au niveau du collège, dans le cas de l'école publique des îles Féroé. Afin d'identifier quelles sont les connaissances algébriques testées à l'examen écrit en mathématiques à la fin de la neuvième année, et comment cela pourrait changer si les logiciels symboliques étaient autorisés à cet examen, nous avons analysé tous les exercices de l'examen des 10 dernières années en termes de techniques nécessaires pour résoudre les exercices avec et sans logiciels symbolique. La comparaison suggère que certaines structures spécifiques et fondamentales de l'algèbre ne seraient peut-être pas apprises si les étudiants utilisent de façon consistante des logiciels symboliques pour les tâches rencontrées à l'examen.

Introduction

The students of lower secondary school (grade 7 - 9) are introduced to the formalism of algebra by syntactically-guided manipulation, such as factorization, or simplification of simple algebraic expressions, or solving a first order equation (Kaput & Blanton, 2001; Måsøval, 2011). These techniques play a crucial role in the students learning of mathematics; through these techniques, the students learn the fundament of algebraic structures, work with and manipulations of these. The techniques are later used to further study mathematics including formalistic algebra and algebra as a tool for generalization, modelling and problem solving. How will the implementation of CAS in lower secondary school influence these fundamental techniques?

To study the potential influence of CAS on traditional algebra exercises we have chosen examine how the use of CAS applies to standard exam exercises. In the literature, two studies consider this problem: Flynn and McCrae (2001); Kokol-Voljc (1999). The studies conclude that for traditional exercises mathematics is devaluated to some extent. However, the studies do not give an explicit and exact answer to what mathematical knowledge is no longer present. Such answers are sought, in the present studies, through praxeological analysis.

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1. Notes on praxeology

We assume that the reader is familiar with the concept of praxeologies, a model suggested by the Anthropological Theory of the Didactic to study human activity (Bosch & Gascón, 2014; Chevallard, 1999).

We will adopt the notation *T* for types of tasks and τ for techniques. Furthermore we will distinguish between techniques in a paper-and-pencil environment and in a CAS environment and will refer to them as non-instrumented techniques and instrumented techniques respectively (Trouche, 2005).

Techniques change over time as students' activities become more routinized. When introduced to the formalism of manipulation of an equation, the technique of solving $3x - 1 = 2$ would be to first add 1 on both sides of the equation: $3x - 1 = 2 \rightarrow 3x - 1 + 1 = 2 + 1$. Later on, when students are acquainted with solving this type of tasks, the technique of adding the same constant to both sides of the equation will change into a technology. Instead the technique regrouping the constants on one side will emerge: $3x - 1 = 2 \rightarrow 3x = 2 + 1$ and even later on directly merging the constants on one side: $3x - 1 = 2 \rightarrow 3x = 3$.

For our praxeological model we will consider the techniques on elementary level such as adding a constant to both sides of the equation. We will define these as techniques that are described by and based on definitions and axioms. For example, a technique could be to apply the distributive field axiom rewriting the expression $3x^2 + 21$ into $3(x^2 + 7)$.

We can now, with the notions of praxeology formulate our research goals and questions:

- What are the algebraic non-instrumented techniques of lower secondary school?
- What will happen to the algebraic non-instrumented techniques in a CAS environment?
- How are the algebraic non-instrumented techniques related to the instrumented techniques?

2. Context and rational

Our data material is the set of exercises from the last ten years of the final written exam of lower secondary school on the Faroe Islands. We see the exam exercises as a representation of the minimal requirements of lower secondary school students.

From the set of exams we consider only a subset of exercises. We study the exercises in which variables or unknowns are used, either in manipulation of algebraic expressions, solving of equations or inequalities, in modelling or problem based exercises. This means that several exercises pose a geometric problem but are solved with algebraic techniques.

First all selected exercises were solved by the author using paper-and-pencil, and all solutions have been documented. The solutions for the exercises were made with techniques supposedly known by students of lower secondary school, and thus the technique chosen can be considered as a minimum level of actions required to solve the exercises. In the cases where several different solutions were possible, a ninth grade teacher was consulted or the solution requiring the least number of techniques chosen, and if still undecided a minimum set of techniques were chosen.

Following, the same set of exercises were solved using GeoGebra and the input, the command and the output documented. GeoGebra was chosen as the CAS, since it is the most

frequently used CAS program on the Faroe Islands, based on a questionnaire 2015 (unpublished). A few of the exercises were more easily solved using the geometric environment of GeoGebra. They are therefore not a part of the exercises forming the basis for the development of our praxeological model involving the instrumented techniques.

3. Praxeological reference model

The praxeological model we developed is not only a tool for our study, but also one of the main results for our study in order to answer our research questions. Our praxeological model includes both types of tasks and instrumented and non-instrumented techniques.

3.1. Types of tasks

The first part of the practice block of a praxeology, and what is observable to us, is the types of tasks. The type of tasks is constituted by the form of the tasks.

Though the students of lower secondary school are supposed to operate in the field of real numbers, in our set of selected exercises only the field of the rational numbers was in play.

The types of tasks and following the elementary techniques identified are not exhaustive for 9'TH grade, but what are present in the last ten years of written exams.

Let $T_{\text{solve.eqn}}$ denote the type of tasks of **solving a first order equation**.

Example: A simple example of $T_{\text{solve.eqn}}$, is exercise 18 from 2014: $x + 3 = 24$, a more advanced example of such type of tasks is exercise 6a) from 2013: Solve the equation: $6x - 30 = 3(x - 4)$.

Let $T_{\text{solve.stm}}$ denote the type of tasks of **solving a system of two linear first order equations.**

Example: A standard example of $T_{\text{solve.stm}}$ is exercise 6d) from 2013: Solve the system of equations: $y = -3x - 4\ and\ y = 2x + 6$.

Let $T_{\text{solve.scnd}}$ denote the type of tasks of **solving a second-degree equation** of the form $ax^2 + bx + c = d$, where a, b, c and d are in $\mathbb{N}$.

Example: An example of $T_{\text{solve.scnd}}$ is exercise 7b) from 2012: Solve the equation: $4x^2 - 28x = 0$.

Let $T_{\text{eval.ineql}}$ denote the type of tasks of, given a finite set of given values, **evaluating an inequality** of the form $ax + b \leq c$ where a, b and c are in $\mathbb{N}$.

Example: An example of $T_{\text{eval.ineql}}$ is exercise 45 from 2010: Which of the numbers 2,3,4,5 and 6 are solutions of the inequality: $3x - 2 \leq 10$.

Let $T_{\text{solve.ineql}}$ denote the type of tasks of **solving an inequality** with one variable and constant and coefficients in $\mathbb{N}$.

Example: An example of $T_{\text{solve.ineql}}$ is exercise 6c) from 2011: Solve the inequality $8 + 3x > 2(x - 2)$.

Let $T_{\text{eval.expr}}$ denote the type of tasks of **evaluating an algebraic expression** for given values of the variables.

Example: An example of $T_{\text{eval.expr}}$ is exercise 26 from 2011: $a = -2$ and $b = 4, 3a + 3b =$ _____.

Let $T_{\text{reduce.expr}}$ denote the type of tasks of **reducing an algebraic expression**.

An advanced type of tasks $T_{\text{reduce.expr}}$ is exercise 5a) from 2011: Reduce the expression: $(a + 3b)^2 - (a - 2b)^2$. A simpler example of such is exercise 23 from 2010: $5a - 2b - 4a + 3b =$ _____.

Let $T_{\text{factor.expr}}$ denote the type of tasks of **factoring an algebraic expression**.

Example: An example of $T_{\text{factor.expr}}$ is exercise 6c) from 2008: Put as much as possible outside of brackets: $28x^2 - 14x + 21x^2$.

Let T_{text} denote the type of tasks that begins with a text description of a real world situation. The students are then asked a question in which they should define a variable and relations to information given in the text.

Let T_{geom} denote the type of tasks containing **geometric problem**.

Example: Exercise 4d) from 2013: Are the triangles ABC and DEF similar?

Let T_{other} denote all other of the selected exercises, which do not fall into other types of tasks.

Example: Exercise is 17 from 2014: Mark which of the following expressions have the greatest value for $p = 3$: $p \cdot 4, p^2 + 5, 5p - 4$.

3.2. $T_{\text{reduce.expr}}$ and $T_{\text{solve.eqn}}$

The two most frequent occurring types of tasks are $T_{\text{solve.eqn}}$ and $T_{\text{reduce.expr}}$. We therefore further divide these types of tasks into more fine grained types of tasks. We define the following four types of tasks based on $T_{\text{solve.eqn}}$, due to notational reasons we have introduced the notation $T_{1.1}$, $T_{1.2}$, $T_{1.3}$ and $T_{1.4}$:

Type of tasks	Description
$T_{1.1}$	Solve first order equation of the form $x + a = b$, where a and b are non-zero numbers in $\mathbb{N}$.
$T_{1.2}$	Solve first order equation of the form $cx + a = b$, where a, b and c are non-zero numbers in $\mathbb{N}$.
$T_{1.3}$	Solve first order equation of the form $d(cx + a) = b$, where a, b, c and d are non-zero numbers in $\mathbb{N}$, or d of the form $\frac{1}{e}$ where e is a non-zero number in $\mathbb{Z}$.
$T_{1.4}$	Solve first order equation of different form with constants in $\mathbb{Q}$.

Table 1. Types of tasks within $T_{\text{solve.eqn}}$

Example: An example of a task of type $T_{1.4}$ is exercise 5b) from 2011: Solve the equation $\frac{x}{2} + 3x = 7$.

For the type of tasks $T_{\text{reduce.expr}}$ we get the following five types of tasks, for notational reasons we have introduced the notation $T_{7.1}$, $T_{7.1}$, $T_{7.3}$,..., $T_{7.6}$:

Type of tasks	Description
$T_{7.1}$	Reduce an algebraic expression of the form $ax + by + c + dx + ey + f$, where a, b, c, d, e and f are numbers in $\mathbb{N}$.
$T_{7.2}$	Reduce an algebraic expression of the form $a(bx + cy) + dy$, where a, b, c and d are numbers in $\mathbb{N}$.
$T_{7.3}$	Reduce an algebraic expression of the form $ax + y + b(cy + s) + d(ey + t)$, where a, b, c, d and e are numbers in $\mathbb{N}$ and s and t are numbers in $\mathbb{Q}$.
$T_{7.4}$	Reduce an algebraic expression containing a squared variable with constants in $\mathbb{N}$.
$T_{7.5}$	Reduce an algebraic expression of the form $x^n x^m x^l y^s y^t$, where n, m, l and t are numbers in $\mathbb{N}$ and n and m are different from zero.

$T_{7.6}$	Reduce an algebraic expression of other form.

Table 2. Type of tasks within $T_{\text{reduce.expr}}$

Example: An example of an exercise of the type $T_{7.6}$ is exercise 6a) from 2008: Reduce the expression $\frac{4+3a}{3a} - \frac{2a+a^2}{3a} + \frac{a}{3}$.

3.3. Non-instrumented techniques

To reduce the expression $3a + 4b + a - 2b$ we group terms by applying the additive commutative axiom and the distributive axiom from right to left:

$$3a + 4b + a - 2b \rightarrow (3 + 1)a + (4 - 2)b \rightarrow 2a + 2b.$$

We are only interested in the techniques including letters, thus we do not consider the arithmetic techniques such as rewriting $4 - 2$ into 2 using the ring axioms to rewrite $(((1 + 1) + 1) + 1) - (1 + 1)$ into $(1 + 1)$.

3.4. Non-instrumented techniques based on the field axioms

A field is a fundamental algebraic structure consisting of a set of elements, including a neutral and zero-element, together with two compatible operations satisfying the field axioms. In our study we will be referencing the following axioms:

- The distributive axiom: $a(b + c) = ab + ac$,
- The additive inverse axiom: $a + (-a) = 0$,
- The multiplicative inverse axiom: $aa^{-1} = 1$, whenever a $\neq 0$,

for all a, b and c in the field. The students of lower secondary school operate on the field of real polynomials in two variables $\mathbb{R}[x, y]$.

For the distributive axiom we will not distinguish between the right and the left distributive axiom, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ respectively. Nevertheless we will distinguish between applying the axiom from the left to the right or from the right to the left, $a(b + c) \rightarrow ab + ac$ and $ab + ac \rightarrow a(b + c)$ respectively.

> Let $\tau_{\text{right.left}}$ denote the technique of applying the distributive field axiom from the right to the left. That is, an expression of the form $ab + ac$ is rewritten into the form $a(b + c)$.

Example: The technique, $\tau_{\text{right.left,}}$ is used such as in exercise 30 from 2013: Reduce the expression: $3a - 2b - 6a + 5b$. As part of the solution the students will have to apply $\tau_{\text{right.left}}$ in order to arrive at $(3 - 6)a + (-2 + 5)b$. $\tau_{\text{right.left}}$ is also used such as in exercise 6c) from 2013: put outside of brackets: $6x^2 + 21$. Here the students will have to apply the technique $\tau_{\text{right.left}}$ to arrive at $3(2x^2 + 7)$.

> Let $\tau_{\text{left.right}}$ denote the technique of applying the distributive axiom from left to the right. That is, an expression of the form $a(b + c)$ is rewritten into the form $ab + ac$.

Example: The technique, $\tau_{\text{left.right,}}$ is used in exercises such as exercise 31 from 2013: Reduce the expression: $2(-2a + b) + 7a$. Here the technique $\tau_{\text{left.right}}$ is applied in order to arrive at the expression $2(-2)a + 2b + 7a$. In other types of task the technique, $\tau_{\text{left.right}}$, is used nine times such as in 5a) from 2011: Reduce the expression: $(a + 3b)^2 - (a - 2b)^2$ to arrive at the expression $a^2 + 3ab + 3ab + 9b^2 - a^2 + 2ab + 2ab - 4b^2$.

Let $\tau_{add.inv}$ denote the technique of applying the additive inverse field axiom from left to right. That is, an expression of the form $a + (-a)$ is rewritten into 0.

Example: This technique is used to solve an exercise such as exercise 24 from 2010: $2(-2a + b) + 4a$. The technique computes the following step: $-4a + 2b + 4a \rightarrow 2b$. Thus, $\tau_{add.inv}$ substitutes the technique $\tau_{right.left}$ in cases where the coefficients are additive inverses of each other.

3.5. Non-instrumented techniques based on the axiom of substitution

To substitute a variable by a number in any relation is often referred to by the substitution property in introductory courses at universities. Further, the following was found at a scholarly discussion forum (theage, 2015):

> If $\phi(x)$ is a statement and if $\phi(a)$ is true and $a = b$ is true, then $\phi(b)$ is true. An example of this axiom is if we have the statement $\phi(x)$: x *is red* and if for an object a the statement a is red is true and another object b is identical to a then we can conclude that the object b is red.

By applying the axiom of substitution and introducing functions, we get that if $\phi(x)$ is the statement and $f(x) = f(a)$ then $\phi(a)$ is true. Since $a = b$ it follows from the axiom of substitution that $\phi(b)$ is true and thus $f(a) = f(b)$.

Thus, the non-instrumented techniques of this section can be deduced from the axiom of substitution:

Let $\tau_{add.eqn}$ denote the technique of adding a real number or a variable on both sides of an equation. That is, an expression $a = b$ is rewritten into $a + c = b + c$.

Example: The technique, τ_{add}, is applied in exercises, where the object of the exercise is to find a solution for a first order equation or inequality, such as in exercise 18 from 2014: $x + 3 = 24$. The technique τ_{add} is applied in the following computation $x + 3 - 3 = 24 - 3$.

Let $\tau_{multi.eqn}$ denote the technique of multiplying on both sides of a first order equation with a real number.

Example: The technique, $\tau_{multi.eqn}$, computes the following step $5x = 30 \rightarrow 5x \cdot \frac{1}{5} = 30 \cdot \frac{1}{5}$ in exercise 29 from 2006: $5x = 30$.

Let $\tau_{sub.num}$ denote the technique of substituting a variable with a number in a first order equation or inequality. That is, given an algebraic expression $ax + by$, where a and b are in R, and values s and t for x and y, respectably, then we have the rewriting into $as + bt$.

Example: The technique, $\tau_{sub.num}$, is applied in exercises such as 35 from 2007: $a = -2$, $b = 4$, $-5a - 2b =$ _____ and computes the following step $-5a - 2b \rightarrow (-5)(-2) - 2 \cdot 4$.

Let $\tau_{sub.expr}$ denote the technique of substituting a variable with an algebraic expression. That is, given a system equations $y = ax + b$ and $y = cx + d$ then we have the computation $ax + b = cx + d$.

Example: The technique, $\tau_{sub.expr}$, is applied in exercises of the type where students are asked to find the solution of a system of two linear equations such as exercise 6b) from 2008: Solve the

system of equations: $y = 2x - 1$ and $y = -\frac{2}{3}x + 7$ and computes the following expression $2x - 1 = -\frac{2}{3}x + 7$.

> Let $\tau_{add.ineq}$ denote the technique of adding a real number or a variable on both sides of an inequality. That is, an expression of the form $a \leq b$ is rewritten into $a + c \leq b + c$.

Example: The technique, $\tau_{add.ineq}$, to compute the following step $8 + 3x > 2(2 - x) \rightarrow 8 + 3x - 8 > 2(2 - x) - 8$ in order to solve the exercise 6c) from 2013: Solve the inequality: $8 + 3x > 2(2 - x)$.

Note that the technique, $\tau_{add.ineq}$, does not extend to include multiplying of variables.

3.6. Non-instrumented techniques based on the definition of exponents

Exponentiation of a natural number b to the n'th power is defined by $b^n = b \cdot b \cdots b$ (n times multiplication of b by itself). The following technique is justified based on this definition.

> Let τ_{power} denote the technique of multiplying one variable raised to a power with another variable raised to a power, where both variables are denoted with the same letter. That is an expression of the form $a^n a^m$ is rewritten into a^{n+m}.

Example: The technique, τ_{power}, computes the following step $a^2 \cdot a^3 = a^{2 \cdot 3}$ in exercise 36 from 2014: $a^2 \cdot a^3 \cdot a^{-1} =$ _____. τ_{power} is also applied when doing the rewriting of $b \cdot b$ into b^2 such as in exercise 6 b) from 2005: Reduce the expression: $3(b - 1) - (b + 1)(b - 2) + b^2$.

3.7. Example

To exemplify the non-instrumented techniques defined earlier we consider again exercise 7a) from 2005: Solve the system of equations: $y = x + 4$ and $y = -\frac{1}{2}x + 1$:

$x + 4$	=	$-\frac{1}{2}x + 1$	($\tau_{sub.expr}$)
$x + 4 + \frac{1}{2}x$	=	$-\frac{1}{2}x + 1 + \frac{1}{2}x$	(τ_{add})
$\left(1 + \frac{1}{2}\right)x + 4$	=	1	($\tau_{right.left}$, $\tau_{add.inv}$)
$\frac{3}{2}x + 4$	=	1	
$\frac{3}{2}x + 4 - 4$	=	$1 - 4$	(τ_{add})
$\frac{3}{2}x$	=	-3	
$\frac{2}{3}\frac{3}{2}x$	=	$\frac{2}{3}(-3)$	(τ_{multi})
x	=	-2	
x	=	$-2 + 4$	($\tau_{sub.num}$)
x	=	2	

Table 3. Exercise 7a) from 2005

Note that the techniques are disjoint and that they do not describe every elementary step in order to solve an exercise. Instead, they aim at describing every elementary step involving a letter.

3.8. Instrumented techniques

We categorize the instrumented techniques based on the command used, the type of input and the type of output. These criteria are based on GeoGebra, thus if one was to use e.g. Maple instead, one might use the criteria only of the command used or even a class of commands. GeoGebra is a piece of software designed for teaching and learning mathematics and science from the level of primary school to university. In the GeoGebra window for conducting CAS work there are twelve commands. Relevant for our level of mathematics and the exercises are the four commands: Evaluate, Factor, Expand and Solve. We note that we did not need to use the command Substitute due to the effectiveness of other commands and that substitution of a variable with a number is done, not by a command, but when entering the expression, equation or inequality such as in exercise 35 from 2007: $a = -2$, $b = 4$, $-5a - 2b =$ _____.

Let $\tau_{\text{solve. eqn}}$ denote the technique of using the command **Solve** on a first order equation.

Example: The technique, $\tau_{\text{solve. eqn}}$, is used in exercises such as 18 from 2014: $x + 3 = 24$. The input is $x + 3 = 24$, the command Solve giving the output Solve: $\{x = 21\}$.

Let $\tau_{\text{solve. ineqn}}$ denote the technique of the command **Solve** on a first order inequality.

Example: The technique, $\tau_{\text{solve. ineqn}}$, is used in exercises such as exercise 6c) from 2013: Solve the inequality: $8 + 3x > 2(2 - x)$. The input is $8 + 3x > 2(2 - x)$, the command Solve giving the output $\left\{x > \frac{(-4)}{5}\right\}$.

Let $\tau_{\text{solve.system}}$ denote the technique of using the command **Solve** with an input of a system of two linear first order equations.

Example: The technique, $\tau_{\text{solve.system}}$, is used in exercises such as 7a) from 2005: Solve the system of equations: $y = x + 4$ and $y = -\frac{1}{2}x + 1$. The exercise is solved by entering each linear equation followed by pressing enter, such that GeoGebra stores each linear equation as an equation. Then both equations need to be highlighted before pressing the button "Solve" resulting in the output: Solve: $\{x = -2, y = 2\}$.

Let $\tau_{\text{eval.num}}$ denote the technique of using the command **Evaluate** with an input of only a numerical expression.

Example: This technique, $\tau_{\text{eval.num}}$, is used in exercises such as 35 from 2007: $a = -2$, $b = 4$, $-5a - 2b =$ _____, where the substitution of the variables with numbers are completed while entering the expression $-5 * (-2) - 2 * 4$. Note that the technique is not used in exercises such as 38 from 2008: Which of the numbers $-2, -1, 0, 1, 2, 3\ and\ 4$ are solutions for the inequality: $4x - 2 < 2$ because it would require the technique seven times, and the command Solve produces the solution with less effort.

Let $\tau_{\text{eval.expr}}$ denote the technique of employing the command **Simplify** with an input of an algebraic expression.

Example: The technique, $\tau_{\text{eval.expr,}}$ is employed in exercises such as 31 from 2013: Reduce the expression: $2(-2a + b) + 7a$. The exercise is solved by entering the expression followed by the command "Symbolic Evaluation" which results in the output $\rightarrow 3a + 2b$.

Let τ_{factor} denote the technique of employing the command **Factor** with an input of an algebraic expression.

Example: The technique, τ_{factor}, is used in exercises such as 6b) from 2013: Put outside of brackets: $6x^2 + 21$. The exercise is solved by entering the expression followed by the command "Factor", which results in the output: " Factor: $3(2x^2 + 7)$".

Let $\tau_{brackets}$ denote the technique of **inserting brackets** into an expression in order for CAS to correctly read the expression.

Example: The technique, $\tau_{brackets}$,is used in exercises such as 7a) from 2014: Solve the equation: $\frac{2x-4}{5} = 6$. In order for GeoGebra to correctly read and distinguish between the numerator and denominator brackets must be inserted: $(2x - 4)/5 = 6$.

Let $\tau_{interpret}$ denote the technique of **interpreting the output**.

Example: The technique, $\tau_{interpret}$, is used in exercises such as 45 from 2012: Which of the numbers -2, 0, 2, 6 and 7 are solutions for the inequality: $5x - 2 \leq 10$, where GeoGebra returns the output " Solve: $\left\{\frac{12}{5} \geq x\right\}$". The student must then further interpret the output from GeoGebra in order to reach a solution for the exercise.

In all of the exercises, a solution can be reached with only one technique as it is necessary to employ only one command in order to solve an exercise.

4. Analysis and results

In this section we will give a short overview of the quantitative result of our praxeological reference model on the selected exercises, followed by establishing relations between non-instrumented and instrumented techniques.

4.1. Types of tasks

For the selected exercises in our study, we get the following distribution of types of tasks:

Type of tasks	Frequency
$T_{\text{solve.eqn}}$	25
$T_{\text{solve.stm}}$	7
$T_{\text{solve.scnd}}$	5
$T_{\text{eval.ineql}}$	9
$T_{\text{solve.ineql}}$	1
$T_{\text{eval.expr}}$	8
$T_{\text{reduce.expr}}$	30
$T_{\text{factor.expr}}$	2
T_{text}	18
T_{geom}	4
T_{other}	1

Table 4. Frequency of types of tasks

We see that the most frequent occurring types of tasks are $T_{\text{solve.eqn}}$, $T_{\text{reduce.expr}}$ and T_{text} constituting more than 66% percent of the exercises.

By considering types of task within the $T_{\text{solve.eqn}}$ we get the following distribution:

Type of tasks	Frequency
$T_{1.1}$	5
$T_{1.2}$	12
$T_{1.3}$	5
$T_{1.4}$	3

Table 5. Frequency of types of tasks $T_{1.1}$, $T_{1.2}$, $T_{1.3}$ and $T_{1.4}$

By considering types of task within $T_{\text{reduce.expr}}$ we get the following distribution:

Type of tasks	Number of occurrences
$T_{7.1}$	12
$T_{7.2}$	10
$T_{7.3}$	2
$T_{7.4}$	3
$T_{7.5}$	2
$T_{7.6}$	2

Table 6. Frequency of the types of tasks $T_{7.1}$, $T_{7.2}$, $T_{7.3}$, $T_{7.4}$, $T_{7.5}$ and $T_{7.6}$

4.2. Structure of types of tasks

When solving the tasks using paper and pencil several of the types of tasks are relational. For example, the task $T_{\text{solve.stm}}$ includes the task $T_{\text{solve.eqn}}$ and $T_{\text{solve.eqn}}$ can include the task $T_{\text{reduce.expr}}$, thus we can draw the follwoing diagram of relations between types of tasks when solving using paper and pencil, see Table 7.

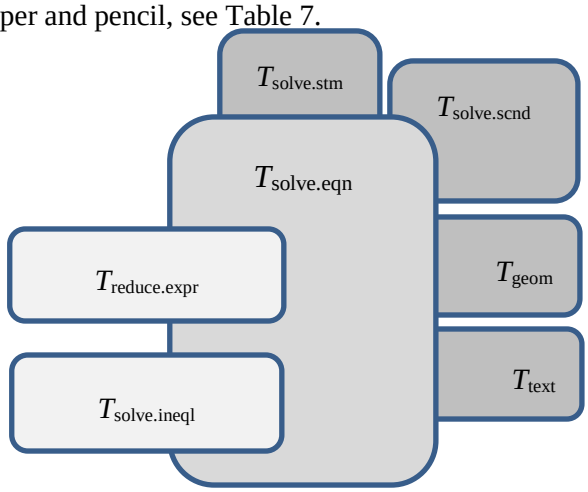

Table 7. Relation between tasks in a non-instrumented environment.

However, when we solve the same set of exercises using GeoGebra only two types of tasks are relational, the T_{text} and the $T_{\text{solve.eqn}}$. Thus the relation of traditional algebraic exercises is considerable weakened when solved using GeoGebra.

4.3. Techniques

Applying our praxeological model for non-instrumented techniques we get the following distribution of non-instrumented techniques:

Non-instrumented technique	Number of uses in solutions
$\tau_{right.left}$	59
$\tau_{left.right}$	49
$\tau_{add.inv}$	23
τ_{add}	47
τ_{multi}	46
$\tau_{sub.num}$	98
$\tau_{sub.expr}$	7
τ_{power}	15
τ_{text}	11

Table 8. Frequency of non-instrumented techniques

Furthermore, we get the following distribution of number of non-instrumented techniques used per exercise:

Number of non-instrumented techniques per exercise	Frequency
1	17
2	44
3	20
4	7
5	5
6	8
7	4
8	1
9	3
10	3
12	1

Table 9. Frequency of number of non-instrumented techniques per exercise

It follows from the table that most exercises require a composition of non-instrumented techniques. If we consider the praxeology, then the technology is the explanation for and justification of techniques. Thus in exercises where a composition of two or more elementary atomic techniques are required to reach a solution, then a richer technology is present in order to successfully choose the non-instrumented techniques.

Applying our model for the instrumented techniques, we get the following distribution of instrumented techniques:

$\tau_{solve.\ eqn}$	$\tau_{solve.\ ineqn}$	$\tau_{solve.system}$	$\tau_{eval.num}$	$\tau_{eval.expr}$	τ_{factor}
48	10	7	8	30	2

Table 10. Frequency of instrumented techniques

Furthermore, in 105 out of 110 exercises only one of the instrumented techniques was necessary to obtain the solution. In 4 of the remaining 5 exercises the geometric environment of GeoGebra was preferable to obtain the solution for the exercises and has therefor been left out. The last exercise we consider an exception, and we are uncertain of what instrumented technique that would most effortlessly solve the exercise.

4.4. Relations between non-instrumented and instrumented techniques

In this section we will present our study of the relations between the non-instrumented techniques and the instrumented techniques. We have selected two different approaches to investigate this relation. The first investigation is a direct correspondence between the non-instrumented techniques and the instrumented techniques. The second investigation considers the relations between the non-instrumented techniques and the instrumented techniques via types of tasks to get a more explicit relation that relies on exercises.

4.5. Relations between non-instrumented and instrumented techniques through definitions

In our first analysis we begin with the non-instrumented techniques and determine what instrumented technique(s) are capable of accomplishing the same action as the non-instrumented technique. Thus, if considering applying the distributive field axiom, what instrumented techniques could return the same result?

Consider the non-instrumented technique $\tau_{\text{right.left}}$, equivalent to the action of applying the distributive field axiom from the right to the left: $ab + ac = a(b + c)$. The same result can be achieved by applying the instrumented technique τ_{factor}. However none of the other instrumented techniques yields the output $a(b + c)$. For the non-instrumented technique $\tau_{\text{left.right}}$, we establish a relation to the instrumented technique $\tau_{\text{eval.expr}}$, with similar method.

For the non-instrumented techniques $\tau_{\text{add.inv}}$ and τ_{power} corresponding respectively to the technique of applying the additive inverse axiom from the left to the right and applying the definition of exponentiation, we reach the same results with applying the instrumented technique $\tau_{\text{eval.expr}}$.

For the non-instrumented techniques τ_{add}, τ_{multi}, $\tau_{\text{sub.num}}$, $\tau_{\text{sub.expr}}$ τ_{text} we are not able to obtain identical outcome with any of our instrumented techniques from our praxeological reference model. However GeoGebra still accommodates methods and commands to carry out these non-instrumented techniques. Furthermore, other methods and commands not included in our praxeological reference model will be able to execute the same actions as the previous mentioned non-instrumented techniques. This means that though GeoGebra affords instrumented techniques to accomplish non-instrumented techniques, because of the types of tasks and the presence of other instrumented techniques, they are not used.

We get the following visualization based on a direct relation between non-instrumented fundamental techniques and instrumented techniques:

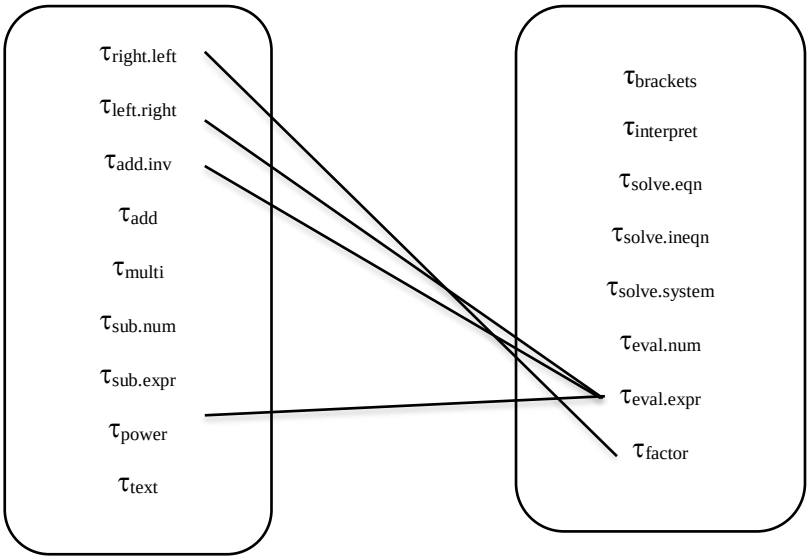

Table 11. Relations by definition.

4.6. Relation of non-instrumented and instrumented techniques through types of task

Due to the relations (or lack of) between instrumented and non-instrumented techniques our second analysis relates the non-instrumented and the instrumented techniques through types of task. By looking at exercises within $T_{\text{solve.eqn}}$ we determine the relation between the instrumented and non-instrumented techniques. Since only one instrumented technique is applied per exercise, one could also see the relation as the relation between a composition of non-instrumented techniques to an instrumented technique.

Consider the type of tasks $T_{\text{solve.eqn}}$. All exercises within $T_{\text{solve.eqn}}$ can be solved applying the instrumented technique $\tau_{\text{solve. eqn}}$. Regarding the non-instrumented techniques, we get the following relations between types of task and series of non-instrumented techniques for $T_{1.1}$, $T_{1.2}$ and $T_{1.3}$:

$T_{1.1}$	⟺	(τ_{add})
$T_{1.2}$	⟺	$(\tau_{\text{add}}, \tau_{\text{multi}})$
$T_{1.3}$	⟺	$(\tau_{\text{left.right}}, \tau_{\text{add}}, \tau_{\text{multi}})$ $(\tau_{\text{multi}}, \tau_{\text{add}}, \tau_{\text{multi}})$

Table 12. Relation between types of task and non-instrumented techniques

For the $T_{1.3}$ we have two cases of series of non-instrumented techniques. The series is dependent on whether the number d is written as a fraction or a whole number, in the expression $d(cx + a) = b$, where a, b and c are non-zero numbers in $\mathbb{N}$.

For $T_{1.4}$ we have less uniformity of the order and types of the non-instrumented techniques applied, nonetheless all exercises in $T_{1.4}$ can be solved by applying a composition of the non-instrumented techniques: $\tau_{\text{right.left}}$, $\tau_{\text{left.right}}$, τ_{add}, and τ_{multi}.

Thus within $T_{\text{solve.eqn}}$ we get that a composition of the non-instrumented fundamental techniques $\tau_{\text{right.left}}$, $\tau_{\text{left.right}}$, τ_{add}, and τ_{multi} is replaceable with the instrumented technique $\tau_{\text{solve.eqn}}$. We also get that the instrumented technique $T_{\text{solve.eqn}}$ can replace several different compositions of non-instrumented techniques.

By similar analysis of non-instrumented and instrumented techniques via types of tasks, except for T_{text}, we get that any composition of non-instrumented techniques is replaceable with an instrumented technique. But also that one instrumented technique can replace several different compositions of non-instrumented techniques. Furthermore we see that one instrumented technique can solve several different types of task, which is not the case with non-instrumented techniques.

5. Conclusion and reflection

In section 5, we observe that direct relations between non-instrumented and instrumented techniques via definitions can, for some cases of non-instrumented techniques, not be established. Furthermore, we observe that the instrumented technique $\tau_{eval.expr}$ can replace all of the non-instrumented techniques $\tau_{left.right}$, $\tau_{add.inv}$ and τ_{power}. This means, that with the current exercises within the domain of algebra, it is not possible to distinguish between applying the distributive field axiom, the additive inverse field axiom or applying the definition of exponents when using GeoGebra.

Furthermore when considering relations between non-instrumented and instrumented techniques through types of task, we saw that the four series of non-instrumented techniques: (τ_{add}), (τ_{add}, τ_{multi}), ($\tau_{left.right}$, τ_{add}, τ_{multi}) and (τ_{multi}, τ_{add}, τ_{multi}) can all be replaced by the instrumented technique $\tau_{solve.eqn}$. Therefore, it is not possible to explicitly distinguish what series of non-instrumented techniques the instrumented technique $\tau_{solve.eqn}$ is substituting.

The conclusion of section 5 being that it is not possible, when using GeoGebra on traditional algebra exercises, to distinguish between individual non-instrumented techniques or distinguishing between different series of non-instrumented techniques.

In addition, we consider the relation among the types of tasks. The relation between the types of tasks are considerable weaker when solving using GeoGebra, compared to paper and pencil.

But what occurs? One type of exercise, when solved in the CAS environment, causes a new technique to emerge: having to determine the intersection of two sets of numbers. For example exercise 48 from 2006: Which of the numbers -3, -2, -1, 0, 1, 2 and 3 are solutions for the inequality: $2x - 3 > -2$, where students applying the instrumented technique τ_{10} to the given inequality and get the output: Solve: $\left\{x > \frac{1}{2}\right\}$. The students then have to find the intersection of the set {-3, -2, -1, 0, 1, 2, 3} and the set $[\frac{1}{2}; \infty[$.

Using CAS does not exclude the presence of non-instrumented techniques as seen in several results from the literature (Hitt & Kieran, 2009; Lagrange, 2005; Pierce, 2001). The non-instrumented techniques might not be part of the praxis, but they can be part of the logos for solving an exercise. Thus it becomes a question of task design.

We suggest that more work on the transition to and interplay between non-instrumented and instrumented environments are necessary such as (Chaachoua, 2010).

With the current algebraic praxeology one non-instrumented technique was unaffected by the instrumented techniques: τ_{text} present in the task type of T_{text}. Thus the future of algebra in lower secondary schools might lie as a tool in modelling activities that goes across the sectors of mathematics and as a process of algebraization (Bosch, 2012).

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